Comm. Korean Math. Soc. **12** (1997), No. 3, pp. 797–812

ERROR BOUNDS FOR GAUSS-RADAU AND GAUSS-LOBATTO RULES OF ANALYTIC FUNCTIONS

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ABSTRACT. For analytic functions we give an expression for the kernel K_n of the remainder terms for the Gauss–Radau and the Gauss–Lobatto rules with end points of multiplicity r and prove the convergence of the kernel we obtained. The error bound are obtained for the type $|R_n(f)| \leq \frac{1}{2\pi} l(\Gamma) \max_{z\in\Gamma} |K_n(z)| \max_{z\in\Gamma} |f(z)|$, where $l(\Gamma)$ denotes the length of contour Γ.

1. Introduction

Gaussian quadrature formulae for special functions, especially analytic functions with Chebyshev weight functions, have been known for a long time. In this paper we study the kernels and prove the convergence and error bounds for the remainder terms of Gauss–Radau and Gauss–Lobatto rules with end points of multiplicity r. It is well–known that the remainder term of Gaussian quadrature can be expressed in terms of contour integral representation. Gautschi and Varga [4] studied the problem to determine where the kernel in the contour integral representation of the remainder precisely attains its maximum modulus along the contours for the Jacobi weight functions. Martin and Stamp [7] derived an explicit expression for kernel K_n by the method of Laurent series expansion. They developed methods for computing the coefficients (in terms of the moments) for the Laurent series of kernel K_n.

Let f be a single–valued analytic function in a domain D which contains $[-1,1]$ and Γ be a closed contour in D surrounding $[-1,1]$. Let the nonnegative weight function $w(x)$ be defined on the interval $[-1,1]$,

Received october 6, 1996. Revised April 10, 1997.

1991 Mathematics Subject Classification: Primary 65D32.

Key words and phrases: Gaussian quadrature, orthogonal polynomial. This work was partially supported by the Dongseo University Research Fund.

where the moments $\int_{-1}^{1} x^k w(x)\,dx$ exist for $k = 0, 1, 2, \cdots$. The Gauss–Radau rule with weight function w and end point -1 of multiplicity r is given by

$$\int_{-1}^{1} f(t)w(t)\,dt = \sum_{\rho=0}^{r-1} k_\rho f^{(\rho)}(-1) + \sum_{\nu=1}^{n} \lambda_\nu f(\tau_\nu) + R_n^R(f), \tag{1.1}$$

where τ_ν are the zeros of the nth degree orthogonal polynomial $\pi_n(\cdot : w^R)$ with respect to the weight function $w^R(t) = (t+1)^r w(t)$ and λ_ν, k_ρ are the interpolatory weights. It is known that the remainder $R_n^R(f)$ is zero whenever f is a polynomial of degree $\le 2n+r-1$, i.e., $R_n^R(f) = 0$ for all $f \in \mathbb{R}_{2n+r-1}$. Similarly, the Gauss–Lobatto rule with weight function w and end points -1 and 1 of multiplicity r is given by

$$\begin{aligned}\int_{-1}^{1} f(t)w(t)\,dt &= \sum_{\rho=0}^{r-1} k_\rho f^{(\rho)}(-1) + \sum_{\nu=1}^{n} \lambda_\nu f(\tau_\nu) \\ &+ \sum_{\rho=0}^{r-1} (-1)^\rho \mu_\rho f^{(\rho)}(1) + R_n^L(f),\end{aligned} \tag{1.2}$$

where τ_ν are the zeros of the nth degree orthogonal polynomial $\pi_n(\cdot : w^L)$ with respect to the weight function $w^L(t) = (t^2-1)^r w(t)$ and k_ρ, λ_ν, μ_ρ are the interpolatory weights and $R_n^L(f) = 0$ for all $f \in \mathbb{R}_{2n+2r-1}$.

When f is an analytic function in a domain D containing $[-1, 1]$ and Γ is a contour in D surrounding $[-1, 1]$, the remainder term $R_n^{R,L}(\cdot)$ can be represented as a contour integral

$$R_n^{R,L}(f) = \frac{1}{2\pi i} \int_\Gamma K_n^{R,L}(z) f(z)\,dz, \tag{1.3}$$

where the kernel $K_n^{R,L}$ is given by

$$K_n^{R,L}(z) = R_n^{R,L}\left(\frac{1}{z - \cdot}\right), \tag{1.4}$$

or

$$K_n^{R,L}(z : w) = \frac{\rho_n^{R,L}(z : w)}{\omega_n^{R,L}(z : w)}, \quad z \in \Gamma. \tag{1.5}$$

Here, $\omega_n(z:w)$ is the polynomial in the form $\omega_n^R(z:w) = (z+1)^r\pi_n(z:w^R)$ for the Gauss-Radau rule, and $\omega_n^L(z:w) = (z^2-1)^r\pi_n(z:w^L)$ for the Gauss-Lobatto rule, and $\rho_n^{R,L}(z:w)$ is defined by

$$\rho_n^{R,L}(z:w) = \int_{-1}^{1} \frac{\omega_n^{R,L}(t:w)}{z-t} w(t)\,dt. \tag{1.6}$$

Superscripts "R" and "L" denote Radau and Lobatto rule, respectively. For details, we refer to [5].

The approximation integration formulas of Gauss-Radau and Gauss-Lobatto rules are of use in the following situations. When we know the value $f(\pm 1) = 0$, or any other known value, the Gauss-Radau formula is useful for solving the ordinary differential equation $y' = f(x, y)$. And the Gauss-Lobatto rule has been applied to the numerical solution of linear differential equations and integral equations.

During the last ten years, an interest in Gauss-type quadrature rules has been intensified, partly because of their potential use in quadrature routines, but also, because of the fascinating mathematical problems they pose. It is expected that such Gauss–type quadrature will be widely used in many applications.

In Section 2 we give an expression for the kernels of Gauss–Radau and Gauss–Lobatto rules. In Section 3 we prove the convergence of the kernel K_n we obtained. In Section 4 we obtain the error bounds for the Gauss–Radau and Gauss–Lobatto rules on the circle and ellipse. Finally, we give an example in Section 5.

2. Kernel Form

The Gauss-Radau rule with (nonnegative) weight function w and end point -1 of multiplicity r is given by

$$\int_{-1}^{1} f(t)w(t)\,dt = \sum_{\rho=0}^{r-1} k_\rho f^{(\rho)}(-1) + \sum_{\nu=1}^{n} \lambda_\nu f(\tau_\nu) + R_n^R(f),$$

where τ_ν's are the zeros of the nth degree orthogonal polynomial $\pi_n(\cdot : w^R)$ with respect to the weight function $w^R(t) = (t+1)^r w(t)$ and λ_ν, k_ρ are the interpolatory weights.

For a weight $w^R(t) = (t+1)^r w(t)$, $t \in [-1,1]$, consider a contour Γ satisfying

$$\Gamma \subset \left\{ z \in \mathbb{C} : \left| \frac{t+1}{z+1} \right| < 1 \quad \text{for all} \quad t \in [-1,1] \right\}. \tag{2.1}$$

This set is equal to

$$\Gamma = \{ z \in \mathbb{C} : |z+1| > 2 \}.$$

Now if we assume that f is an analytic function in a simply connected region D containing Γ, then we obtain the following theorem.

THEOREM 1. *For $z \in \Gamma$, we have the following expression for kernel $K_n^R(z)$*

$$K_n^R(z) = \sum_{k=2n}^{\infty} \frac{\beta_k}{(z+1)^{k+r+1}}, \quad z \in \Gamma \tag{2.2}$$

where β_k can be obtained recursively by

$$\beta_{2n+k} = \sum_{j=0}^{n-1} \frac{a_j}{a_n} (m_{k+n+j}^R - \beta_{k+n+j}) + m_{2n+k}^R, \quad k = 0,1,2,\ldots. \tag{2.3}$$

PROOF. From the equation (1.6) we have

$$\rho_n^R(z) = \int_{-1}^{1} \frac{\omega_n^R(t)}{z-t} w(t)\, dt = \int_{-1}^{1} \frac{(t+1)^r \pi_n(t : w^R)}{z-t} w(t)\, dt.$$

For any $z \in \Gamma$ and $t \in [-1,1]$, the sum $\sum_{k=0}^{\infty} \frac{(t+1)^k}{(z+1)^{k+1}}$ is uniformly convergent on $[-1,1]$. Hence we get

$$\begin{aligned}
\rho_n^R(z) &= \int_{-1}^{1} \sum_{k=0}^{\infty} \frac{(t+1)^k}{(z+1)^{k+1}} (t+1)^r \pi_n(t : w^R) w(t)\, dt \\
&= \sum_{k=0}^{\infty} \frac{1}{(z+1)^{k+1}} \int_{-1}^{1} (t+1)^k \pi_n(t : w^R) w^R(t)\, dt \\
&= \sum_{k=n}^{\infty} \frac{1}{(z+1)^{k+1}} \int_{-1}^{1} (t+1)^k \pi_n(t : w^R) w^R(t)\, dt.
\end{aligned}$$

The last equation is due to orthogonality.

Letting $\alpha_k = \int_{-1}^{1}(t+1)^k \pi_n(t : w^R) w^R(t)\,dt$, we see from (1.5) that

$$K_n^R(z) = \frac{1}{\omega_n^R(z)} \sum_{k=0}^{\infty} \frac{\alpha_k}{(z+1)^{k+1}}. \tag{2.4}$$

Since the orthogonal polynomial $\pi_n(z : w^R)$ has n simple zeros in $(-1, 1)$, and since $1/\pi_n(z : w^R)$ has the form $\sum_{k=0}^{\infty} \frac{c_k}{(z+1)^{k+1}}$ which is valid for all $z \in \Gamma$, we have formally

$$\begin{aligned} K_n^R(z) &= \frac{1}{(z+1)^r \pi_n(z : w^R)} \sum_{k=0}^{\infty} \frac{\alpha_k}{(z+1)^{k+1}} \\ &= \frac{1}{(z+1)^r} \left(\sum_{k=0}^{\infty} \frac{c_k}{(z+1)^{k+1}} \right) \left(\sum_{k=0}^{\infty} \frac{\alpha_k}{(z+1)^{k+1}} \right) \\ &= \sum_{k=0}^{\infty} \frac{\beta_k}{(z+1)^{k+r+1}}, \qquad z \in \Gamma. \end{aligned} \tag{2.5}$$

Our aim is to find how to calculate the β_k. The nth degree orthogonal polynomial $\pi_n(t : w^R)$ is of the form $\pi_n(t : w^R) = \sum_{j=0}^{n} a_j (t+1)^j$ and from the definition of α_k, it follows that

$$\begin{aligned} \alpha_k &= \int_{-1}^{1} (t+1)^k \left(\sum_{j=0}^{n} a_j (t+1)^j \right) w^R(t)\,dt \\ &= \sum_{j=0}^{n} \int_{-1}^{1} a_j (t+1)^{k+j} w^R(t)\,dt. \end{aligned}$$

Therefore, we have

$$\alpha_k = \sum_{j=0}^{n} a_j m_{k+j}^R, \tag{2.6}$$

where $m_k^R = \int_{-1}^{1}(t+1)^k w^R(t)\,dt$. On the other hand, from (2.5), we have

$$(z+1)^r \pi_n(z : w^R) \sum_{k=0}^{\infty} \frac{\beta_k}{(z+1)^{k+r+1}} = \sum_{k=0}^{\infty} \frac{\alpha_k}{(z+1)^{k+1}}.$$

Since the Taylor expansion of the nth degree orthogonal polynomial at $z=-1$ is in the form $\pi_n(z:w^R)=\sum_{j=0}^{n}a_j(z+1)^j$, we see that

$$\left(\sum_{j=0}^{n}a_j(z+1)^j\right)\sum_{k=0}^{\infty}\frac{\beta_k}{(z+1)^{k+1}}=\sum_{k=0}^{\infty}\frac{\alpha_k}{(z+1)^{k+1}}.$$

Hence by comparing the coefficients of the above equation we obtain

$$\alpha_k=\sum_{j=0}^{n}\beta_{k+j}a_j,\qquad k=n,n+1,\dots \tag{2.7}$$

and by using the fact that $\alpha_0=\alpha_1=\cdots=\alpha_{n-1}=0$, we find $\beta_0=\beta_1=\cdots=\beta_{2n-1}=0$. From the above results (2.6) and (2.7), we see that

$$\sum_{j=0}^{n}\beta_{k+n+j}a_j=\sum_{j=0}^{n}a_jm^R_{k+n+j},\qquad k=0,1,2,\dots. \tag{2.8}$$

This is in the form

$$\beta_{2n+k}a_n+\sum_{j=0}^{n-1}\beta_{k+n+j}a_j=\sum_{j=0}^{n-1}a_jm^R_{k+n+j}+a_nm^R_{2n+k}.$$

So we obtain β_k recursively by

$$\beta_{2n+k}=\sum_{j=0}^{n-1}\frac{a_j}{a_n}(m^R_{k+n+j}-\beta_{k+n+j})+m^R_{2n+k},\qquad k=0,1,2,\dots$$

where $m^R_k=\int_{-1}^{1}(t+1)^kw^R(t)\,dt$ and $a_j's$ are obtained from the fact $a_j=\frac{\pi_n^{(j)}(-1:w^R)}{j!}$. □

We have a closed form of m_k^R for various common measures. For example for Jacobi weight i.e., $w(t) = (1-t)^\alpha(1+t)^\beta$, we have

$$m_k^R = \int_{-1}^{1}(1-t)^\alpha(1+t)^{k+r+\beta}\,dt$$
$$= 2^{k+r+\alpha+\beta+1}\frac{\Gamma(k+r+\beta+1)\Gamma(\alpha+1)}{\Gamma(k+r+\alpha+\beta+2)}.$$

We have seen that the Gauss-Lobatto rule with the (nonnegative) weight function w and end points -1 and 1 of multiplicity r is given by

$$\begin{aligned}\int_{-1}^{1} f(t)w(t)\,dt &= \sum_{\rho=0}^{r-1} k_\rho f^{(\rho)}(-1) + \sum_{\nu=1}^{n}\lambda_\nu f(\tau_\nu)\\ &\quad+\sum_{\rho=0}^{r-1}(-1)^\rho\mu_\rho f^{(\rho)}(1) + R_n^L(f),\end{aligned} \tag{2.9}$$

where τ_ν's are the zeros of the nth degree orthogonal polynomial $\pi_n(\cdot : w^L)$ with respect to the weight function $w^L(t) = (t^2-1)^r w(t)$ and k_ρ, λ_ν, μ_ρ are the interpolatory weights.
For a weight $w^L(t) = (t^2-1)^r w(t), t \in [-1,1]$, consider the same contour Γ (2.1) and if we assume that f is an analytic function in a simply connected region D containing Γ, then we get the following theorem.

THEOREM 2. *For $z \in \Gamma$, we have the following expression for kernel $K_n^L(z)$*

$$K_n^L(z) = \frac{1}{(z-1)^r}\sum_{k=2n}^{\infty}\frac{\beta_k}{(z+1)^{k+r+1}}, \quad z \in \Gamma \tag{2.10}$$

where β_k can be obtained recursively by

$$\beta_{2n+k} = \sum_{j=0}^{n-1}\frac{a_j}{a_n}(m_{k+n+j}^L - \beta_{k+n+j}) + m_{2n+k}^L, \quad k = 0,1,2,\ldots. \tag{2.11}$$

PROOF. From the definition of (1.6), we have

$$\rho_n^L(z) = \int_{-1}^{1} \frac{\omega_n^L(t)}{z-t} w(t)\,dt = \int_{-1}^{1} \frac{(t^2-1)^r \pi_n(t:w^L)}{z-t} w(t)\,dt.$$

For any $z \in \Gamma$ and $t \in [-1,1]$, we have

$$\begin{aligned}\rho_n^L(z) &= \sum_{k=0}^{\infty} \frac{1}{(z+1)^{k+1}} \int_{-1}^{1} (t+1)^k \pi_n(t:w^L) w^L(t)\,dt \\ &= \sum_{k=n}^{\infty} \frac{1}{(z+1)^{k+1}} \int_{-1}^{1} (t+1)^k \pi_n(t:w^L) w^L(t)\,dt.\end{aligned}$$

The last equation comes from the orthogonality.
Letting $\alpha_k = \int_{-1}^{1} (t+1)^k \pi_n(t:w^L) w^L(t)\,dt$, we see that

$$K_n^L(z) = \frac{1}{\omega_n^L(z)} \sum_{k=0}^{\infty} \frac{\alpha_k}{(z+1)^{k+1}}. \tag{2.12}$$

Since $1/\pi_n(z:w^L)$ has Laurent series in the form $\sum_{k=0}^{\infty} \frac{d_k}{(z+1)^{k+1}}$ which is valid for all $z \in \Gamma$, we have formally

$$\begin{aligned}K_n(z) &= \frac{1}{(z^2-1)^r \pi_n(z:w^L)} \sum_{k=0}^{\infty} \frac{\alpha_k}{(z+1)^{k+1}} \\ &= \frac{1}{(z^2-1)^r} \left(\sum_{k=0}^{\infty} \frac{d_k}{(z+1)^{k+1}} \right) \left(\sum_{k=0}^{\infty} \frac{\alpha_k}{(z+1)^{k+1}} \right) \\ &= \frac{1}{(z-1)^r} \sum_{k=0}^{\infty} \frac{\beta_k}{(z+1)^{k+r+1}}, \qquad z \in \Gamma.\end{aligned} \tag{2.13}$$

By the same process of Gauss-Radau rule, i.e., by the fact that the Taylor expansion of the $\pi_n(z:w^L)$ at $z=-1$ is in the form $\pi_n(z: w^L) = \sum_{j=0}^{n} a_j (z+1)^j$, we get from the definition of a_k

$$\alpha_k = \sum_{j=0}^{n} a_j m_{k+j}^L, \tag{2.14}$$

where $m_k^L = \int_{-1}^{1}(t+1)^k w^L(t)\,dt$, and $a_j = \frac{\pi_n^{(j)}(-1:w^L)}{j!}$. We obtain by comparing the coefficients of (2.13)

$$\alpha_k = \sum_{j=0}^{n} \beta_{k+j} a_j, \qquad k = n, n+1, \dots. \tag{2.15}$$

Since $\alpha_0 = \alpha_1 = \cdots = \alpha_{n-1} = 0$, we find $\beta_0 = \beta_1 = \cdots = \beta_{2n-1} = 0$. On the other hand, from the above results (2.14) and (2.15), it follows that

$$\sum_{j=0}^{n} \beta_{k+n+j} a_j = \sum_{j=0}^{n} a_j m_{k+n+j}^L, \qquad k = 0, 1, 2, \dots. \tag{2.16}$$

So, we obtain β_k recursively by

$$\beta_{2n+k} = \sum_{j=0}^{n-1} \frac{a_j}{a_n}(m_{k+n+j}^L - \beta_{k+n+j}) + m_{2n+k}^L, \qquad k = 0, 1, 2, \dots$$

where $m_k^L = \int_{-1}^{1}(t+1)^k w^L(t)\,dt$. □

3. Convergence of Kernel

In the previous section we represent the kernels of remainder term for Gauss–Radau and Gauss–Lobatto rules as an infinite series containing β_k. Now we consider the convergence of kernel K_n. The coefficients β_k which we obtained in the previous chapters are not bounded, but we will shortly show that it is not important. The purpose of this section is to show the convergence of kernel K_n which we obtained from Theorem 1 and Theorem 2.

THEOREM 3. *For $z \in \Gamma$, the kernel $K_n^R(z)$ in Theorem 1 converges absolutely.*

PROOF. By Theorem 1, we have the following expression for kernel $K_n^R(z)$

$$K_n^R(z) = \sum_{k=0}^{\infty} \frac{\beta_k}{(z+1)^{k+r+1}}, \quad z \in \Gamma. \tag{3.1}$$

On the other hand, since $K_n^R(z) = R_n^R((z-\cdot)^{-1})$, we have for $|\frac{t+1}{z+1}| < 1$

$$K_n^R(z) = \sum_{k=0}^{\infty} \frac{R_n^R((t+1)^k)}{(z+1)^{k+1}}, \quad z \in \Gamma. \tag{3.2}$$

Therefore, we represent β_k in the following form

$$\beta_k = \frac{R_n^R((t+1)^k)}{(z+1)^r},. \quad z \in \Gamma. \tag{3.3}$$

Consequently, we have

(3.4)
$$\begin{aligned}\left|\frac{\beta_k}{(z+1)^{k+r+1}}\right| &\le \left|\frac{R_n^R((t+1)^k)}{(z+1)^r(z+1)^{k+r+1}}\right| \le \frac{1}{|z+1|^{2r+1}}\left|\frac{R_n^R(t+1)^k}{(z+1)^k}\right| \\ &\le C\|R_n^R\|_\infty \left|\frac{t+1}{z+1}\right|^k,\end{aligned}$$

where C is a constant and we used the fact that the remainder term R_n^R is continuous linear functional on $(C[-1,1], \|\cdot\|_\infty)$. □

4. Error Bound

We have the remainder term of Gauss–Radau rule with end point -1 of multiplicity r

$$\begin{aligned} R_n^R(f) &= \int_{-1}^{1} f(t)w(t)\,dt - \sum_{\nu=1}^{n} \lambda_\nu f(\tau_\nu) - \sum_{\rho=0}^{r-1} k_\rho f^{(\rho)}(-1) \\ &= \frac{1}{2\pi i}\int_\Gamma K_n^R(z) f(z)\,dz. \end{aligned} \tag{4.1}$$

From the Theorem 1, we have

$$R_n^R(f) = \sum_{k=2n}^{\infty} \beta_k \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{(z+1)^{k+r+1}}\, dz. \tag{4.2}$$

When f is an analytic function in Γ, we have by the Cauchy integral formula

$$R_n^R(f) = \sum_{k=2n}^{\infty} \beta_k \frac{f^{(k+r)}(-1)}{(k+r)!}, \tag{4.3}$$

where β_k is computed using equation (2.3). The above representation (4.3) for $R_n^R(f)$ is of little practical value since the derivatives $f^{(k)}$ are usually too difficult to obtain. So we obtain the error bound from (4.2) in the final form

$$|R_n^R(f)| \leq \sum_{k=2n}^{\infty} |\beta_k| \frac{l(\Gamma)}{2\pi} \max_{z\in\Gamma} |f(z)| \max_{z\in\Gamma} \frac{1}{|z+1|^{k+r+1}}, \tag{4.4}$$

Now, we will get the error bound (4.4) for the case where Γ is a circle and an ellipse which satisfies the condition (2.1).

Case 1: Circle

If the contour Γ is a circle in the form $C_R = \{z \in \mathbb{C} : |z+1| = R, R > 2\}$, we have

$$\max_{z\in C_R} \frac{1}{|z+1|^{k+r+1}} = \frac{1}{R^{k+r+1}}.$$

Since $l(C_R) = 2\pi R$, we get the error bound on a circle

$$|R_n^R(f)| \leq \sum_{k=2n}^{\infty} |\beta_k| \max_{z\in C_R} |f(z)| \frac{1}{R^{k+r}}. \tag{4.5}$$

Case 2: Ellipse

In case where the contour is an ellipse of the form $\Lambda = \{z \in \mathbb{C} : z+1 = \frac{1}{2}(\rho e^{i\theta} + \rho^{-1}e^{-i\theta}),\quad 0 \leq \theta \leq 2\pi,\quad \rho > 2+\sqrt{5}\}$, we get

$$\max_{z\in\Lambda} \frac{1}{|z+1|^{k+r+1}} = \frac{1}{[\frac{1}{2}(\rho - \rho^{-1})]^{k+r+1}}.$$

Since the ellipse Λ has length $l(\Lambda) = 4\epsilon^{-1}E(\epsilon)$, where $\epsilon = \frac{2}{\rho+\rho^{-1}}$ is the eccentricity of Λ and $E(\epsilon) = \int_0^{\frac{\pi}{2}} \sqrt{1-\epsilon^2 \sin^2\theta}\, d\theta$, we get the error bound on an ellipse

$$(4.6) \qquad |R_n^R(f)| \le \sum_{k=2n}^{\infty} |\beta_k| \frac{2}{\pi}\epsilon^{-1}E(\epsilon)\frac{1}{[\frac{1}{2}(\rho-\rho^{-1})]^{k+r+1}} \max_{z\in\Lambda}|f(z)|.$$

Similarly we have the remainder term of Gauss-Lobatto rule with end points -1 and 1 of multiplicity r

$$(4.7) \qquad \begin{aligned} R_n^L(f) &= \frac{1}{2\pi i}\int_\Gamma K_n^L(z) f(z)\, dz \\ &= \sum_{k=2n}^{\infty} \beta_k \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{(z-1)^r (z+1)^{k+r+1}}\, dz. \end{aligned}$$

So we obtain the error bound of the form

$$(4.8) \qquad |R_n^L(f)| \le \sum_{k=2n}^{\infty} |\beta_k| \frac{l(\Gamma)}{2\pi} \max_{z\in\Gamma}|f(z)| \max_{z\in\Gamma}\frac{1}{|z-1|^r|z+1|^{k+r+1}},$$

We will get the error bound (4.8) for the case where Γ is a circle or an ellipse.

Case 1: Circle

If the contour is a circle of the form $C_R = \{z \in \mathbb{C} : |z+1| = R, R > 2\}$, we have the error bound on the circle

$$(4.9) \qquad |R_n^L(f)| \le \sum_{k=2n}^{\infty} |\beta_k| \max_{z\in C_R}|f(z)| \frac{1}{(R-2)^r R^{k+r}},$$

the fact $\max_{z\in C_R}\frac{1}{|z-1|^r}\max_{z\in C_R}\frac{1}{|z+1|^{k+r+1}} = \frac{1}{(R-2)^r R^{k+r+1}}$

Case 2: Ellipse

In case where the contour is an ellipse of the form $\Lambda = \{z \in \mathbb{C} : z+1 = \frac{1}{2}(\rho e^{i\theta}+\rho^{-1}e^{-i\theta}),\quad 0\le\theta\le 2\pi,\quad \rho > 2+\sqrt{5}\}$, we get

$$(4.10) \qquad \begin{aligned} |R_n^L(f)| &\le \sum_{k=2n}^{\infty} |\beta_k| \frac{2}{\pi}\epsilon^{-1}E(\epsilon) \\ &\quad \times \frac{1}{[\frac{1}{2}(\rho-\rho^{-1})]^{k+r+1}[\frac{1}{2}(\rho+\rho^{-1})-2]^r} \max_{z\in\Lambda}|f(z)|, \end{aligned}$$

where $\epsilon = \frac{2}{\rho+\rho^{-1}}$ is the eccentricity of Λ and $E(\epsilon) = \int_0^{\frac{\pi}{2}} \sqrt{1-\epsilon^2 \sin^2\theta}\, d\theta$, by using $\max_{z\in\Gamma} \frac{1}{|z-1|^r |z+1|^{k+r+1}} \le \max_{z\in\Gamma} \frac{1}{|z-1|^r} \max_{z\in\Gamma} \frac{1}{|z+1|^{k+r+1}}$.

5. Example

All the computations were performed in double precision on the HP 715 computer (machine precision approximately 15 decimal digits). We apply the results of section 3 to obtain the error estimates for Gauss-Radau and Gauss-Lobatto rules on the circle and the ellipse.

EXAMPLE. For the given

$$\int_{-1}^{1} \frac{\cos[a(t+1)]}{\sqrt{5+t}} \sqrt{\frac{1-t}{1+t}}\, dt, \quad a > 0 \tag{5.1}$$

we consider $w(t) = \sqrt{\frac{1-t}{1+t}}$ as the Jacobi weight with parameters $\alpha = -\beta = \frac{1}{2}$. Accordingly,

$$f(z) = \frac{\cos[a(z+1)]}{\sqrt{5+z}},$$

the square root being understood in the sense of the principal value.

To bound f on the circle $C_R = \{z \in \mathbb{C} : |z+1| = R, R > 2\}$, note that

$$\begin{aligned} |\cos[a(z+1)]| &= \frac{1}{2} \left| e^{-aR\sin\theta} e^{iaR\cos\theta} + e^{aR\sin\theta} e^{-iaR\cos\theta} \right| \\ &\le \frac{1}{2} \left(e^{-aR\sin\theta} + e^{aR\sin\theta} \right), \quad z \in C_R \end{aligned}$$

and

$$|\sqrt{5+z}| \ge \sqrt{4 - |z+1|}, \quad z \in C_R.$$

We obtain

$$|f(z)| \le \frac{\cosh(aR)}{\sqrt{4-R}}, \quad z \in C_R. \tag{5.2}$$

Thus we have following error bound on the circle. First, for Gauss-Radau rule the inequality (4.5) yields on the circle

$$(5.3) \qquad |R_n^R(f)| \le \sum_{k=2n}^{\infty} |\beta_k| \frac{\cosh(aR)}{\sqrt{4-R}} \frac{1}{R^{k+r}},$$

where β_k is computed by (2.3). For Gauss-Lobatto rule we have from (4.9)

$$(5.4) \qquad |R_n^L(f)| \le \sum_{k=2n}^{\infty} |\beta_k| \frac{\cosh(aR)}{\sqrt{4-R}} \frac{1}{(R-2)^r R^{k+r}},$$

where β_k is computed using (2.11).

Table 1 (Radau on the circle)

n	a	bound	R	True error
5	1	2.533(-7)	3.812	-4.898(-9)
	2	1.098(-5)	3.716	-2.520(-6)
	4	1.439(-2)	3.119	8.221(-4)
10	1	3.296(-12)	3.906	6.662(-16)
	2	1.622(-10)	3.886	5.413(-13)
	4	3.586(-7)	3.801	-3.795(-12)

Table 2 (Lobatto on the circle)

n	a	bound	R	True error
5	1	3.798(-8)	3.864	3.613(-11)
	2	1.771(-6)	3.818	-2.310(-7)
	4	3.036(-3)	3.579	-2.304(-4)
10	1	2.446(-15)	3.922	-1.110(-15)
	2	1.226(-13)	3.907	5.398(-13)
	4	2.909(-10)	3.858	2.184(-12)

Using an elliptic contour $\Lambda = \{z \in \mathbb{C} : z+1 = \frac{1}{2}(\rho e^{i\theta}+\rho^{-1}e^{-i\theta}), \quad 0 \le \theta \le 2\pi, \quad \rho > 2+\sqrt{5}\}$, we have in place of (5.2)

$$|f(z)| \le \frac{\cosh(\frac{1}{2}a(\rho-\rho^{-1}))}{\sqrt{4-\frac{1}{2}(\rho+\rho^{-1})}}, \quad z \in \Lambda. \tag{5.5}$$

Thus we have following error bound on the ellipse. The inequality (4.6) yields the error bound for Gauss-Radau rule in the form

$$|R_n^R(f)| \le \sum_{k=2n}^{\infty} |\beta_k| \frac{2}{\pi}\epsilon^{-1}E(\epsilon)\frac{1}{[\frac{1}{2}(\rho-\rho^{-1})]^{k+r+1}}\frac{\cosh[\frac{1}{2}a(\rho-\rho^{-1})]}{\sqrt{4-\epsilon^{-1}}}, \tag{5.6}$$

where $\epsilon = \frac{2}{\rho+\rho^{-1}}$ is the eccentricity of Λ and $E(\epsilon) = \int_0^{\frac{\pi}{2}} \sqrt{1-\epsilon^2\sin^2\theta}\, d\theta$ and β_k is computed using equation (2.3). For Gauss-Lobatto rule we have from (4.10) the error bound

$$\begin{aligned} |R_n^L(f)| \le &\sum_{k=2n}^{\infty} |\beta_k| \frac{2}{\pi}\epsilon^{-1}E(\epsilon)\frac{\cosh[\frac{1}{2}a(\rho-\rho^{-1})]}{\sqrt{4-\epsilon^{-1}}} \\ &\times \frac{1}{[\frac{1}{2}(\rho-\rho^{-1})]^{k+r+1}[\frac{1}{2}(\rho+\rho^{-1})-2]^r}. \end{aligned} \tag{5.7}$$

Our results are shown in Table 1-2. We have expressed the error bound as a function of ρ or/and R. Numbers in parentheses of the Table 1-2 indicate decimal exponents. For simplicity we had the error bound for multiplicity two. We sum the five terms of infinite series which is the right hand side of 5.3-4. In the last column we give the true error using the package. We have several interesting features. As a decreases and n increases, the number R approaches to $R = 4$ and ρ approaches to $\rho = 4+\sqrt{15}$. This is because of the nature of weak singularity of the denominator factor. On the other hand, for increasing a, we have bad error bound. We can easily guess this phenomenon before we calculate the error bound. Because for a large value of a, error bound (4.3) grows like exponential function a^k.

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