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A NOTE ON BETTI NUMBERS AND RESOLUTIONS

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ABSTRACT. We study the Betti numbers, the Bass numbers and the resolution of modules under the change of rings. For modules of finite homological dimension, we study the Euler characteristic of them.

1. Introduction

Throughout this paper, every ring is assumed to be commutative and noetherian with identity.

Let $(R, \mathbf{m})$ be a local ring with residue class field k and M be a finitely generated R-module. Then the *i-th Betti number* $b_i^R(M)$ is rank of the ith-module in the minimal R-free resolution of M. Also the *ith Bass number* $\mu_i^R(M)$ of M is the number of copies of the injective envelope $E_R(k)$ in the ith module of the minimal R -injective resolution of M. Generally, for each prime ideal p of R, the *ith Bass number* $\mu_i^R(p, M)$ is the number of copies of $E_R(R/p)$ in the ith module of the minimal R-injective resolution of M. Thus

$$b_i^R(M) = dim_k \mathrm{T}or_i^R(M, k) = dim_k \mathrm{E}xt_i^R(M, k),$$

$$\mu_i^R(p, M) = dim_{\kappa(p)} \mathrm{E}xt^i_{R_p}(\kappa(p), M_p),\ \ \kappa(p) = R_p/pR_p.$$

This paper considers the problem of computing the Betti numbers and the Bass numbers under the change of rings. There are two formulas of change of rings introduced in the text [6, p140 Lemma 2]: Let R be a

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ring and the M, N be R-modules. Suppose that $x \in R$ is R-regular, and M-regular with $xN = 0$. Put $\bar{R} = R/xR$ and $\bar{M} = M/xM$, then

$$\mathrm{T}or_i^R(M,\ N) \cong \mathrm{T}or_i^{\bar{R}}(\bar{M},\ N),$$

$$\mathrm{E}xt_R^{i+1}(N,\ M) \cong \mathrm{E}xt_{\bar{R}}^i(N,\ \bar{M}).$$

Some of the useful homological properties can be computed from the above formulas. For example, let $(R,\ \mathbf{m})$ be local and substitute $R/\mathbf{m}$ for N in the formulas. Then the formulas can be rephrased as

$$b_i^R(M) = b_i^{\bar{R}}(\bar{M}) \text{ and } \mu_{i+1}^R(M) = \mu_i^{\bar{R}}(\bar{M}).$$

The spectral sequence of homological algebra can be used to prove the above formulas and it gives more information concerning the structure of resolution of modules. The purpose of this paper is to investigate the Betti numbers, the Bass numbers and the resolutions using the spectral sequence.

In section 2, we introduce the spectral sequence of a double complex and prove the 5-term exact sequence in a general form(cf. [7, Theorem 11.2]).

In section 3, we focus on the Grothendieck spectral sequences([7, Theorem 11.38, 11.39]) and adopt the tensor and hom functor to investigate Tor and Ext modules. We are interested in modules of finite projective dimension and compute the Euler characteristic of them.

2. Spectral Sequences

Let R be a commutative ring with identity. Then (X, d', d'') is called *a double complex of* R if $X = (X^{i,\ j})$ for $i, j \in Z$, $d'_{i,\ j} : X^{i,\ j} \longrightarrow X^{i+1,\ j}$ and $d''_{i,\ j} : X^{i,\ j} \longrightarrow X^{i,\ j+1}$ satisfying $d'_{i+1,\ j} \circ d'_{i,\ j} = 0$, $d''_{i,\ j+1} \circ d''_{i,\ j} = 0$ and $d''_{i+1,\ j} \circ d'_{i,\ j} + d'_{i,\ j+1} \circ d''_{i,\ j} = 0$. When X is a double complex, we define *the total complex* $X = \oplus X^n$ by

$$X^n = \oplus_{i+j=n} X^{i,\ j}$$

with the boundary map $d = d' + d''$. That is, for $x_{i,\ j} \in X^{i,\ j}$ with $i + j = n$, $d^n(x_{i,\ j}) = d'_{i,\ j} x_{i,\ j} + d''_{i,\ j}(x_{i,\ j})$. Note that $d^{n+1} \circ d^n = 0$. So $(X,\ d)$ is a complex. Put $H^n(X) = \ker d^n / \operatorname{im} d^{n-1}$.

NOTATION. We write $d'(x_{i,\ j})$ and $d''(x_{i,\ j})$ for $x_{i,\ j} \in X^{i,\ j}$ instead of $d'_{i,\ j}(x_{i,\ j})$ and $d''_{i,\ j}(x_{i,\ j})$.

We define two more homologies $H_I^{i,\ j}(= H_I^{i,\ j}(X))$ =ker $d'_{i,\ j}/\mathrm{im} d'_{i-1,\ j}$ and $H_{II}^{i,\ j}(= H_{II}^{i,\ j}(X))$ =ker$d''_{i,\ j}/\mathrm{im} d''_{i,\ j-1}$. Note that for each i $(H_I^{i,\ j}, d'')$ is a complex as well as $(H_{II}^{i,\ j},\ d')$ is a complex for each j, where $d' : H_I^{i,\ j} \longrightarrow H_I^{i+1,\ j}$ and $d'' : H_{II}^{i,\ j} \longrightarrow H_{II}^{i,\ j+1}$ are the induced maps from the original ones. Then we define $H_{II}H_I^{i,\ j}(= H_{II}H_I^{i,\ j}(X))$ =ker $d''_{i,\ j}/\mathrm{im}\ d''_{i,\ j-1}$ and $H_I H_{II}^{i,\ j}(= H_I H_{II}^{i,\ j}(X))$ =ker$d'_{i,\ j}/\mathrm{im}\ d'_{i-1,\ j}$. Let $X = (X^{i,\ j})$ be a double comlex. We say that X is *bounded* if for each n there are only finitely many nonzero $X^{i,\ j}$ with $i + j = n$. In this case we say that the spectral sequence $H_{II}H_I^{i,\ j}$ as well as $H_I H_{II}^{i,\ j}$ *converges to* H^n. Denoted by $H_{II}H_I^{i,\ j} \Longrightarrow H^n$ and $H_I H_{II}^{i,\ j} \Longrightarrow H^n$. And if $H_{II}H_I^{i,\ j} = 0$ (or if $H_I H_{II}^{i,\ j} = 0$) except for only one i (or j), then we say that the spectral sequence *collapes*.

We denote the elements of $X^{i,\ j}$ by $x_{i,\ j}$, $y_{i,\ j}$ and $z_{i,\ j}$.

DEFINITION 2.1. Let $X = (X^{i,\ j})$ be a double complex. Define $Z^{i,\ j} = \{x_{i,\ j} | d'x_{i,\ j} = d''x_{i+1,\ j-1}\ \exists\ x_{i+1,\ j-1} \in X_{i+1,\ j-1}$ and $d''x_{i,\ j} = 0\}$ and $B^{i,\ j} = \{x_{i,\ j} | x_{i,\ j} = d'x_{i-1,\ j} + d''x_{i,\ j-1}$ with $d''x_{i-1,\ j} = 0\}$. Note that $B^{i,\ j} \subseteq Z^{i,\ j}$ and we define $H^{i,\ j} = Z^{i,\ j}/B^{i,\ j}$.

Consider

$$H_{II}^{i,\ j} = \{\bar{x}_{i,\ j} = x_{i,\ j} + d''(X_{i,\ j-1}) | d''x_{i,\ j} = 0\}.$$

Then

$$\begin{aligned} H_I H_{II}^{i,\ j} &= \{\bar{\bar{x}}_{i,\ j} = \bar{x}_{i,\ j} + d'(H_{II}^{i-1,\ j}) | \bar{x}_{i,\ j} \in H_{II}^{i,\ j}, d'\bar{x}_{i,\ j} = 0\} \\ &= \{\bar{\bar{x}}_{i,\ j} = \bar{x}_{i,\ j} + d'(H_{II}^{i-1,\ j}) | d'x_{i,\ j} = d''x_{i+1,\ j-1}, d''x_{i,\ j} = 0\}. \end{aligned}$$

Therefore,

$$H_I H_{II}^{i,\ j} \cong H^{i,\ j} = Z^{i,\ j}/B^{i,\ j}.$$

Under the following five conditions we connect the total homology and the spectral sequences.

For a double complex $X = (X^{i,\,j})$, name the following conditions (a) to (e).

(a) $X^{i,\,-i} = 0,\ i \geq k$ for some $k \geq 0$.
(b) $H^{i,\,-i} = 0,\ 1 \leq i \leq k-1$.
(c) $H^{i+1,\,-i} = 0,\ 1 \leq i \leq k-1$.
(d) $H^{-i,\,i} = 0,\ i \geq 1$.
(e) $H^{-i-1,\,i} = 0,\ i \geq 1$.

If (a) and (c) are satisfied, then any element $z_{0,\,0} \in Z^{0,\,0}$ can be extended to a 4th quadrant element $z = z_{0,\,0} + \cdots + z_{k-1,\,-k+1} \in Z^0$. If (a) and (b) hold, and $z = z_{0,\,0} + \cdots + z_{k-1,\,-k+1} \in Z^0$ is a 4th quadrant element such that $z_{0,\,0} \in B^{0,\,0}$. Then $z \in B^0$. So if we assume the conditions (a), (b) and (c). Then there is a homomorphism $\alpha_{0,\,0} : H^{0,\,0} \longrightarrow H^0$ by $\alpha_{0,\,0}(\bar{z}_{0,\,0}) = \bar{z}$, for $z_{0,\,0} \in Z^{0,\,0}$ and $z \in Z^0$ is a 4th quadrant element obtained from $z_{0,\,0}$. Moreover, (a), (b), (c) together with (d) guarantees that $\alpha_{0,\,0}$ is surjective and (e) implies that $\alpha_{0,\,0}$ is one-to-one. Conversely, if (d) and (e) hold, then there exists a homomorphism $\beta_{0,\,0} : H^0 \longrightarrow H^{0,\,0}$ and (d), (e) together with (a), (c) guarantees that $\beta_{0,\,0}$ is onto and (a) and (b) imply that $\beta_{0,\,0}$ is one-to-one. ([7, Theorem 11.42]).

Definition 2.2. For each spot $(i,\ j)$ we assume the corresponding conditions (a) $\sim$ (c) and construct, $\alpha_{i,\,j} : H^{i,\,j} \longrightarrow H^{i+j}$. Assume (d) and (e) and construct $\beta_{i,\,j} : H^{i+j} \longrightarrow H^{i,\,j}$. Each homomorphism $\alpha_{i,\,j}$ and $\beta_{i,\,j}$ are called *an edge homomorphism.* We also define a homomorphism (called *a knight homomorphism*) $\delta = \delta_{i,\,j} : H^{i,\,j} \longrightarrow H^{i+2,\,j-1}$ as follows. For $z_{i,\,j} \in Z^{i,\,j}$, $d''z_{i,\,j} = 0$ and $d'z_{i,\,j} + d''z_{i+1,\,j-1} = 0$ for some $z_{i+1,\,j-1} \in X^{i+1,\,j-1}$. Define $\delta(z_{i,\,j}) = d'z_{i+1,\,j-1}$.

It is a routine process to check that $\delta_{i,\,j}$ is a well-defined homomorphism. Moreover, we construct edge homomorphisms between $H_{II}H_I(X)$ and $H(X)$ and knight homomorphisms for $H_{II}H_I(X)$ under the symmetric conditions.

In the following theorems, we assume the appropriate conditions of (a) $\sim$ (e) for the maps α and β to be defined.

Theorem 2.3. (5-term exact sequence [7, 11.42, 11.43, 11.44]) *Sup-*

pose that $\alpha_{1,\,0}$, $\beta_{0,\,1}$ *and* $\alpha_{2,\,0}$ *are defined. Then*

$$H^{1,\,0} \xrightarrow{\alpha_{1,\,0}} H^1 \xrightarrow{\beta_{0,\,1}} H^{0,\,1} \xrightarrow{\delta} H^{2,\,0} \xrightarrow{\alpha_{2,\,0}} H^2$$

is an exact squence.

PROOF. Note that if $H^{-1,\,1} = 0$, then $\alpha_{1,\,0}$ is one-to-one and if $H^{0,\,1} = 0$, then $\alpha_{1,\,0}$ is onto.

Due to the definition of β, $\beta_{0,\,1} \circ \alpha_{1,\,0} = 0$. If $z = z_{0,\,1} + z_{1,\,0} + \cdots + z_{k,\,-k+1} \in Z^1$ and $\beta(z) = z_{0,\,1} \in B^{0,\,1}$. Then $z_{0,\,1} + d'x_{-1,\,1} + d''x_{0,\,0} = 0$ and $d''x_{-1,\,1} = 0$ for some $x_{-1,\,1}$and $x_{0,\,0}$. Instead of z, $z + d(x_{-1,\,1} + x_{0,\,0}) = (d'x_{0,\,0} + z_{1,\,0}) + \cdots + z_{k,\,-k+1} \in \operatorname{im} \alpha_{1,\,0}$. This shows the exactness at H^1.

Let $z \in Z^1$, and write $z = z_{0,\,1} + z_{1,\,0} + \cdots + z_{k,\,-k+1} \in Z^1$. Then $\beta_{0,\,1}(z) = z_{0,\,1}$, $\delta \circ \beta_{0,\,1}(z) = \delta(z_{0,\,1})$. Since $d''z_{0,\,1} = 0$ and $d'z_{0,\,1} + d''z_{1,\,0} = 0$, $\delta(z_{0,\,1}) = d'z_{1,\,0}$. Note that $d'z_{1,\,0} = d'0 + d''(-z_{2,\,-1})$ with $d''0 = 0$. Hence $d'z_{1,\,0} \in B^{2,\,0}$. Therefore, $\operatorname{im} \beta_{0,\,1} \subseteq \ker \delta$.

Now let $z_{0,\,1} \in Z^{0,\,1}$ such that $\delta(z_{0,\,1}) \in B^{2,\,0}$. Then $d''z_{0,\,1} = 0$ and $d'z_{0,\,1} + d''z_{1,\,0} = 0$ such that $\delta(z_{1,\,0}) = d'z_{1,\,0} \in B^{2,\,0}$. Hence $d'z_{1,\,0} + d'y_{1,\,0} + d''x_{2,\,-1} = 0$ and $d''y_{1,\,0} = 0$ for some $y_{1,\,0}$ and $x_{2,\,-1}$.

Inductively, we have found $y_{i+1,\,-i}$, $x_{i+2,\,-i-1} (i \geq 0)$ satisfying

$$d'z_{i+1,\,-i} + d'y_{i+1,\,-i} + d''x_{i+2,\,-i-1} = 0$$

and $d''y_{i+1,\,-i} = 0$. Note that $d'x_{i+2,\,-i-1} \in Z_{i+3,\,-i-1} = B_{i+3,\,-i-1}$. Because $d''(d'x_{i+2,\,-i-1}) = d'(-d''x_{i+2,\,-i-1}) = d'(d'z_{i+1,\,-i} + d'y_{i+1,\,-i}) = 0$, and $d'd'x_{i+2,\,-i-1} = 0 = d''0$.

Now $0 = d'x_{i+2,\,-i-1} + d'y_{i+2,\,-i-1} + d''x_{i+3,\,-i-2}$ and $d''y_{i+2,\,-i-1} = 0$ for some $y_{i+2,\,-i-1}$ and $x_{i+3,\,-i-2}$. Let $x_{1,\,0} = z_{1,\,0}$ and put

$$\begin{aligned} z &= z_{0,\,1} + \sum_{i=0}^{k} (x_{i+1,\,-i} + y_{i+1,\,-i}) \\ &= z_{0,\,1} + (z_{1,\,0} + y_{1,\,0}) + (x_{2,\,-1} + y_{2,\,-1}) + \cdots + (x_{k+1,\,-k} + y_{k+1,\,-k}). \end{aligned}$$

Then

$$\begin{aligned} dz &= d''z_{0,\,1} + (d'z_{0,\,1} + d''z_{1,\,0}) + (d'z_{1,\,0} + d'y_{1,\,0} + d''y_{1,\,0} + d''x_{2,\,-1}) + \cdots \\ &= 0 \end{aligned}$$

Therefore, $z \in Z^1$ and $\beta_{0,\,1}(z) = z_{1,\,0}$. Let $z_{0,\,1} \in Z^{0,\,1}$. Then $d''z_{0,\,1} = 0$ and $d'z_{0,\,1} + d''z_{1,\,0} = 0$ for some $z_{1,\,0} \in X^{1,\,0}$. Hence $\delta(z_{0,\,1}) = d'(z_{1,\,0}) = d(z_{0,\,1} + z_{1,\,0})$. Trivially, $\alpha(\delta(z_{0,\,1})) = \alpha(d'z_{1,\,0}) = d(z_{0,\,1} + z_{1,\,0}) \in B^2$. Therefore, im $\delta \subseteq \ker \alpha_{2,\,0}$.

Suppose that $z_{2,\,0} \in Z^{2,\,0}$ such that $\alpha(z_{2,\,0}) = z_{2,\,0} + z_{3,\,-1} + \cdots + z_{k+2,\,-k} = z \in B^2$. Then $z = dx$ for some $x = x_{-l+1,\,l} + \cdots + x_{0,\,1} + x_{1,\,0} + \cdots + x_{k+1,\,-k} \in X^1$. Now $d''x_{-l+1,\,l} = 0$ and $d'x_{-l+1,\,l} + d''x_{-l,\,l-1} = 0$. So that $x_{-l+1,\,l} \in Z^{-l+1,\,l} = B^{-l+1,\,l}$. There are $y_{-l,\,l}$ and $y'_{-l+1,\,l-1}$ such that $x_{-l+1,\,l} + d'y_{-l,\,l} + d''y'_{-l+1,\,l-1} = 0$. Put

$$x' = x + d(y_{-l,\,l} + y'_{-l+1,\,l-1}).$$

Finally, we have $x' = x'_{0,\,1} + x_{1,\,0} + \cdots + x_{k+1,\,-k}$ such that $z = dx = dx'$. From the equation $z = dx'$, $d''x'_{0,\,1} = 0$, $d'x'_{0,\,1} + d''x_{1,\,0} = 0$ and $d'x_{1,\,0} + d''x_{2,\,-1} - z_{2,\,0} = 0$. Thus $x'_{0,\,1} \in Z^{0,\,1}$ and $\delta(x'_{0,\,1}) = d'x_{1,\,0} = z_{2,\,0} - d''x_{2,\,-1} = z_{2,\,0}$ (modulo $B^{2,\,0}$). Therefore, $\ker \alpha_{2,\,0} \subseteq$ im δ and this concludes the proof of the therem. □

If X is a bounded double complex and the spectral sequence collapses, that is, $H_I H_{II}^{i,\,j}(X) = 0$ except for one i (or one j), then the conditions (a) $\sim$ (e) are satisfied at each spot with $x = i$ (or with $y = j$) and $\alpha_{i,\,j} : H^{i,\,j} \longrightarrow H^{i+j}$ is an isomorphism. If $H_I H_{II}^{i,\,j} = 0$ except for i and $i+1$, then conditions (a), (b), (c) and (e) are satified at each spot with $x = i+1$ and (d) and (e) are satisfied at each spot with $x = i$. Hence we obtain following short exact sequences.

Corollary 2.4. *Let $X = (X^{i,\,j})$ be a bounded complex. Suppose that $H_I H_{II}^{i,\,j} = H^{i,\,j} = 0$, for any i and j except for two columns (say $i = 0, 1$). Then we have an exact sequence*

$$0 \xrightarrow{\delta} H^{1,\,n} \xrightarrow{\alpha_{1,\,n}} H^{n+1} \xrightarrow{\beta_{0,\,n+1}} H^{0,\,n+1} \xrightarrow{\delta} 0.$$

If $H_I H_{II}^{i,\,j} = 0$ except for j and $j+1$, then conditions (a), (b) and (c) are satisfied at each spot with $y = j$ and (d) and (e) are satisfied at each spot with $y = j+1$. Therefore, we have a long exact sequence as follows.

COROLLARY 2.5. *Let $X = (X^{i,\,j})$ be a bounded complex. Suppose that $H_I H_{II}^{i,\,j} = 0$, for any i and j except for two rows (say $j = 0, 1$). Then there is a long exact sequence*

$$H^{n,\,0} \xrightarrow{\alpha} H^n \xrightarrow{\beta} H^{n-1,\,1} \xrightarrow{\delta} H^{n+1,\,0} \xrightarrow{\alpha} H^{n+1}.$$

3. Betti Numbers and the Resolutions

There are several types of Grothendieck spectral sequences in the text (see [7, 11.38 $\sim$ 11.41]). If we substitute the tensor functor for the right exact functor and the hom functor for the left exact (contravariant) functor in the Grothendieck spectral sequences ([7, 11.38 $\sim$ 11.40]), then we have the spectral sequences in term of Tor and Ext modules. Let $\phi : R \longrightarrow S$ be a homomorphism of rings and M be an R-module. Give an R-projective resolution $P.$ of M. Let $Q^{\cdot\cdot}$ be a proper projective resolution (cf. [7]) of the complex $(S \otimes_R P.)$. That is, That is , $P^{\cdot,\,j}$ is an R-projective resolution of $(S \otimes_R P_j)$ and columns are complexes. Now consider the double complex $X^{\cdot\cdot} = N \otimes_S Q^{\cdot\cdot}$ for an S-module N. Note that if P is a projective R-module, then $S \otimes_R P$ is a projective S-module. Hence $H_I^{i,\,j}(X) = 0$ for $i \neq 0$ and the spectral sequence collapes. Thus $H_{II} H_I^{0,\,n}(X) = \mathrm{T}or_n^R(N,\ M)$. However from the construction of proper resolution $Q^{\cdot\cdot}$,

$$H_I H_{II}^{i,\,j}(X) = \mathrm{T}or_i^S(N,\ \mathrm{T}or_j^R(S,\ M)).$$

Therefore, we have the spectral sequence

$$\mathrm{T}or_i^S(N,\ \mathrm{T}or_j^R(S,\ M)) \Longrightarrow \mathrm{T}or_{i+j}^R(N,\ M).$$

The following three Grothendieck spectral sequence will be used for the purpose of us.

THEOREM 3.1. *Let $\phi : R \longrightarrow S$ be a homomorphism of rings. M be an R-module and N be an S-module. Then*
(1) [7. 11.38] $Ext_S^i(N,\ Ext_R^j(S,\ M)) \Longrightarrow Ext_R^{i+j}(N,\ M)$.
(2) [7, 11.39] $Tor_i^S(N,\ Tor_j^R(S,\ M)) \Longrightarrow Tor_{i+j}^R(N,\ M)$.

Rotman gave a proof of (1) and we sketched a proof of (2).

Let M be an R-module with a finite free resoulution $F_{.} \longrightarrow M \longrightarrow 0$. Then we define the *Euler characteristic* $\chi_R(M)$ of M by

$$\chi_R(M) := \sum(-1)^i \mathrm{r}kF_i$$

where F_i is the ith-module of F. This is well-defined ([4, sec 4.3]). For an R-module M of finite injective dimension and a prime ideal p of R define the *Bass-Euler characteristic* $\chi_R(p,\ M)$ of M relative to p by

$$\mathrm{X}_R(p,\ M) := \sum(-1)^i \mu_i(p,\ M).$$

This does not depend on the injective resolutions of M. If $(R,\ \mathbf{m})$ is local, then we write $\mathrm{X}_R(M)$ instead of $\mathrm{X}_R(\mathbf{m},\ M)$.

THEOREM 3.2. *Let $(R,\ \mathbf{m})$ be a local ring and M be an R-module. Suppose that $x \in \mathbf{m}$ is R-regular and put $\bar{R} = R/xR$, $\bar{M} = M/xM$ and $M_1 = (0 :_x M)$. Then*

(1) $\mu_i^R(M) \le \mu_i^{\bar{R}}(M_1) + \mu_{i-1}^{\bar{R}}(\bar{M})$, *and* $\mu_1^{\bar{R}}(M_1) \le \mu_1^R(M)$.
(2) *Assume that M is a finite R-module with $depth_R M = d$ and $injdim_R M = n < \infty$. Then*

$$\mu_{i-2}^{\bar{R}}(\bar{M}) = \mu_i^{\bar{R}}(M_1)\ (i \le d-1,\ n+2 \le i),$$

$$\mu_{d-2}^{R}(\bar{M}) \le \mu_d^{\bar{R}}(M_1) \text{ and } \mu_{n-1}^{\bar{R}}(\bar{M}) \ge \mu_{n+1}^{\bar{R}}(M_1).$$

(3) *If both $injdim_R M$ and $injdim_{\bar{R}} \bar{M}$ are finite, then $injdim_{\bar{R}} M_1$ is finite and*

$$X_{\bar{R}}(M_1) = X_R(M) + X_{\bar{R}}(\bar{M}).$$

PROOF. Substitute $k = R/\mathbf{m}$ for N and $\bar{R}$ for S in the spectral sequence of Theorem 3.1 (1), we obtain

$$\mathrm{E}xt^i_{\bar{R}}(k,\ \mathrm{E}xt^j_R(\bar{R},\ M)) \Longrightarrow \mathrm{E}xt^{i+j}_R(k,\ M).$$

Since x is R-*regular*, $\mathrm{pd}_R \bar{R} = 1$, $Ext^0_R(\bar{R}, M) = (0 :_x M) = M_1$, $Ext^1_R(\bar{R}, M) = \bar{M}$ and $Ext^j_R(\bar{R}, M) = 0$, $j \geq 2$. Hence we obtain 5-term exact sequence (Corollary 2.5)

$$k^{\mu_i^{\bar{R}}(M_1)} \xrightarrow{\alpha} k^{\mu_i^R(M)} \xrightarrow{\beta} k^{\mu_{i-1}^{\bar{R}}(\bar{M})} \xrightarrow{\delta} k^{\mu_{i+1}^{\bar{R}}(M_1)} \xrightarrow{\alpha} k^{\mu_{i+1}^R(M)}$$

(1) It follows from the above exact sequence.

(2) Note that $\mathrm{depth}_R M = \inf\{i \mid Ext^i_R(k,\ M) \neq 0\}$ and $\mathrm{injdim}_R M = \sup\{i \mid Ext^i_R(k,\ M) \neq 0\}$. Reviewing the exact sequence outside d and n, we can conclude (2).

(3) From the exact sequence, take the alternating sum of the dimension of each k-vector space, we obtain $\mathrm{X}_{\bar{R}}(M_1) = \mathrm{X}_R(M) + \mathrm{X}_{\bar{R}}(\bar{M})$. □

Using the Tor spectral sequence (Theorem 3.1 (2)) we obtain a result parallel with Theorem 3.2

THEOREM 3.3. *Let $(R, \mathbf{m})$ be a local ring, $x \in \mathbf{m}$ be R-regular. For a finite R-module M, put $\bar{M} = M/xM$ and $M_1 = (0 :_M x)$. Then*

(1) $b_i^R(M) \leq b_i^{\bar{R}}(\bar{M}) + b_{i-1}^{\bar{R}}(M_1)$, $i \geq 1$, $b_1^{\bar{R}}(\bar{M}) \leq b_1^R(M)$.

(2) *If $\mathrm{pd}_R M = n < \infty$, then $b_{n-1}^{\bar{R}}(M_1) \geq b_{n+1}^{\bar{R}}(\bar{M})$ and $b_i^{\bar{R}}(M_1) = b_{i+2}^{\bar{R}}(\bar{M})$, $i \geq n$.*

(3) *If both $\mathrm{pd}_R M$ and $\mathrm{pd}_{\bar{R}}(\bar{M})$ are finite, then so is $\mathrm{pd}_{\bar{R}}(M_1)$ and $\chi_{\bar{R}}(\bar{M}) = \chi_R(M) + \chi_{\bar{R}}(M_1)$.*

PROOF. Consider the Tor spectral sequence

$$Tor_i^{\bar{R}}(k,\ Tor_j^R(\bar{R},\ M)) \Longrightarrow Tor_{i+j}^R(M,\ k),\ k = R/\mathbf{m}.$$

Note that $Tor_0^R(\bar{R},\ M) = \bar{R} \otimes_R M = \bar{M}$, $Tor_1^R(\bar{R},\ M) = M_1$ and $Tor_j^R(M,\ \bar{R}) = 0$, $j \geq 2$. Hence we obtain 5-term exact sequence

$$\xrightarrow{\delta} k^{b_i^{\bar{R}}(\bar{M})} \xrightarrow{\alpha} k^{b_i^R(M)} \xrightarrow{\beta} k^{b_{i-1}^{\bar{R}}(M_1)} \xrightarrow{\delta} k^{b_{i+1}^{\bar{R}}(\bar{M})} \xrightarrow{\alpha} k^{b_{i+1}^R(M)} \xrightarrow{\beta}.$$

Note that $\mathrm{pd}_R M = \sup\{i \mid b_i^R(M) \neq 0\}$ and a similar process for Bass numbers as in the proof of Theorem 3.2 conclude the theorem. □

It is due to Auslander and Buchsbaum [2] that for an R-module M with a finite free resolution, if $\chi(M) = 0$, then annM contains a nonzero-divisor and if $\chi(M) \neq 0$, then ann$M = 0$. The converse is also true [4, Theorem 196]. Howerever, we can hardly find any reference concering the properties of Bass-Euler characteristic. In the corollary below we determine when $X_R(M) = 0$.

COROLLARY 3.4. *If an R-module M is annihilated by a nonzero divisor x in R and if both injdim$_R M$ and injdim$_{\bar{R}} M (\bar{R} = R/xR)$ are finite. Then for any prime ideal p of R containing x*

$$X_{R_P}(M_p) = 0.$$

PROOF. Note that $x/1 \in R_p$ is a nonzero-divisor. So we may assume that $(R, \mathbf{m})$ is a local ring and it is enough to show that $X_R(M) = 0$. Since $xM = 0$, $\overline{M} = M/xM = M$ and $M_1 = (0 :_M x) = M$. From the equation $\mathrm{X}_{\overline{R}}(M) = \mathrm{X}_R(M) + \mathrm{X}_{\overline{R}}(M)$ in Theorem 3.2, we conclude that $\mathrm{X}_R(M) = 0$. □

If we replace the condition, injdim$_{\bar{R}} M < \infty$ by $\mu_m^{\bar{R}}(M) = \mu_m^{\bar{R}}(M)$. For some $m \geq$ injdim$_R^M$. Then Corollary 3.4 is true. Take the alternating sum of the dimension of k-vector spaces up to $k^{\mu_{n+1}^R(M)}$. Then this is conceivable.

QUESTION 3.5. (1) Is it true that injdim$_R M < \infty$, then either ann $(M) = 0$ or ann(M) contains a nonzero-divisor?

(2) If ann$(M) = 0$, then $\mathrm{X}_R(M) > 0$?

(3) If ann(M) contains a nonzero-divisor, then $\mathrm{X}_R(M) = 0$?

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