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THE UNITS AND IDEMPOTENTS IN THE GROUP RING OF A FINITE CYCLIC GROUP

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ABSTRACT. Let K be a algebraically closed field of characteristic 0 and G a cyclic group of order n. We find the units and idempotent elements of the group ring KG by using the basic group table matrix of G.

1. Introduction

Let $G = \{g_0 = 1, g_1, g_2, \cdots, g_{n-1}\}$ be a finite group with the fixed order $g_0, g_1, g_2, \cdots, g_{n-1}$ of elements.

From the group table

	g_0	g_1	$\cdots$	g_j	$\cdots$	g_{n-1}
g_0				$\vdots$		
g_1				$\vdots$		
$\vdots$				$\vdots$		
g_i	$\cdots$	$\cdots$	$\cdots$	g_ig_j		
$\vdots$						
g_{n-1}						

we obtain the group table matrix

$$(g_ig_j).$$

A basic group table matrix is a matrix with the diagonal entries 1 obtaining from the group table matrix (g_ig_j) by elementary row operations

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interchanging two rows. The elements of the first column of the basic group table matrix are inverses of the elements $g_0, g_1, g_2, \cdots, g_{n-1}$ of G.

Let R be a ring with unity and $G = \{g_0 = 1, g_1, g_2, \cdots, g_{n-1}\}$ a finite group with the fixed order $g_0, g_1, g_2, \cdots, g_{n-1}$ of elements. From the element $\alpha = \sum_{i=0}^{n-1} r(g_i)g_i$ of the group ring RG, we obtain a following matrix M_α by putting $r(g_i)$ in the place of g_i in the basic group table matrix of G.

$$M_\alpha = \begin{pmatrix} r(1) & r(g_1) & \cdots & r(g_{n-1}) \\ r(g_1^{-1}) & r(1) & \cdots & \cdot \\ \vdots & & \ddots & \vdots \\ r(g_{n-1}^{-1}) & \cdot & \cdots & r(1) \end{pmatrix}$$

This matrix M_α is called a represented matrix of $\alpha = \sum_{i=0}^{n-1} r(g_i)g_i$.

Let R be a ring with unity, G be a finite group and $M(R, G) = \{M_\alpha | \alpha \in RG\}$. Then followings are trivial

(1) $M_{\alpha+\beta} = M_\alpha + M_\beta$
(2) $M_{\alpha\beta} = M_\alpha M_\beta$
(3) $M_{r\alpha} = rM_\alpha$ for $r \in R$.

And we can see that $RG \cong M(R, G)$ by an algebra isomorphism.

DEFINITION. A element $\sum_{i=0}^{n-1} r(g_i)g_i$ of RG is called a symmetrix element of RG if whenever $g_ig_j = 1$, $r(g_i) = r(g_j)$.

If a group G is a Klein's four group, then every element of RG is symmetrix.

In this paper, let a field K be a algebraically closed field of characteristic 0 and a group G be cyclic of order n.

2. The units in the group ring of a finite cyclic group

Hughes and Pearson found the units in the group ring $\mathbb{Z}S_3$ of the symmetrix group S_3. And Passman and Sehgal proved that if G is an up-group and K is a field of characteristic 0, then all units in KG are trivial.

We shall find the units in the group ring KG by using the represented matrix where K is a algebraically closed field of characteristic 0 and G is a finite cyclic group of order n.

DEFINITION. Let $\alpha = \sum_{i=0}^{n-1} r_i g^i$ be an element of KG where $r_t \neq 0$ $(0 \leq t \leq n-1)$ and $r_j = 0$ $(j > t)$. Then the polynomial $p(x) = r_0 + r_1 x + \cdots + r_t x^t \in K[x]$ is called a represented polynomial of α.

Let p_σ be a permutation matrix corresponding to the cyclic permutation $\sigma = (12 \cdots n)$. Then for a element $\alpha = \sum_{i=0}^{n-1} r_i g^i$ of KG, the represented matrix M_α of α is as following

$$M_\alpha = \sum_{i=0}^{n-1} r_i p_\sigma^i.$$

Let ξ be a primitive nth root of unity in K. Consider the Vandermonde matrix $V(1\xi \cdots \xi^{n-1})$ and the represented polynomial $p(x)$ of $\alpha = \sum_{i=0}^{n-1} r_i g^i$. Since

$$p_\sigma = \frac{1}{\sqrt{n}} V(1\xi \cdots \xi^{n-1})\ diag(1\xi \cdots \xi^{n-1}) \frac{1}{\sqrt{n}} V(1\bar{\xi} \cdots \bar{\xi}^{n-1})$$

where $\bar{\xi}$ is conjugate to ξ, we have

$$\begin{aligned} M_\alpha &= \sum_{i=0}^{n-1} r_i p_\sigma^i \\ &= \frac{1}{\sqrt{n}} V(1\xi \cdots \xi^{n-1}) diag(p(1)p(\xi) \cdots p(\xi^{n-1})) \frac{1}{\sqrt{n}} V(1\bar{\xi} \cdots \bar{\xi}^{n-1}). \end{aligned}$$

Hence

$$det M_\alpha = p(1)p(\xi) \cdots p(\xi^{n-1}).$$

Thus $\alpha \in KG$ is a unit if and only if $det M_\alpha \neq 0$. If $\alpha = \sum_{i=0}^{n-1} r_i g^i$ is a unit, then from $M_{\alpha^{-1}} = M_\alpha^{-1}$,

$$\alpha^{-1} = \frac{1}{n} \sum_{i=0}^{n-1} \sum_{k=0}^{n-1} \bar{\xi}^{-ik} p(\xi^k)^{-1} g^i.$$

Therefore we have the following theorem.

THEOREM 2.1. *Let* $\alpha = \sum_{i=0}^{n-1} r_i g^i \in KG$ *where* $r_t \neq 0$ $(0 \leq t \leq n-1)$ *and* $r_j = 0$ $(j > t)$. *Then* α *is a unit if and only if all* n*th root of unity in* K *are not equal to the roots of the represented polynomial* $p(x)$ *of* α. *And if* $\alpha = \sum_{i=0}^{n-1} r_i g^i \in KG$ *is a unit, then*

$$\alpha^{-1} = \frac{1}{n} \sum_{i=0}^{n-1} \sum_{k=0}^{n-1} \bar{\xi}^{-ik} p(\xi^k)^{-1} g^i$$

where ξ *is a* n*th root of unity in* K *and* $\bar{\xi}$ *is conjugate to* ξ.

Now if $\alpha \in KG$ is a unit, then we shall express α^{-1} by the roots of the represented polynomial of α.

Let $\alpha = \sum_{i=0}^{n-1} r_i g^i \in KG$ where $r_t \neq 0$ $(0 \leq t \leq n-1)$ and $r_j = 0$ $(j > t)$. Let u_k $(k = 1, 2, \cdots, t)$ be all roots of the represented polynomial $p(x)$ of α. Then

$$p(x) = r_t \prod_{k=1}^{t} (x - u_k)$$

and thus

$$M_\alpha = p(p_\sigma) = r_t \prod_{k=1}^{t} (p_\sigma - u_k I).$$

If α is a unit, then M_α is a unit and $u_k^n \neq 1$ $(k = 1, 2, \cdots, t)$. Therefore

$$M_\alpha^{-1} = r_t^{-1} \prod_{k=1}^{t} (p_\sigma - u_k I)^{-1}$$

and

$$(p_\sigma - u_k I)^{-1} = (1 - u_k^n)^{-1} (u_k^{n-1} I + u_k^{n-2} p_\sigma + \cdots + p_\sigma^{n-1}).$$

Let $\beta_k = (1 - u_k^n)^{-1} \sum_{i=0}^{n-1} u_k^{n-1-i} g^i$. Then

$$M_{\beta_k} = (p_\sigma - u_k I)^{-1}.$$

Therefore

$$M_\alpha^{-1} = r_t^{-1} \prod_{k=1}^{t} M_{\beta_k}.$$

Since

$$M_{\alpha^{-1}} = M_\alpha^{-1} = r_t^{-1} \prod_{k=1}^{t} M_{\beta_k} = M_{r_t^{-1} \prod_{k=1}^{t} \beta_k},$$

we have

$$\alpha^{-1} = r_t^{-1} \prod_{k=1}^{t} \beta_k.$$

Therefore we have the following theorem

THEOREM 2.2. *Let* $\alpha = \sum_{i=0}^{n-1} r_i g^i \in KG$ *where* $r_t \neq 0$ $(0 \leq t \leq n-1)$ *and* $r_j = 0$ $(j > t)$. *Let* u_k $(k = 1, 2, \cdots, t)$ *be the roots of the represented polynomial of* α. *If* α *is a unit, then*

$$\alpha^{-1} = r_t^{-1} \prod_{k=1}^{t} \beta_k$$

where $\beta_k = (1 - u_k^n)^{-1} \sum_{k=0}^{n-1} u_k^{n-1-i} g^i$.

We can see that the symmetrix element $\alpha = r + ag + ag^2 + \cdots + ag^{n-1} (a \neq 0)$ of KG is a unit if and only if $r \neq a$ and $r \neq (1-n)a$.

Thus if n is prime and α is a unit, then

$$\alpha^{-1} = r' + a'g + a'g^2 + \cdots + a'g^{n-1}$$

where

$$r' = \frac{1}{n}[\{r + (n-1)a\}^{-1} + (n-1)(r-a)^{-1}] \quad \text{and}$$
$$a' = \frac{1}{n}[\{r + (n-1)a\}^{-1} - (r-a)^{-1}]$$

from Theorem 2.1.

3. The idempotents in the group ring of a finite cyclic group

By Kaplansky and Zalesskii, it was proved that if $\alpha = \sum_{i=0}^{n} r_i g_i \in KG$ is a nontrivial idempotent element where g_0 is an identity, then r_0 is a rational number lying strictly between 0 and 1 when K is a field of characteristic 0 and G is a finite group.

We shall find the r_0 of an idempotent element $\alpha = \sum_{i=0}^{n-1} r_i g^i \in KG$ and nontrivial idempotent elements in KG when K is a algebraically closed field of characteristic 0 and G is a cyclic group of order n.

The represented matrix M_α of $\alpha = \sum_{i=0}^{n-1} r_i g^i \in KG$ is following.

$$M_\alpha = \frac{1}{\sqrt{n}} V(1 \xi \cdots \xi^{n-1}) diag(p(1) p(\xi) \cdots p(\xi^{n-1})) \frac{1}{\sqrt{n}} V(1 \bar{\xi} \cdots \bar{\xi}^{n-1})$$

where ξ is a primitive nth root of unity in K. Therefore α is an idempotent element if and only if $p(1) = 1$ or 0, $p(\xi) = 1$ or 0, $\cdots$ and $p(\xi^{n-1}) = 1$ or 0. Hence KG has 2^n-2 nontrivial idempotent elements.

From M_α, we have followings

$$r_0 = \frac{1}{n} \sum_{i=0}^{n-1} p(\xi^i)$$

$$r_1 = \frac{1}{n} \sum_{i=0}^{n-1} \xi^{n-i} p(\xi^i)$$

$$\vdots$$

$$r_j = \frac{1}{n} \sum_{i=0}^{n-1} \xi^{n-ji} p(\xi^i)$$

$$\vdots$$

$$r_{n-1} = \frac{1}{n} \sum_{i=0}^{n-1} \xi^i p(\xi^i).$$

Hence we have the following theorem.

THEOREM 3.1. *KG has $2^n - 2$ nontrivial idempotent elements and if $\alpha = \sum_{i=0}^{n-1} r_i g^i \in KG$ is an idempotent element, then*

(1) $r_0 = 0, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{n-1}{n}, 1.$

(2) *The values of $r_0, r_1, \cdots$ and r_{n-1} are followings.*

r_0	r_1	r_2	$\cdots$	r_{n-1}
0	0	0	$\cdots$	0
$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	$\cdots$	$\frac{1}{n}$
$\frac{1}{n}$	$\frac{1}{n}\xi^{n-1}$	$\frac{1}{n}\xi^{n-2}$	$\cdots$	$\frac{1}{n}\xi$
$\frac{1}{n}$	$\frac{1}{n}\xi^{n-2}$	$\frac{1}{n}\xi^{n-4}$	$\cdots$	$\frac{1}{n}\xi^2$
$\vdots$	$\vdots$	$\vdots$		$\vdots$
$\frac{1}{n}$	$\frac{1}{n}\xi$	$\frac{1}{n}\xi^2$	$\cdots$	$\frac{1}{n}\xi^{n-1}$
$\frac{2}{n}$	$\frac{1}{n}(1+\xi^{n-1})$	$\frac{1}{n}(1+\xi^{n-2})$	$\cdots$	$\frac{1}{n}(1+\xi)$
$\frac{2}{n}$	$\frac{1}{n}(1+\xi^{n-2})$	$\frac{1}{n}(1+\xi^{n-4})$	$\cdots$	$\frac{1}{n}(1+\xi^2)$
$\vdots$	$\vdots$	$\vdots$		$\vdots$
$\frac{2}{n}$	$\frac{1}{n}(1+\xi)$	$\frac{1}{n}(1+\xi^2)$	$\cdots$	$\frac{1}{n}(1+\xi^{n-1})$
$\frac{2}{n}$	$\frac{1}{n}(\xi^{n-1}+\xi^{n-2})$	$\frac{1}{n}(\xi^{n-2}+\xi^{n-4})$	$\cdots$	$\frac{1}{n}(\xi+\xi^2)$
$\vdots$	$\vdots$	$\vdots$		$\vdots$
$\frac{2}{n}$	$\frac{1}{n}(\xi^{n-1}+\xi)$	$\frac{1}{n}(\xi^{n-2}+\xi^2)$	$\cdots$	$\frac{1}{n}(\xi+\xi^{n-1})$
$\vdots$	$\vdots$	$\vdots$		$\vdots$
$\frac{2}{n}$	$\frac{1}{n}(\xi^2+\xi)$	$\frac{1}{n}(\xi^4+\xi^2)$	$\cdots$	$\frac{1}{n}(\xi^{n-2}+\xi^{n-1})$
$\frac{3}{n}$	$\frac{1}{n}(1+\xi^{n-1}+\xi^{n-2})$	$\frac{1}{n}(1+\xi^{n-2}+\xi^{n-4})$	$\cdots$	$\frac{1}{n}(1+\xi+\xi^2)$
$\vdots$	$\vdots$	$\vdots$		$\vdots$
1	0	0	$\cdots$	0

where ξ is a primitive nth root of unity in K.

THEOREM 3.2. *KG has $3^n - 2$ nontrivial tripotent elements and if $\alpha = \sum_{i=0}^{n-1} r_i g^i \in KG$ is a tripotent element, then $r_0 = 0, \pm\frac{1}{n}, \pm\frac{2}{n}, \cdots, \pm\frac{n}{n}$.*

PROOF. From $M_\alpha = \frac{1}{\sqrt{n}} V(1\xi \cdots \xi^{n-1}) diag(p(1)p(\xi) \cdots p(\xi^{n-1})) \frac{1}{\sqrt{n}} V(1\bar{\xi} \cdots \bar{\xi}^{n-1})$ and $M_\alpha^3 = M_\alpha$,

$$p(1) = -1, 0, \text{ or } 1, \ \ p(\xi) = -1, 0, \text{ or } 1, \ \ \cdots, \ \ p(\xi^{n-1}) = -1, 0, \text{ or } 1.$$

Therefore, we have done. □

EXAMPLE 1. *In the case that G is a cyclic group of order 3. Let $\alpha = \sum_{i=0}^{2} r_i g^i \in KG$. Then*

$$r_0 = \frac{1}{3}\{p(1) + p(\xi) + p(\xi^2)\}$$
$$r_1 = \frac{1}{3}\{p(1) + \xi^2 p(\xi) + \xi p(\xi^2)\}$$
$$r_2 = \frac{1}{3}\{p(1) + \xi p(\xi) + \xi^2 p(\xi^2)\}$$

where ξ is a primitive 3th root of unity in K and $p(x) = r_0 + r_1 x + r_2 x^2$. Therefore if α is an idempotent element, then the values of r_0, r_1 and r_2 are followings.

r_0	r_1	r_2
0	0	0
$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
$\frac{1}{3}$	$\frac{-1+\sqrt{3}i}{6}$	$\frac{-1-\sqrt{3}i}{6}$
$\frac{1}{3}$	$\frac{-1-\sqrt{3}i}{6}$	$\frac{-1+\sqrt{3}i}{6}$
$\frac{2}{3}$	$\frac{-1}{3}$	$\frac{-1}{3}$
$\frac{2}{3}$	$\frac{1+\sqrt{3}i}{6}$	$\frac{1-\sqrt{3}i}{6}$
$\frac{2}{3}$	$\frac{1-\sqrt{3}i}{6}$	$\frac{1+\sqrt{3}i}{6}$
1	0	$0.$

EXAMPLE 2. In the case that G is a cyclic group oforder 4. Let $\alpha = \sum_{i=0}^{3} r_i g^i \in KG$. Then

$$r_0 = \frac{1}{4}\{p(1) + p(\xi) + p(\xi^2) + p(\xi^3)\}$$
$$r_1 = \frac{1}{4}\{p(1) + \xi^3 p(\xi) + \xi^2 p(\xi^2) + \xi p(\xi^3)\}$$
$$r_2 = \frac{1}{4}\{p(1) + \xi^2 p(\xi) + p(\xi^2) + \xi^2 p(\xi^3)\}$$
$$r_3 = \frac{1}{4}\{p(1) + \xi p(\xi) + \xi^2 p(\xi^2) + \xi^3 p(\xi^3)\}$$

where ξ is a primitive 4th root of unity in K and $p(x) = r_0 + r_1 x + r_2 x^2 + r_3 x^3$.

Therefore α is an idempotent element, then the values of r_0, r_1, r_2 and r_3 are followings.

r_0	r_1	r_2	r_3
0	0	0	0
$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
$\frac{1}{4}$	$\frac{-1}{4}$	$\frac{1}{4}$	$\frac{-1}{4}$
$\frac{1}{4}$	$\frac{1}{4}i$	$\frac{-1}{4}$	$\frac{-1}{4}i$
$\frac{1}{4}$	$\frac{-1}{4}i$	$\frac{-1}{4}$	$\frac{1}{4}i$
$\frac{1}{2}$	0	$\frac{1}{2}$	0
$\frac{1}{2}$	0	$\frac{-1}{2}$	0
$\frac{1}{2}$	$\frac{1}{4}(i-1)$	0	$\frac{-1}{4}(i+1)$
$\frac{1}{2}$	$\frac{-1}{4}(i-1)$	0	$\frac{1}{4}(i+1)$
$\frac{1}{2}$	$\frac{1}{4}(i+1)$	0	$\frac{-1}{4}(i-1)$
$\frac{1}{2}$	$\frac{-1}{4}(i+1)$	0	$\frac{1}{4}(i-1)$
$\frac{3}{4}$	$\frac{1}{4}$	$\frac{-1}{4}$	$\frac{1}{4}$
$\frac{3}{4}$	$\frac{-1}{4}$	$\frac{-1}{4}$	$\frac{-1}{4}$
$\frac{3}{4}$	$\frac{1}{4}i$	$\frac{1}{4}$	$\frac{-1}{4}i$
$\frac{3}{4}$	$\frac{-1}{4}i$	$\frac{1}{4}$	$\frac{1}{4}i$
1	0	0	0.

In the case that G is a group of order 4, we can see that KG has 14 idempotent elements and, if $\alpha = \sum_{i=0}^{3} r_i g_i \in KG$ is an idempotent element, then $r_0 = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$ from the following theorem.

THEOREM 3.3. *Let K be a algebraically closed field of characteristic 0 and G be a group of order 4. Then*

(1) *KG has 14 nontrivial idempotent elements.*
(2) *if $\alpha = \sum_{i=0}^{3} r_i g_i \in KG$ is an idempotent element, then $r_0 = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$.*

PROOF. If G is a group of order 4, then either G is isomorphic to the cyclic group or G is isomorphic to the Klein's four group.

In 4, we found all idempotent elements in the group ring of Klein's four group.

In example 2, we found all idempotent elements in the group ring of a cyclic group of order 4. □

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