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DIRECT PROJECTIVE MODULES WITH THE SUMMAND SUM PROPERTY

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ABSTRACT. Let R be a ring with a unity and let M be a unitary left R-module. In this paper, we establish [5, Proposition 2.8] by showing the proof of it. Moreover, from the above result, we obtain some properties of direct projective modules which have the summand sum property.

1. Introduction

Let R be a ring with a unity and let M be a unitary left R-module. Throughout this note, every map is an R-homomorphism. We denote $\mathrm{End}(M)$ the endomorphism ring of M. A module M is said to be direct projective if, given any direct summand N of M and the projection map $p : M \longrightarrow N$, for each epimorphism $f : M \longrightarrow N$, there exists $g \in \mathrm{End}(M)$ such that the following diagram

$$\begin{array}{ccc} & & M \\ & {\scriptstyle g}\swarrow & \downarrow {\scriptstyle p} \\ M & \xrightarrow{f} & N \rightarrow O \end{array}$$

commutes, i.e. $f \circ g = p$. This concept which is the generalization of quasi-projectivity was introduced by W. K. Nicholson [4] in 1976 and J. Hausen [2] established some properties of direct projective modules. Moreover, W. Xue [7] investigated direct projective modules over hereditary, semihereditary and semisimple rings. A module M is said to have the summand sum property if the sum of any two direct summands of M

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is again a direct summand of M. The summand sum property was studied by J. L. Garcia [1], who characterized modules with the summand sum property. Similarly, a module M is said to have the summand intersection property if the intersection of any two direct summands of M is a direct summand of M. The summand sum property and the summand intersection property have been mostly considered for arbitrary R-modules.

We found an obscure part in the proof of [5, Proposition 2.8], and in this note, we establish [5, Proposition 2.8] by showing the proof of it. Moreover, from the above result, we obtain some properties of direct projective modules which have the summand sum property.

A. K. Tiwary and P. C. Bharadwaj [5] showed that every direct summand of a direct projective module is direct projective and J. Hausen [2] proved the following.

THEOREM 1.1 [2]. (i), (ii) *and* (iii) *of the module* M *are equivalent:*

(i) *A module* M *is direct projective.*

(ii) *If* S *is a submodule of* M *such that* M/S *is isomorphic to a direct summand of* M, *then* S *is a direct summand of* M.

(iii) *Every exact sequence* $N \longrightarrow A \longrightarrow O$ *with* N *an epimorphic image of* M *and* A *a direct summand of* M *splits.*

THEOREM 1.2 [1]. *A module* M *has the summand sum property if and only if for every pair* A, B *of direct summands of* M, *the image of the restriction to* A *of the canonical projection* $p : M \longrightarrow B$ *is a direct summand of* B.

THEOREM 1.3 [6]. *A module* M *has the summand intersection property if and only if for every pair of direct summands* A *and* B *with* $p : M \longrightarrow A$ *the projection map, the kernel of the restricted map* $p|_B$ *is a direct summand.*

2. Main results

THEOREM 2.1. *If a direct projective module* M *has the summand sum property, then* M *has the summand intersection property.*

PROOF. Assume that a direct projective module M has the summand sum property. Let A, B be direct summands of M and $p: M \longrightarrow B$ be the projection map. It is sufficient to show that Ker $p|_A$ is a summand of A([6, Proposition 1]). Since the module M has the summand sum property, by Theorem 1.2, Im $p|_A$ is a direct summand of B and $A \oplus$ Im $p|_A$ is a direct summand of M. It is already known that every direct summand of a direct projective module is direct projective. So $A \oplus \mathrm{Im}\, p|_A$ is a direct projective module. Since A is an epimorphic image (the projection map) of $A \oplus \mathrm{Im}\, p|_A$, by applying Theorem 1.1

$$O \longrightarrow \mathrm{Ker}\, p|_A \longrightarrow A \longrightarrow \mathrm{Im}\, p|_A \longrightarrow O$$

the above exact sequence splits, that is, Ker $p|_A$ is a summand of A. Hence, M has the summand intersection property. □

COROLLARY 2.2. *A direct projective module M has the summand sum property if and only if End(M) has the summand sum property.*

PROOF. If a direct projective module M has the summand sum property, then by Theorem 2.1, M has the summand intersection property. In [1, Lemma 4.1], a module M has both the summand sum property and the summand intersection property if and only if End(M) has the summand sum property. Hence, it is proved. □

COROLLARY 2.3. *Let M be a module which is a direct sum of indecomposable modules. If a module M is direct projective, then M has the summand sum property if and only if End(M) is a product of indecomposable rings and a regular ring.*

PROOF. It is an immediate consequence of [1] and Theorem 2.1. □

THEOREM 2.4. *Let M be a direct projective module. If $M \oplus M$ has the summamnd sum property, then End(M) is a Von Neuman regular ring.*

PROOF. Suppose that $M \oplus M$ has the summand sum property. Then for arbitrary map $f : M \longrightarrow M$, Im f and Ker f are direct summands of M. Since the module M is direct projective, there exists a map $g : M \longrightarrow M$ such that the following diagram

$$\begin{array}{ccc} & M & \\ g \swarrow & \downarrow p & \\ M \xrightarrow{f} & \text{Im } f & \rightarrow O \end{array}$$

commutes, i.e., $f \circ g = p$ for the projection map $p : M \longrightarrow \text{Im } f$.

$$f \circ g \circ i = p \circ i = I_{\text{Im f}} \text{ and } f \circ g \circ i \circ f = f$$

Therefore, for every map $f \in \text{End}(M)$, there exists a map $h \in \text{End}(M)$ defined by

$$h = g \circ i \circ p$$

such that $f \circ g \circ i \circ f = f$ where $h|_{\text{Im } f} = g \circ i$. Hence $\text{End}(M)$ is a Von Neuman regular ring. □

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