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A GENERALIZATION OF SILVIA CLASS OF FUNCTIONS

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ABSTRACT. E. M. Silvia introduced the class S_α^λ of α-λ-spirallike functions $f(z)$ satisfying the condition

$$\text{(A)} \qquad Re[(e^{i\lambda}-\alpha)\frac{zf'(z)}{f(z)}+\alpha\frac{(zf'(z))'}{f'(z)}]>0,$$

where $\alpha \geq 0, |\lambda| < \frac{\pi}{2}$ and $|z| < 1$. We will generalize Silvia class of functions by formally replacing $f(z)$ in the denominator of (A) by a spirallike function $g(z)$. We denote the new class of functions by $Y(\alpha,\lambda)$.

In this note we obtain some results for the class $Y(\alpha,\lambda)$ including integral representation formula, relations between our class $Y(\alpha,\lambda)$ and Ziegler class Z_λ, the radius of convexity problem, a few coefficient estimates and a covering theorem for the class $Y(\alpha,\lambda)$.

1. Introduction

Let $f(z)$ belong to the class S of normalized univalent and holomorphic functions in the open unit disk E. Let $S_p(\lambda)$ denote the class of λ-spirallike functions in E with $f(0)=0, f'(0)=1$. These satisfy

$$\text{(1.1)} \qquad Re[e^{i\lambda}\frac{zf'(z)}{f(z)}]>0 \text{ for } z\in E, |\lambda|<\frac{\pi}{2}.$$

Spaček [9] showed that each f in $S_p(\lambda)$ is univalent in E.

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Ziegler [11] generalized the concept of close-to-convexity by formally replacing $f(z)$ in the denominator in (1.1) by a λ-spirallike function. That is, f lies in the Ziegler class Z_λ if there is a $g(z) \in S_p(\lambda)$ such that

$$Re[e^{i\lambda}\frac{zf'(z)}{g(z)}] > 0 \text{ for } z \in E. \tag{1.2}$$

When $\lambda = 0, Z_\lambda$ is the class of close-to-convex functions in E.

A holomorphic function $f(z)$ satisfying $f(z)\cdot f'(z) \neq 0$ for $0 < |z| < 1$ is said to be α-convex in E if

$$Re[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{(zf'(z))'}{f'(z)}] > 0 \text{ for } z \in E, \alpha \geq 0. \tag{1.3}$$

Chichra [1] introduced the class C_α of α-close-to-convex functions in E by formally replacing $f(z)$ in the denominator of (1.3) by a starlike function $\phi(z)$ in S^*. That is, $f \in C_\alpha$ if there exists a $\phi(z) \in S^*$ such that

$$Re[(1-\alpha)\frac{zf'(z)}{\phi(z)} + \alpha\frac{(zf'(z))'}{\phi'(z)}] > 0 \text{ for } z \in E, \alpha \geq 0. \tag{1.4}$$

Silvia [8] generalized the definition of α-convexity to α-λ-spirallikeness as follows ;

Let S_α^λ denote the class of α-λ-spirallike functions in E, where $f \in S_\alpha^\lambda$ if

$$Re[(e^{i\lambda}-\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{(zf'(z))'}{f'(z)}] > 0 \text{ for } z \in E, \alpha \geq 0, |\lambda| < \frac{\pi}{2}. \tag{1.5}$$

He proved that $S_\alpha^\lambda \subset S_p(\lambda) \subset S$.

Now we will generalize Silvia class by formally replacing $f(z)$ in the denominator of (1.5) by a spirallike function $g(z)$ in $S_p(\lambda)$. We denote this new class of functions by $Y(\alpha, \lambda)$.

Definition. Let $f(z)$ be holomorphic in E with $f(0) = f'(0) - 1 = 0$. $f(z)$ belongs to the class $Y(\alpha, \lambda)$ if there exists a $g(z) \in S_p(\lambda)$ such that

$$Re[(e^{i\lambda}-\alpha)\frac{zf'(z)}{g(z)} + \alpha\frac{(zf'(z))'}{g'(z)}] > 0 \tag{1.6}$$

for $z \in E, \alpha \geq 0, |\lambda| < \frac{\pi}{2}$.

Note that if α=0 , then $Y(\alpha, \lambda)$ is equal to Ziegler class Z_λ. If $\lambda = 0, Y(\alpha, \lambda)$ is equal to Chichra class C_α. If α=0, λ=0, then $Y(\alpha, \lambda)$ is the class of close-to-convex functions in E.

In this note we prove some geometric properties for the functions $f(z)$ in $Y(\alpha, \lambda)$. We obtain the integral representation formula for the functions in $Y(\alpha, \lambda)$. We prove that for every admissible α and λ, each function $f(z)$ in $Y(\alpha, \lambda)$ lies in Ziegler class Z_λ and find the disk $|z| < r_0$ so that functions in Ziegler class Z_λ may satisfy the defining inequality (1.6) for the class $Y(\alpha, \lambda)$. Moreover,we solve the radius of convexity problem for the class $Y(\alpha, \lambda)$ and a covering theorem as well as a few coefficient estimates for functions in $Y(\alpha, \lambda)$.

2. Main Results for the class $Y(\alpha, \lambda)$

We now obtain an integral representation for the functions in the class $Y(\alpha, \lambda)$

THEOREM 2.1. *$f(z)$ is in the class $Y(\alpha, \lambda)$ if and only if there exists a regular function $p(z)$ with $p(0) = 1, Re\ p(z) > 0$ for $z \in E$ such that*

$$(2.1)\quad f'(z) = \frac{1}{\alpha z[g(z)]^{\frac{e^{i\lambda}}{\alpha}-1}} \int_0^z (\cos\lambda p(\zeta) + i\sin\lambda) g'(\zeta)[g(\zeta)]^{\frac{e^{i\lambda}}{\alpha}-1} d\zeta,$$

where the powers are taken as principal values and $g(z) \in S_p(\lambda), \alpha \neq 0$.
If $\alpha = 0$, then

$$f'(z) = e^{-i\lambda} z^{-1} g(z)[\cos\lambda p(z) + i\sin\lambda].$$

PROOF. If $f(z) \in Y(\alpha, \lambda)$, then for some regular function $p(z)$ with $p(0)$=1 and $Re\ p(z) > 0$ for $z \in E$, we can write

$$(2.2)\qquad (e^{i\lambda} - \alpha)\frac{zf'(z)}{g(z)} + \alpha\frac{(zf'(z))'}{g'(z)} = \cos\lambda p(z) + i\sin\lambda$$

where $g(z) \in S_p(\lambda)$ and $|\lambda| < \frac{\pi}{2}$.
Multiplying both sides of (2.2) by $\alpha^{-1} g'(z)[g(z)]^{\frac{e^{i\lambda}}{\alpha}-1}$, we have

$$(2.3)\qquad (\frac{e^{i\lambda}}{\alpha} - 1)zf'(z)g'(z)[g(z)]^{\frac{e^{i\lambda}}{\alpha}-2} + (zf'(z))'[g(z)]^{\frac{e^{i\lambda}}{\alpha}-1}$$

$$= (\cos\lambda p(z) + i\sin\lambda)\alpha^{-1}g'(z)[g(z)]^{\frac{e^{i\lambda}}{\alpha}-1}.$$

The left hand side of (2.3) is the exact differential of $zf'(z)[g(z)]^{\frac{e^{i\lambda}}{\alpha}-1}$. So integrating both sides of (2.3) with respect to z, we obtain

$$zf'(z)[g(z)]^{\frac{e^{i\lambda}}{\alpha}-1} = \alpha^{-1}\int_0^z (\cos\lambda p(\zeta) + i\sin\lambda)g'(\zeta)[g(\zeta)]^{\frac{e^{i\lambda}}{\alpha}-1}d\zeta,$$

and get

$$f'(z) = \frac{1}{\alpha z[g(z)]^{\frac{e^{i\lambda}}{\alpha}-1}}\int_0^z (\cos\lambda p(\zeta) + i\sin\lambda)g'(\zeta)[g(\zeta)]^{\frac{e^{i\lambda}}{\alpha}-1}.$$

In particular if $\alpha = 0$, we have

$$e^{i\lambda}\frac{zf'(z)}{g(z)} = \cos\lambda p(z) + i\sin\lambda,$$

and hence

$$f'(z) = \frac{e^{-i\lambda}}{z}g(z)[\cos\lambda p(z) + i\sin\lambda].$$

Conversely, if $f(z)$ satisfies (2.1), we get,

$$\frac{zf'(z)}{g(z)} = \frac{1}{\alpha[g(z)]^{\frac{e^{i\lambda}}{\alpha}}}\int_0^z (\cos\lambda p(\zeta) + i\sin\lambda)g'(\zeta)[g(\zeta)]^{\frac{e^{i\lambda}}{\alpha}-1}d\zeta$$

and

$$\begin{aligned}&\frac{(zf'(z))'}{g'(z)}\\ &= \frac{\alpha^{-1}}{[g(z)]^{\frac{e^{i\lambda}}{\alpha}}}(1 - \frac{e^{i\lambda}}{\alpha})\cdot\int_0^z (\cos\lambda p(\zeta) + i\sin\lambda)g'(\zeta)[g(\zeta)]^{\frac{e^{i\lambda}}{\alpha}-1}d\zeta\\ &+ \alpha^{-1}[\cos\lambda p(z) + i\sin\lambda].\end{aligned}$$

It follows that

$$(e^{i\lambda} - \alpha)\frac{zf'(z)}{g(z)} + \alpha\frac{(zf'(z))'}{g'(z)} = \cos\lambda p(z) + i\sin\lambda$$

which implies $f(z) \in Y(\alpha, \lambda)$, since $Re\ p(z) > 0$ for $z \in E$. □

In order to verify that a function $f(z)$ in $Y(\alpha, \lambda)$ belongs to the Ziegler's class Z_λ, we need the following lemma which is due to I. S. Jack [3].

LEMMA 1. [3]. *Let $\omega(z)$ be regular in E with $\omega(0) = 0$. If there exists a ζ in E such that*

$$max_{|z|\leq|\zeta|}\ |\omega(z)| = |\omega(\zeta)|,$$

then $\zeta\omega'(\zeta) = k\omega(\zeta)$ for some $k \geq 1$.

THEOREM 2.2. *For $\alpha \geq 0, |\lambda| < \frac{\pi}{2}$, let $f(z)$ be in the class $Y(\alpha, \lambda)$. Then $f(z)$ satisfies the condition*

$$Re\ [e^{i\lambda}\frac{zf'(z)}{g(z)}] > 0, \quad (z \in E) \quad \text{for some } g(z) \in S_p(\lambda),$$

and hence lies in Ziegler class Z_λ.

PROOF. If $f(z) \in Y(\alpha, \lambda)$, let us set

$$(2.4) \qquad e^{i\lambda}\frac{zf'(z)}{g(z)} = \cos\lambda\frac{1-\omega(z)}{1+\omega(z)} + i\sin\lambda \quad (|\lambda| < \frac{\pi}{2}, g(z) \in S_p(\lambda))$$

where $\omega(z)$ is regular in E with $\omega(0) = 0$ and $\omega(z) \neq -1$ for $z \in E$. Since $Re\ \frac{1-\omega(z)}{1+\omega(z)} > 0$ whenever $|\omega(z)| < 1$ for $z \in E$, it suffices to show that $|\omega(z)| < 1$ for $z \in E$ in (2.4). Simplifying (2.4), it follows that

$$(2.5) \qquad e^{i\lambda}\frac{zf'(z)}{g(z)} = \frac{e^{i\lambda} - \omega(z)e^{-i\lambda}}{1+\omega(z)}.$$

Differentiating (2.5) and using the condition (1.6), we have

$$\begin{aligned}&(e^{i\lambda} - \alpha)\frac{zf'(z)}{g(z)} + \alpha\frac{(zf'(z))'}{g'(z)}\\ &= \cos\lambda\frac{1-\omega(z)}{1+\omega(z)} + i\sin\lambda + \alpha e^{-i\lambda}\cos\lambda\{\frac{g(z)}{g'(z)} \cdot \frac{-2\omega'(z)}{(1+\omega(z))^2}\}.\end{aligned}$$

Now, suppose that $|\omega(z)| \geq 1$ for $z \in E$. Then there exists a ζ in E such that $max_{|z|\leq|\zeta|}\ |\omega(z)| = |\omega(\zeta)| = 1$.

By Lemma 1, $\zeta\omega'(\zeta) = k\omega(\zeta)$ for some $k \geq 1$. For this ζ we have

$$Re\ \frac{1-\omega(\zeta)}{1+\omega(\zeta)} = 0$$

and

$$\frac{\omega(\zeta)}{(1+\omega(\zeta))^2} = \frac{1}{4\cos^2\frac{\theta}{2}}, \qquad \text{where } \omega(\zeta) = e^{i\theta}.$$

Then

$$\begin{aligned} &Re\{(e^{i\lambda}-\alpha)\frac{\zeta f'(\zeta)}{g(\zeta)} + \alpha\frac{(\zeta f'(\zeta))'}{g'(\zeta)}\} \\ &= Re\{\alpha\cos\lambda\frac{e^{-i\lambda}g(\zeta)}{\zeta g'(\zeta)}\cdot\frac{-2k\omega(\zeta)}{(1+\omega(\zeta))^2}\} \\ &= -\frac{\alpha k\cos\lambda}{2\cos^2\frac{\theta}{2}} Re\{\frac{e^{-i\lambda}g(\zeta)}{\zeta g'(\zeta)}\} \quad (|\lambda| < \frac{\pi}{2}). \end{aligned}$$

Since $g(z) \in S_p(\lambda)$, $Re\frac{e^{-i\lambda}g(\zeta)}{\zeta g'(\zeta)} > 0$ for $\zeta \in E$.
Therefore, we have

$$Re\{(e^{i\lambda}-\alpha)\frac{\zeta f'(\zeta)}{g(\zeta)} + \alpha\frac{(\zeta f'(\zeta))'}{g'(\zeta)}\} \le 0 \quad (\zeta \in E).$$

But this contradicts (1.6), since $f \in Y(\alpha,\lambda)$.
Hence,

$$Re\ e^{i\lambda}\frac{zf'(z)}{g(z)} > 0 \text{ for } z \in E.$$

This completes the proof. □

COROLLARY. *If* $\alpha > \beta \ge 0$ *and* $|\lambda| < \frac{\pi}{2}$, *then* $Y(\alpha,\lambda) \subset Y(\beta,\lambda)$.

PROOF. For $\beta = 0$, Theorem 2.2 shows that

$$Y(\alpha,\lambda) \subset Y(0,\lambda) = Z_\lambda.$$

For $\beta \neq 0$

$$\begin{aligned} &(e^{i\lambda}-\beta)\frac{zf'(z)}{g(z)} + \beta\frac{(zf'(z))'}{g'(z)} \\ &\qquad = \frac{\beta}{\alpha}[e^{i\lambda}(\frac{\alpha}{\beta}-1)\frac{zf'(z)}{g(z)} + (e^{i\lambda}-\alpha)\frac{zf'(z)}{g(z)} + \alpha\frac{(zf'(z))'}{g'(z)}]. \end{aligned}$$

For $\alpha > \beta$, if $f(z) \in Y(\alpha, \lambda)$, then

$$Re\{(e^{i\lambda} - \alpha)\frac{zf'(z)}{g(z)} + \alpha\frac{(zf'(z))'}{g'(z)}\} > 0 \quad (z \in E).$$

Now $f(z) \in Y(\alpha, \lambda) \subset Z_\lambda$ implies that $Re\ e^{i\lambda}\frac{zf'(z)}{g(z)} > 0$. Hence,

$$Re\{(e^{i\lambda} - \beta)\frac{zf'(z)}{g(z)} + \beta\frac{(zf'(z))'}{g'(z)}\} > 0$$

showing that $f(z) \in Y(\beta, \lambda)$. □

We now show that functions in the Ziegler class Z_λ satisfy the defining inequality for the class $Y(\alpha, \lambda)$ on a certain disk $|z| < r_0$.

THEOREM 2.3. *If $f(z)$ belongs to the Ziegler class Z_λ with $f(0) = 0, f'(0) = 1$, then $f(z)$ satisfies the inequality*

$$Re[(e^{i\lambda} - \alpha)\frac{zf'(z)}{g(z)} + \alpha\frac{(zf'(z))'}{g'(z)}] > 0$$

for $|z| < r_0$ where $r_0 = (1 + \alpha) - \sqrt{(1 + \alpha)^2 - 1}$.

PROOF. If $f(z) \in Z_\lambda$, there exists a $g(z) \in S_p(\lambda)$ such that

$$Re\ e^{i\lambda}\frac{zf'(z)}{g(z)} > 0 \text{ for } |\lambda| < \frac{\pi}{2}.$$

Let $\omega(z)$ be the regular function in E with $\omega(0) = 0, |\omega(z)| < 1$ for $z \in E$, given by

$$e^{i\lambda}\frac{zf'(z)}{g(z)} = \cos\lambda\frac{1 + \omega(z)}{1 - \omega(z)} + i\sin\lambda. \tag{2.6}$$

By logarithmic differentiation of (2.6) we get

$$1 + \frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)} = \frac{(1 + e^{-2i\lambda})z\omega'(z)}{(1 + e^{-2i\lambda}\omega(z))(1 - \omega(z))}.$$

Thus,

$$\begin{aligned}
&(e^{i\lambda}-\alpha)\frac{zf'(z)}{g(z)}+\alpha\frac{(zf'(z))'}{g'(z)}\\
&=\frac{e^{i\lambda}+e^{-i\lambda}\omega(z)}{1-\omega(z)}+\alpha\frac{(1+e^{-2i\lambda})z\omega'(z)}{(1+e^{-2i\lambda}\omega(z))(1-\omega(z))}\frac{f'(z)}{g'(z)}\\
&=\frac{e^{i\lambda}+e^{-i\lambda}\omega(z)}{1-\omega(z)}+\alpha\frac{(1+e^{-2i\lambda})z\omega'(z)}{(1-\omega(z))^2}\frac{g(z)}{zg'(z)},
\end{aligned}$$

where the last equality is obtained by (2.6).

In order to show that $f(z)\in Y(\alpha,\lambda)$ for $|z|<r_0$, where $r_0=(1+\alpha)-\sqrt{(1+\alpha)^2-1}$, we observe that

$$\begin{aligned}
&Re\ [(e^{i\lambda}-\alpha)\frac{zf'(z)}{g(z)}+\alpha\frac{(zf'(z))'}{g'(z)}]\\
&\geq Re\frac{e^{i\lambda}+e^{-i\lambda}\omega(z)}{1-\omega(z)}-\alpha\left|\frac{(1+e^{-2i\lambda})z\omega'(z)}{(1-\omega(z))^2}\right|\left|\frac{g(z)}{zg'(z)}\right|.
\end{aligned}$$

By using the well known inequalities [2]

$$|\omega'(z)|\leq\frac{1-|\omega(z)|^2}{1-r^2},\quad \left|\frac{g(z)}{zg'(z)}\right|\leq\frac{1+r}{1-r}$$

and

$$|1-\omega(z)|\leq 1+|\omega(z)|\leq 1+r,\quad (|z|=r),$$

we have

$$\begin{aligned}
&Re\ \frac{e^{i\lambda}+e^{-i\lambda}\omega(z)}{1-\omega(z)}-\alpha\left|\frac{(1+e^{-2i\lambda})z\omega'(z)}{(1-\omega(z))^2}\right|\left|\frac{g(z)}{zg'(z)}\right|\\
&\geq\frac{\cos\lambda(1-|\omega(z)|^2)}{|1-\omega(z)|^2}-\alpha\frac{|1+e^{-2i\lambda}|(1-|\omega(z)|^2)r}{|1-\omega(z)|^2(1-r)^2}\\
&\geq\frac{\cos\lambda(1-2r-2\alpha r+r^2)(1-|\omega(z)|^2)}{(1-r)^2|(1-\omega(z))|^2}\\
&\geq\frac{\cos\lambda(1-2r-2\alpha r+r^2)}{(1-r)^2}.
\end{aligned}$$

Since the smallest positive root of $1-2(1+\alpha)r+r^2=0$ is $r_0=(1+\alpha)-\sqrt{(1+\alpha)^2-1}$, $f(z)$ satisfies the inequality (1.6) for $|z|<r_0$.□

THEOREM 2.4. *Let $f(z) \in Y(\alpha,\lambda)$. Then $w = f(z)$ maps the disk $|z| < r_1$ onto a convex region where r_1 is the smallest positive root of the equation*

$$\alpha|\alpha+e^{i\lambda}| - r\{|\alpha+e^{i\lambda}|(2\alpha\cos\lambda + 4\cos\lambda+\alpha) + 2\alpha\cos\lambda\}$$
$$+ r^2\{|\alpha+e^{i\lambda}|(2\alpha\cos^2\lambda + 2\cos\lambda - 2\ cos\lambda - \alpha) - 2\alpha\cos\lambda\}$$
$$- r^3|\alpha+e^{i\lambda}|(2\alpha\cos^2\lambda - 2\cos\lambda - \alpha) = 0, \quad |z| = r.$$

PROOF. If $f(z) \in Y(\alpha,\lambda)$ the integral representation formula shows that

$$(2.7)\quad f'(z) = \frac{1}{\alpha z[g(z)]^{\frac{e^{i\lambda}}{\alpha}-1}} \int_0^z (\cos\lambda p(\zeta) + i\sin\lambda) g'(\zeta)[g(\zeta)]^{\frac{e^{i\lambda}}{\alpha}-1} d\zeta$$

for $z \in E$ and $|\lambda| < \frac{\pi}{2}$, where the powers are taken as principal values. By logarithmic differentiation of (2.7), we obtain

$$(2.8)\qquad 1 + \frac{zf''(z)}{f'(z)}$$

$$= (1 - \frac{e^{i\lambda}}{\alpha})\frac{zg'(z)}{g(z)} + \frac{z(\cos\lambda p(\zeta) + i\sin\lambda)g'(z)[g(z)]^{\frac{e^{i\lambda}}{\alpha}-1}}{\int_0^z (\cos\lambda p(\zeta) + i\sin\lambda)g'(\zeta)[g'(\zeta)]^{\frac{e^{i\lambda}}{\alpha}-1} d\zeta}.$$

Thus

$$Re\{1 + \frac{zf''(z)}{f'(z)}\} = Re(1 - \frac{e^{i\lambda}}{\alpha})\frac{zg'(z)}{g(z)} + ReF(z)\ ,$$

where $F(z)$ is the second term of the right hand side of (2.8). Since $g(z) \in S_p(\lambda)$,

$$Re\{e^{i\lambda}\frac{zg'(z)}{g(z)}\} \le \cos\lambda\frac{1+r}{1-r}\ .$$

If we let $g(z) = z + b_2 z^2 + \cdots$ and $p(z) = 1 + p_1 z + \cdots$ in $F(z)$, a brief calculation shows that $F(z)$ has a series expansion

$$F(z) = \frac{e^{i\lambda}}{\alpha} + \frac{\alpha\cos\lambda p_1 + e^{i\lambda}(\alpha + e^{i\lambda})b_2}{\alpha(\alpha+e^{i\lambda})} z + \cdots .$$

Furthermore, $F(z)$ is a regular function in E and hence

$$|F(z)-\frac{e^{i\lambda}}{\alpha}|\le\frac{2\alpha\cos\lambda+2|\alpha+e^{i\lambda}|\cos\lambda}{\alpha|\alpha+e^{i\lambda}|}\frac{r}{(1-r)^2}.$$

This inequality shows that

$$ReF(z)\ge\frac{\cos\lambda}{\alpha}-\frac{2\cos\lambda(\alpha+|\alpha+e^{i\lambda}|)}{\alpha|\alpha+e^{i\lambda}|}\frac{r}{(1-r)^2}.$$

Robertson [6] showed that if $g(z)\in S_p(\lambda)$, then for $|z|=r<1$

(2.9)

$$\frac{(1-\lambda\cos\lambda)^2-r^2\ sin^2\lambda}{1-r^2}\le Re\frac{zg'(z)}{g(z)}\le\frac{(1+\lambda\cos\lambda)^2-r^2\ sin^2\lambda}{1-r^2}.$$

Using (2.9) we have

$$\begin{aligned}&Re\{1+\frac{zf''(z)}{f'(z)}\}\\&\ge\frac{(1-\lambda\cos\lambda)^2-r^2\ sin^2\lambda}{1-r^2}-\frac{\cos\lambda(1+r)}{\alpha(1-r)}\\&\quad+\frac{\cos\lambda}{\alpha}-\frac{2r\cos\lambda(\alpha+|\alpha+e^{i\lambda}|)}{\alpha|\alpha+e^{i\lambda}|(1-r)^2}\\&=\frac{N(r)}{\alpha|\alpha+e^{i\lambda}|(1+r)(1-r)^2},\end{aligned}$$

where

$$\begin{aligned}N(r)&=\alpha|\alpha+e^{i\lambda}|-r|\alpha+e^{i\lambda}|(2\alpha\cos\lambda+4\cos\lambda+\alpha)+2\alpha\cos\lambda\\&\quad+r^2|\alpha+e^{i\lambda}|(2\alpha\cos^2\lambda+2\alpha\cos\lambda-2\cos\lambda-\alpha)-2\alpha\cos\lambda\\&\quad-r^3|\alpha+e^{i\lambda}|(2\alpha\cos^2\lambda-2\cos\lambda-\alpha)\ .\end{aligned}$$

Here, $N(0)=\alpha|\alpha+e^{i\lambda}|>0$ and $N(1)=-4\cos\lambda(\alpha+|\alpha+e^{i\lambda}|)<0$. Hence f(z) is convex for $|z|<r_1$ where r_1 is the smallest positive root of $N(r)=0$. This completes the proof. □

To discuss coefficient estimates for the class $Y(\alpha,\lambda)$ we need the following lemma which is due to Libera.

LEMMA 2. [5]. *If* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *belongs to* $S_p(\lambda)$ *in* E, *then*

$$|a_n| \leq \prod_{k=0}^{n-2} \frac{|2\cos\lambda e^{-i\lambda} + k|}{k+1}, \quad n = 2, 3, 4, \cdots,$$

and these bounds are sharp for all admissible λ *and for each* n.

THEOREM 2.5. *Let* $f(z)$ *be in the class* $Y(\alpha, \lambda)$. *If* $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$, *then*

$$|a_2| \leq \frac{\cos\lambda(1 + |\alpha + e^{i\lambda}|)}{|\alpha + e^{i\lambda}|}$$

and

$$|a_3| \leq \frac{\cos\lambda\sqrt{1 + 8\cos^2\lambda}}{3} + \frac{4\cos^2\lambda|3\alpha + e^{i\lambda}|}{3|\alpha + e^{i\lambda}||2\alpha + e^{i\lambda}|} + \frac{2\cos\lambda}{3|2\alpha + e^{i\lambda}|}.$$

These bounds are sharp for all admissible α *and* λ.

PROOF. Let $\mathcal{P} = \{p(z); p(z)\ \textit{is regular in}\ E\ \textit{with}\ p(0) = 1, \textit{Re}\ p(z) > 0\}$. If $f(z)$ is in $Y(\alpha, \lambda)$, we can write , for some $g(z) \in S_p(\lambda)$ and $p(z) \in \mathcal{P}$,

$$(e^{i\lambda} - \alpha)\frac{zf'(z)}{g(z)} + \alpha\frac{(zf'(z))'}{g'(z)} = \cos\lambda p(z) + i\sin\lambda. \tag{2.10}$$

Let $g(z) = z + b_2 z^2 + b_3 z^3 + \cdots$ and $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$. Then (2.10) can be written as

(2.11)
$$\begin{aligned} &(e^{i\lambda} - \alpha)(z + 2a_2 z^2 + 3a_3 z^3 + \cdots)(1 + 2b_2 z + 3b_3 z^2 + \cdots) \\ &+ \alpha(1 + 4a_2 z + 9a_3 z^2 + \cdots)(z + b_2 z^2 + b_3 z^3 + \cdots) \\ &= [e^{i\lambda} + \cos\lambda(p_1 z + p_2 z^2 + \cdots)](z + b_2 z^2 + \cdots)(1 + 2b_2 z + \cdots). \end{aligned}$$

On equating both sides of (2.11), we get

$$2(\alpha + e^{i\lambda})a_2 = (\alpha + e^{i\lambda})b_2 + \cos\lambda p_1 \tag{2.12}$$

and
(2.13)
$3(2\alpha + e^{i\lambda})a_3 = -4e^{i\lambda}a_2b_2 + (2\alpha + e^{i\lambda})b_3 + 2e^{i\lambda}{b_2}^2 + (3b_2p_1 + p_2)\cos\lambda.$

It is well known [2] that $|p_n| \leq 2$ for $n = 1, 2, 3, \cdots$. By Lemma 2, (2.12) reduces to

$$|\alpha + e^i\lambda||a_2| \leq \cos\lambda(1 + |\alpha + e^i\lambda|).$$

Hence ,

$$|a_2| \leq \frac{\cos\lambda(1 + |\alpha + e^{i\lambda}|)}{|\alpha + e^{i\lambda}|}.$$

Now substituting (2.12) into (2.13) we obtain

$$3(2\alpha + e^{i\lambda})(\alpha + e^{i\lambda})a_3$$
$$= (\alpha + e^{i\lambda})(2\alpha + e^{i\lambda})b_3 + \cos\lambda(3\alpha + e^{i\lambda})b_2p_1 + \cos\lambda(\alpha + e^{i\lambda})p_2 \ .$$

By lemma 2 again , we have

$$|a_3| \leq \frac{\cos\lambda\sqrt{1 + 8\cos^2\lambda}}{3} + \frac{4cos^2\lambda|3\alpha + e^{i\lambda}|}{3|\alpha + e^{i\lambda}||2\alpha + e^{i\lambda}|} + \frac{2\cos\lambda}{3|2\alpha + e^{i\lambda}|}.$$

The functions $g(z) = z(1 - z)^{-2\cos\lambda e^{-i\lambda}}$ and $p(z) = \frac{1+e^{-i\lambda}z}{1-e^{-i\lambda}z}$ show that the results are sharp. □

Using the second coefficient estimate for the class $Y(\alpha, \lambda)$, we obtain the following result similar to Köebe 's covering theorem.

THEOREM 2.6. *Let $f(z)$ be in the class $Y(\alpha, \lambda)$ and let ω be any complex number such that $f(z) \neq \omega$ for z in E. Then*

$$|\omega| \geq \frac{\alpha + 1}{3\alpha + 4}, \quad (\alpha \geq 0).$$

PROOF. Let us write

$$f_1(z) = \frac{\omega f(z)}{\omega - f(z)}.$$

Then $f_1(z)$ belong to S and $f_1(z) = z + (a_2 + \frac{1}{\omega})z^2 + \cdots$. Hence, $|a_2 + \frac{1}{\omega}| \leq 2$. By Theorem 2.5, we obtain $|\frac{1}{\omega}| \leq \frac{(2+\cos\lambda)|\alpha+e^{i\lambda}|+\cos\lambda}{|\alpha+e^{i\lambda}|} \leq \frac{4+3\alpha}{1+\alpha}$. Hence, $|\omega| \geq \frac{\alpha+1}{3\alpha+4}$. □

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