A GENERALIZATION OF SILVIA CLASS OF FUNCTIONS

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ABSTRACT. E. M. Silvia introduced the class $S_{\alpha}^{\lambda}$ of $\alpha$-$\lambda$-spirallike functions $f(z)$ satisfying the condition

$$\text{Re}[(e^{i\lambda} - \alpha) z f'(z) f(z) + \alpha (z f(z)f'(z)')] > 0,$$

where $\alpha \geq 0, |\lambda| < \frac{\pi}{2}$ and $|z| < 1$. We will generalize Silvia class of functions by formally replacing $f(z)$ in the denominator of $(A)$ by a spirallike function $g(z)$. We denote the new class of functions by $Y(\alpha, \lambda)$.

In this note we obtain some results for the class $Y(\alpha, \lambda)$ including integral representation formula, relations between our class $Y(\alpha, \lambda)$ and Ziegler class $Z_{\lambda}$, the radius of convexity problem, a few coefficient estimates and a covering theorem for the class $Y(\alpha, \lambda)$.

1. Introduction

Let $f(z)$ belong to the class $S$ of normalized univalent and holomorphic functions in the open unit disk $E$. Let $S_p(\lambda)$ denote the class of $\lambda$-spirallike functions in $E$ with $f(0) = 0$, $f'(0) = 1$. These satisfy

$$\text{Re}[e^{i\lambda} z f'(z) f(z)] > 0 \text{ for } z \in E, |\lambda| < \frac{\pi}{2}.$$ 

Spaček [9] showed that each $f$ in $S_p(\lambda)$ is univalent in $E$.

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Ziegler [11] generalized the concept of close-to-convexity by formally replacing $f(z)$ in the denominator in (1.1) by a $\lambda$-spiral-like function. That is, $f$ lies in the Ziegler class $Z_\lambda$ if there is a $g(z) \in S_p(\lambda)$ such that

$$(1.2) \quad \text{Re}[e^{i\lambda} \frac{zf'(z)}{g(z)}] > 0 \text{ for } z \in E.$$ 

When $\lambda = 0$, $Z_\lambda$ is the class of close-to-convex functions in $E$.

A holomorphic function $f(z)$ satisfying $f(z) \cdot f'(z) \neq 0$ for $0 < |z| < 1$ is said to be $\alpha$-convex in $E$ if

$$(1.3) \quad \text{Re}[(1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \frac{(zf'(z))'}{f'(z)}] > 0 \text{ for } z \in E, \alpha \geq 0.$$ 

Chichra [1] introduced the class $C_\alpha$ of $\alpha$-close-to-convex functions in $E$ by formally replacing $f(z)$ in the denominator of (1.3) by a starlike function $\phi(z)$ in $S^*$. That is, $f \in C_\alpha$ if there exists a $\phi(z) \in S^*$ such that

$$(1.4) \quad \text{Re}[(1 - \alpha) \frac{zf'(z)}{\phi(z)} + \alpha \frac{(zf'(z))'}{\phi'(z)}] > 0 \text{ for } z \in E, \alpha \geq 0.$$ 

Silvia [8] generalized the definition of $\alpha$-convexity to $\alpha$-$\lambda$-spirallikeness as follows:

Let $S_\alpha$ denote the class of $\alpha$-$\lambda$-spirallike functions in $E$, where $f \in S_\alpha$ if

$$(1.5) \quad \text{Re}[(e^{i\lambda} - \alpha) \frac{zf'(z)}{f(z)} + \alpha \frac{(zf'(z))'}{f'(z)}] > 0 \text{ for } z \in E, \alpha \geq 0, |\lambda| < \frac{\pi}{2}.$$ 

He proved that $S_\alpha \subset S_p(\lambda) \subset S$.

Now we will generalize Silvia class by formally replacing $f(z)$ in the denominator of (1.5) by a spiral-like function $g(z)$ in $S_p(\lambda)$. We denote this new class of functions by $Y(\alpha, \lambda)$.

**DEFINITION.** Let $f(z)$ be holomorphic in $E$ with $f(0) = f'(0) - 1 = 0$. $f(z)$ belongs to the class $Y(\alpha, \lambda)$ if there exists a $g(z) \in S_p(\lambda)$ such that

$$(1.6) \quad \text{Re}[(e^{i\lambda} - \alpha) \frac{zf'(z)}{g(z)} + \alpha \frac{(zf'(z))'}{g'(z)}] > 0$$
for $z \in E, \alpha \geq 0, |\lambda| < \frac{\pi}{2}$.

Note that if $\alpha=0$, then $Y(\alpha, \lambda)$ is equal to Ziegler class $Z_{\lambda}$. If $\lambda = 0, Y(\alpha, \lambda)$ is equal to Chichra class $C_{\alpha}$. If $\alpha=0, \lambda=0$, then $Y(\alpha, \lambda)$ is the class of close-to-convex functions in $E$.

In this note we prove some geometric properties for the functions $f(z)$ in $Y(\alpha, \lambda)$. We obtain the integral representation formula for the functions in $Y(\alpha, \lambda)$. We prove that for every admissible $\alpha$ and $\lambda$, each function $f(z)$ in $Y(\alpha, \lambda)$ lies in Ziegler class $Z_{\lambda}$ and find the disk $|z| < r_0$ so that functions in Ziegler class $Z_{\lambda}$ may satisfy the defining inequality (1.6) for the class $Y(\alpha, \lambda)$. Moreover, we solve the radius of convexity problem for the class $Y(\alpha, \lambda)$ and a covering theorem as well as a few coefficient estimates for functions in $Y(\alpha, \lambda)$.

2. Main Results for the class $Y(\alpha, \lambda)$

We now obtain an integral representation for the functions in the class $Y(\alpha, \lambda)$

**Theorem 2.1.** $f(z)$ is in the class $Y(\alpha, \lambda)$ if and only if there exists a regular function $p(z)$ with $p(0) = 1$, $Re \ p(z) > 0$ for $z \in E$ such that

$$f'(z) = \frac{1}{\alpha z[g(z)]^{\frac{e^{i\lambda}}{\alpha} - 1}} \int_0^z (\cos \lambda p(\zeta) + i \sin \lambda)g'(\zeta)[g(\zeta)]^{\frac{e^{i\lambda}}{\alpha} - 1} d\zeta,$$

where the powers are taken as principal values and $g(z) \in S_p(\lambda), \alpha \neq 0$.

If $\alpha = 0$, then

$$f'(z) = e^{-i\lambda}z^{-1}g(z)[\cos \lambda p(z) + i \sin \lambda].$$

**Proof.** If $f(z) \in Y(\alpha, \lambda)$, then for some regular function $p(z)$ with $p(0)=1$ and $Re \ p(z) > 0$ for $z \in E$, we can write

$$(e^{i\lambda} - \alpha)zf'(z)g(z) + \alpha z f'(z)g'(z) = \cos \lambda p(z) + i \sin \lambda$$

where $g(z) \in S_p(\lambda)$ and $|\lambda| < \frac{\pi}{2}$.

Multiplying both sides of (2.2) by $\alpha^{-1}g'(z)[g(z)]^{\frac{e^{i\lambda}}{\alpha} - 1}$, we have

$$e^{i\lambda} - 1)zf'(z)g'(z)[g(z)]^{\frac{e^{i\lambda}}{\alpha} - 2} + (zf'(z))[g(z)]^{\frac{e^{i\lambda}}{\alpha} - 1}$$
\[ = (\cos \lambda p(z) + i \sin \lambda)\alpha^{-1} g'(z)[g(z)]^{\frac{s\lambda}{\alpha} - 1}. \]

The left hand side of (2.3) is the exact differential of \(zf'(z)[g(z)]^{\frac{s\lambda}{\alpha} - 1}\).

So integrating both sides of (2.3) with respect to \(z\), we obtain

\[ zf'(z)[g(z)]^{\frac{s\lambda}{\alpha} - 1} = \alpha^{-1} \int_0^z (\cos \lambda p(\zeta) + i \sin \lambda)g'(\zeta)[g(\zeta)]^{\frac{s\lambda}{\alpha} - 1} d\zeta, \]

and get

\[ f'(z) = \frac{1}{\alpha z [g(z)]^{\frac{s\lambda}{\alpha} - 1}} \int_0^z (\cos \lambda p(\zeta) + i \sin \lambda)g'(\zeta)[g(\zeta)]^{\frac{s\lambda}{\alpha} - 1}. \]

In particular if \(\alpha = 0\), we have

\[ e^{i\lambda}zf'(z) = \cos \lambda p(z) + i \sin \lambda, \]

and hence

\[ f'(z) = \frac{e^{-i\lambda}}{z}g(z)[\cos \lambda p(z) + i \sin \lambda]. \]

Conversely, if \(f(z)\) satisfies (2.1), we get,

\[ \frac{zf'(z)}{g(z)} = \frac{1}{\alpha [g(z)]^{\frac{s\lambda}{\alpha}}} \int_0^z (\cos \lambda p(\zeta) + i \sin \lambda)g'(\zeta)[g(\zeta)]^{\frac{s\lambda}{\alpha} - 1} d\zeta \]

and

\[ \frac{(zf'(z))'}{g'(z)} = \frac{\alpha^{-1}}{[g(z)]^{\frac{s\lambda}{\alpha}}}(1 - \frac{e^{i\lambda}}{\alpha}) \cdot \int_0^z (\cos \lambda p(\zeta) + i \sin \lambda)g'(\zeta)[g(\zeta)]^{\frac{s\lambda}{\alpha} - 1} d\zeta + \alpha^{-1} [\cos \lambda p(z) + i \sin \lambda]. \]

It follows that

\[ (e^{i\lambda} - \alpha)\frac{zf'(z)}{g(z)} + \alpha\frac{(zf'(z))'}{g'(z)} = \cos \lambda p(z) + i \sin \lambda \]

which implies \(f(z) \in Y(\alpha, \lambda)\), since \(Re \ p(z) > 0\) for \(z \in E\). \(\square\)

In order to verify that a function \(f(z)\) in \(Y(\alpha, \lambda)\) belongs to the Ziegler's class \(Z_\lambda\), we need the following lemma which is due to I. S. Jack [3].
LEMMA 1. [3]. Let $\omega(z)$ be regular in $E$ with $\omega(0) = 0$. If there exists a $\zeta$ in $E$ such that

$$\max_{|z| \leq |\zeta|} |\omega(z)| = |\omega(\zeta)|,$$

then $\zeta \omega'(\zeta) = k \omega(\zeta)$ for some $k \geq 1$.

THEOREM 2.2. For $\alpha \geq 0, |\lambda| < \frac{\pi}{2}$, let $f(z)$ be in the class $Y(\alpha, \lambda)$. Then $f(z)$ satisfies the condition

$$\text{Re} \left[ e^{i\lambda} \frac{zf'(z)}{g(z)} \right] > 0, \quad (z \in E) \quad \text{for some } g(z) \in S_p(\lambda),$$

and hence lies in Ziegler class $Z_{\lambda}$.

PROOF. If $f(z) \in Y(\alpha, \lambda)$, let us set

$$(2.4) \quad e^{i\lambda} \frac{zf'(z)}{g(z)} = \cos \lambda \frac{1 - \omega(z)}{1 + \omega(z)} + i \sin \lambda \quad (|\lambda| < \frac{\pi}{2}, g(z) \in S_p(\lambda))$$

where $\omega(z)$ is regular in $E$ with $\omega(0) = 0$ and $\omega(z) \neq -1$ for $z \in E$. Since $\text{Re} \frac{1 - \omega(z)}{1 + \omega(z)} > 0$ whenever $|\omega(z)| < 1$ for $z \in E$, it suffices to show that $|\omega(z)| < 1$ for $z \in E$ in (2.4). Simplifying (2.4), it follows that

$$(2.5) \quad e^{i\lambda} \frac{zf'(z)}{g(z)} = e^{i\lambda} - \omega(z) e^{-i\lambda} \frac{1 + \omega(z)}{1 + \omega(z)}.$$

Differentiating (2.5) and using the condition (1.6), we have

$$\left( e^{i\lambda} - \alpha \right) \frac{zf'(z)}{g(z)} + \alpha \frac{(zf'(z))'}{g'(z)}$$

$$= \cos \lambda \frac{1 - \omega(z)}{1 + \omega(z)} + i \sin \lambda + \alpha e^{-i\lambda} \cos \lambda \left\{ \frac{g(z)}{g'(z)} \cdot \frac{-2 \omega'(z)}{(1 + \omega(z))^2} \right\}.$$

Now, suppose that $|\omega(z)| \geq 1$ for $z \in E$. Then there exists a $\zeta$ in $E$ such that $\max_{|z| \leq |\zeta|} |\omega(z)| = |\omega(\zeta)| = 1$.

By Lemma 1, $\zeta \omega'(\zeta) = k \omega(\zeta)$ for some $k \geq 1$. For this $\zeta$ we have

$$\text{Re} \left[ \frac{1 - \omega(\zeta)}{1 + \omega(\zeta)} \right] = 0.$$
and 
\[ \frac{\omega(\zeta)}{(1 + \omega(\zeta))^2} = \frac{1}{4 \cos^2 \frac{\theta}{2}}, \quad \text{where} \quad \omega(\zeta) = e^{i\theta}. \]

Then
\[
Re\{ (e^{i\lambda} - \alpha) \frac{\zeta f''(\zeta)}{g(\zeta)} + \alpha \frac{(\zeta f'(\zeta))^'}{g'(\zeta)} \} \\
= Re\{ \alpha \cos \lambda \frac{e^{-i\lambda} g(\zeta)}{\zeta g'(\zeta)} \cdot \frac{-2k\omega(\zeta)}{(1 + \omega(\zeta))^2} \}
\]
\[
= -\frac{\alpha k \cos \lambda}{2 \cos^2 \frac{\theta}{2}} Re\{ \frac{e^{-i\lambda} g(\zeta)}{\zeta g'(\zeta)} \} \quad (|\lambda| < \frac{\pi}{2}).
\]

Since \( g(z) \in S_p(\lambda) \), \( Re\frac{e^{-i\lambda} g(\zeta)}{\zeta g'(\zeta)} > 0 \) for \( \zeta \in E \).
Therefore, we have
\[
Re\{ (e^{i\lambda} - \alpha) \frac{\zeta f''(\zeta)}{g(\zeta)} + \alpha \frac{(\zeta f'(\zeta))^'}{g'(\zeta)} \} \leq 0 \quad (\zeta \in E).
\]

But this contradicts (1.6), since \( f \in Y(\alpha, \lambda) \).

Hence,
\[
Re \frac{e^{i\lambda} zf'(z)}{g(z)} > 0 \quad \text{for} \quad z \in E.
\]

This completes the proof. \( \square \)

**Corollary.** If \( \alpha > \beta \geq 0 \) and \( |\lambda| < \frac{\pi}{2} \), then \( Y(\alpha, \lambda) \subset Y(\beta, \lambda) \).

**Proof.** For \( \beta = 0 \), Theorem 2.2 shows that \( Y(\alpha, \lambda) \subset Y(0, \lambda) = Z_\lambda \).

For \( \beta \neq 0 \)
\[
(e^{i\lambda} - \beta) \frac{zf'(z)}{g(z)} + \beta \frac{(zf'(z))^'}{g'(z)}
\]
\[
= \frac{\beta}{\alpha} [e^{i\lambda} (\frac{\alpha}{\beta} - 1) \frac{zf'(z)}{g(z)} + (e^{i\lambda} - \alpha) \frac{zf'(z)}{g(z)} + \alpha \frac{(zf'(z))^'}{g'(z)}].
\]
For $\alpha > \beta$, if $f(z) \in Y(\alpha, \lambda)$, then

$$\text{Re}\{(e^{i\lambda} - \alpha)\frac{zf''(z)}{g(z)} + \alpha(zf''(z))' + \alpha g'(z)\} > 0 \quad (z \in E).$$

Now $f(z) \in Y(\alpha, \lambda) \subset Z_\lambda$ implies that $\text{Re} \ e^{i\lambda} \frac{zf''(z)}{g(z)} > 0$. Hence,

$$\text{Re}\{(e^{i\lambda} - \beta)\frac{zf''(z)}{g(z)} + \beta(zf''(z))' + \beta g'(z)\} > 0$$

showing that $f(z) \in Y(\beta, \lambda)$.

We now show that functions in the Ziegler class $Z_\lambda$ satisfy the defining inequality for the class $Y(\alpha, \lambda)$ on a certain disk $|z| < r_0$.

**Theorem 2.3.** If $f(z)$ belongs to the Ziegler class $Z_\lambda$ with $f(0) = 0$, $f'(0) = 1$, then $f(z)$ satisfies the inequality

$$\text{Re}\left[(e^{i\lambda} - \alpha)\frac{zf''(z)}{g(z)} + \alpha(zf''(z))' + \alpha g'(z)\right] > 0$$

for $|z| < r_0$ where $r_0 = (1 + \alpha) - \sqrt{(1 + \alpha)^2 - 1}$.

**Proof.** If $f(z) \in Z_\lambda$, there exists a $g(z) \in S_p(\lambda)$ such that

$$\text{Re} \ e^{i\lambda} \frac{zf''(z)}{g(z)} > 0 \text{ for } |\lambda| < \frac{\pi}{2}.$$

Let $\omega(z)$ be the regular function in $E$ with $\omega(0) = 0$, $|\omega(z)| < 1$ for $z \in E$, given by

$$e^{i\lambda} \frac{zf''(z)}{g(z)} = \cos \lambda \frac{1 + \omega(z)}{1 - \omega(z)} + i \sin \lambda. \quad (2.6)$$

By logarithmic differentiation of (2.6) we get

$$1 + \frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)} = \frac{(1 + e^{-2i\lambda})\omega'(z)}{(1 + e^{-2i\lambda}\omega(z))(1 - \omega(z))}. $$
Thus,
\[
(e^{i\lambda} - \alpha) \frac{zf'(z)}{g(z)} + \alpha \frac{(zf'(z))'}{g'(z)}
= \frac{e^{i\lambda} + e^{-i\lambda} \omega(z)}{1 - \omega(z)} + \alpha \frac{(1 + e^{-2i\lambda})z\omega'(z)}{(1 + e^{-2i\lambda} \omega(z))(1 - \omega(z))} \frac{f'(z)}{g'(z)}
= \frac{e^{i\lambda} + e^{-i\lambda} \omega(z)}{1 - \omega(z)} + \alpha \frac{(1 + e^{-2i\lambda})z\omega'(z)}{(1 - \omega(z))^2} \frac{g(z)}{zg'(z)}
\]

where the last equality is obtained by (2.6).

In order to show that \(f(z) \in Y(\alpha, \lambda)\) for \(|z| < r_0\), where \(r_0 = (1 + \alpha) - \sqrt{(1 + \alpha)^2 - 1}\), we observe that

\[
\text{Re} \left[ (e^{i\lambda} - \alpha) \frac{zf'(z)}{g(z)} + \alpha \frac{(zf'(z))'}{g'(z)} \right]
\geq \text{Re} \left[ \frac{e^{i\lambda} + e^{-i\lambda} \omega(z)}{1 - \omega(z)} - \alpha \frac{(1 + e^{-2i\lambda})z\omega'(z)}{(1 - \omega(z))^2} \frac{g(z)}{zg'(z)} \right].
\]

By using the well known inequalities \([2]\)

\[
|\omega'(z)| \leq \frac{1 - |\omega(z)|^2}{1 - r^2}, \quad \left| \frac{g(z)}{zg'(z)} \right| \leq \frac{1 + r}{1 - r}
\]

and

\[
|1 - \omega(z)| \leq 1 + |\omega(z)| \leq 1 + r, \quad (|z| = r),
\]

we have

\[
\text{Re} \left[ \frac{e^{i\lambda} + e^{-i\lambda} \omega(z)}{1 - \omega(z)} - \alpha \frac{(1 + e^{-2i\lambda})z\omega'(z)}{(1 - \omega(z))^2} \frac{g(z)}{zg'(z)} \right]
\geq \frac{\cos \lambda (1 - |\omega(z)|^2)}{|1 - \omega(z)|^2} - \alpha \left[ 1 + e^{-2i\lambda} \right] \frac{(1 - |\omega(z)|^2)r}{(1 - \omega(z))^2(1 - r)^2}
\geq \frac{\cos \lambda (1 - 2r - 2\alpha r + r^2)(1 - |\omega(z)|^2)}{(1 - r)^2(1 - \omega(z))^2}
\geq \frac{\cos \lambda (1 - 2r - 2\alpha r + r^2)}{(1 - r)^2}.
\]

Since the smallest positive root of \(1 - 2(1 + \alpha)r + r^2 = 0\) is \(r_0 = (1 + \alpha) - \sqrt{(1 + \alpha)^2 - 1}\), \(f(z)\) satisfies the inequality (1.6) for \(|z| < r_0\). \(\Box\)
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Theorem 2.4. Let \( f(z) \in Y(\alpha, \lambda) \). Then \( w = f(z) \) maps the disk \(|z| < r_1\) onto a convex region where \( r_1 \) is the smallest positive root of the equation

\[
\alpha|\alpha + e^{i\lambda}| - r\{\alpha + e^{i\lambda}| (2\alpha \cos \lambda + 4 \cos \lambda + \alpha) + 2\alpha \cos \lambda \}
+ r^2\{\alpha + e^{i\lambda}| (2\alpha \cos^2 \lambda + 2 \cos \lambda - 2 \cos \lambda - \alpha) - 2\alpha \cos \lambda \}
- r^3|\alpha + e^{i\lambda}| (2\alpha \cos^2 \lambda - 2 \cos \lambda - \alpha) = 0, \quad |z| = r.
\]

Proof. If \( f(z) \in Y(\alpha, \lambda) \) the integral representation formula shows that

\[
(2.7) \quad f'(z) = \frac{1}{\alpha z[g(z)]^\frac{\alpha}{\alpha}} \int_0^z \frac{\cos \lambda p(\zeta) + i \sin \lambda)g'(\zeta)[g(\zeta)]^{\frac{\alpha}{\alpha}-1} d\zeta
\]

for \( z \in E \) and \(|\lambda| < \frac{\pi}{2}\), where the powers are taken as principal values. By logarithmic differentiation of (2.7), we obtain

\[
(2.8) \quad 1 + \frac{zf''(z)}{f'(z)} = (1 - \frac{e^{i\lambda}}{\alpha}) \frac{zg'(z)}{g(z)} + \frac{z(\cos \lambda p(\zeta) + i \sin \lambda)g'(\zeta)[g(\zeta)]^{\frac{\alpha}{\alpha}-1}}{\int_0^z (\cos \lambda p(\zeta) + i \sin \lambda)g'(\zeta)[g'(\zeta)]^{\frac{\alpha}{\alpha}-1} d\zeta}.
\]

Thus

\[
Re\{1 + \frac{zf''(z)}{f'(z)}\} = Re(1 - \frac{e^{i\lambda}}{\alpha}) \frac{zg'(z)}{g(z)} + ReF(z),
\]

where \( F(z) \) is the second term of the right hand side of (2.8).

Since \( g(z) \in S_p(\lambda) \),

\[
Re\{e^{i\lambda} \frac{zg'(z)}{g(z)}\} \leq \cos \lambda \frac{1 + r}{1 - r}.
\]

If we let \( g(z) = z + b_2z^2 + \cdots \) and \( p(z) = 1 + p_1z + \cdots \) in \( F(z) \), a brief calculation shows that \( F(z) \) has a series expansion

\[
F(z) = \frac{e^{i\lambda}}{\alpha} + \frac{\alpha \cos \lambda p_1 + e^{i\lambda}(\alpha + e^{i\lambda}b_2)}{\alpha(\alpha + e^{i\lambda})}z + \cdots.
\]
Furthermore, $F(z)$ is a regular function in $E$ and hence

$$|F(z) - \frac{e^{i\lambda}}{\alpha}| \leq \frac{2\alpha \cos \lambda + 2|\alpha + e^{i\lambda}| \cos \lambda}{\alpha|\alpha + e^{i\lambda}|} \frac{r}{(1-r)^2}.$$ 

This inequality shows that

$$ReF(z) \geq \frac{\cos \lambda}{\alpha} - \frac{2 \cos \lambda (\alpha + |\alpha + e^{i\lambda}|)}{\alpha|\alpha + e^{i\lambda}|} \frac{r}{(1-r)^2}.$$ 

Robertson [6] showed that if $g(z) \in S_p(\lambda)$, then for $|z| = r < 1$

$$\frac{(1 - \lambda \cos \lambda)^2 - r^2 \sin^2 \lambda}{1 - r^2} \leq Re \frac{zg'(z)}{g(z)} \leq \frac{(1 + \lambda \cos \lambda)^2 - r^2 \sin^2 \lambda}{1 - r^2}.$$ 

Using (2.9) we have

$$Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} \geq \frac{(1 - \lambda \cos \lambda)^2 - r^2 \sin^2 \lambda}{1 - r^2} - \frac{\cos \lambda (1 + r)}{\alpha(1-r)}$$

$$+ \frac{\cos \lambda}{\alpha} - \frac{2r \cos \lambda (\alpha + |\alpha + e^{i\lambda}|)}{\alpha|\alpha + e^{i\lambda}|(1-r)^2}$$

$$= \frac{N(r)}{\alpha|\alpha + e^{i\lambda}|(1+r)(1-r)^2},$$

where

$$N(r) = \alpha|\alpha + e^{i\lambda}| - r|\alpha + e^{i\lambda}|(2\alpha \cos \lambda + 4 \cos \lambda + \alpha) + 2\alpha \cos \lambda$$

$$+ r^2|\alpha + e^{i\lambda}|(2\alpha \cos^2 \lambda + 2\alpha \cos \lambda - 2 \cos \lambda - \alpha) - 2\alpha \cos \lambda$$

$$- r^3|\alpha + e^{i\lambda}|(2\alpha \cos^2 \lambda - 2 \cos \lambda - \alpha).$$

Here, $N(0) = \alpha|\alpha + e^{i\lambda}| > 0$ and $N(1) = -4 \cos \lambda (\alpha + |\alpha + e^{i\lambda}|) < 0$. Hence $f(z)$ is convex for $|z| < r_1$ where $r_1$ is the smallest positive root of $N(r) = 0$. This completes the proof. 

To discuss coefficient estimates for the class $Y(\alpha, \lambda)$ we need the following lemma which is due to Libera.
Lemma 2. [5]. If \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) belongs to \( S_p(\lambda) \) in \( E \), then

\[
|a_n| \leq \prod_{k=0}^{n-2} \frac{|2 \cos \lambda e^{-i\lambda} + k|}{|k + 1|}, \quad n = 2, 3, 4, \ldots,
\]

and these bounds are sharp for all admissible \( \lambda \) and for each \( n \).

Theorem 2.5. Let \( f(z) \) be in the class \( Y(\alpha, \lambda) \). If \( f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \), then

\[
|a_2| \leq \frac{\cos \lambda (1 + |\alpha + e^{i\lambda}|)}{|\alpha + e^{i\lambda}|}
\]

and

\[
|a_3| \leq \frac{\cos \lambda \sqrt{1 + 8 \cos^2 \lambda}}{3} + \frac{4 \cos^2 \lambda |3\alpha + e^{i\lambda}|}{3|\alpha + e^{i\lambda}||2\alpha + e^{i\lambda}|} + \frac{2 \cos \lambda}{3|2\alpha + e^{i\lambda}|}.
\]

These bounds are sharp for all admissible \( \alpha \) and \( \lambda \).

Proof. Let \( P = \{ p(z); p(z) \) is regular in \( E \) with \( p(0) = 1, \Re p(z) > 0 \} \). If \( f(z) \) is in \( Y(\alpha, \lambda) \), we can write \( , \) for some \( g(z) \in S_p(\lambda) \) and \( p(z) \in P \),

\[
(e^{i\lambda} - \alpha) \frac{zf'(z)}{g(z)} + \alpha \frac{zf'(z)'}{g'(z)} = \cos \lambda p(z) + i \sin \lambda.
\]

Let \( g(z) = z + b_2 z^2 + b_3 z^3 + \cdots \) and \( p(z) = 1 + p_1 z + p_2 z^2 + \cdots \). Then (2.10) can be written as

\[
(e^{i\lambda} - \alpha)(z + 2a_2 z^2 + 3a_3 z^3 + \cdots)(1 + 2b_2 z + 3b_3 z^2 + \cdots) + \alpha(1 + 4a_2 z + 9a_3 z^2 + \cdots)(z + b_2 z^2 + b_3 z^3 + \cdots) = [e^{i\lambda} + \cos \lambda(p_1 z + p_2 z^2 + \cdots)](z + b_2 z^2 + \cdots)(1 + 2b_2 z + \cdots).
\]

On equating both sides of (2.11), we get

\[
2(\alpha + e^{i\lambda})a_2 = (\alpha + e^{i\lambda})b_2 + \cos \lambda p_1
\]
and

\[ (2.13) \]

\[ 3(2\alpha + e^{i\lambda})a_3 = -4e^{i\lambda}a_2b_2 + (2\alpha + e^{i\lambda})b_3 + 2e^{i\lambda}b_2^2 + (3b_2p_1 + p_2)\cos \lambda. \]

It is well known [2] that \(|p_n| \leq 2\) for \(n = 1, 2, 3, \cdots\). By Lemma 2, (2.12) reduces to

\[ |\alpha + e^{i\lambda}|a_2| \leq \cos \lambda (1 + |\alpha + e^{i\lambda}|). \]

Hence,

\[ |a_2| \leq \frac{\cos \lambda (1 + |\alpha + e^{i\lambda}|)}{|\alpha + e^{i\lambda}|}. \]

Now substituting (2.12) into (2.13) we obtain

\[ 3(2\alpha + e^{i\lambda})(\alpha + e^{i\lambda})a_3 = (\alpha + e^{i\lambda})(2\alpha + e^{i\lambda})b_3 + \cos \lambda (3\alpha + e^{i\lambda})b_2p_1 + \cos \lambda (\alpha + e^{i\lambda})p_2. \]

By lemma 2 again, we have

\[ |a_3| \leq \frac{\cos \lambda \sqrt{1 + 8\cos^2 \lambda}}{3} + \frac{4\cos^2 \lambda|3\alpha + e^{i\lambda}|}{3|\alpha + e^{i\lambda}| |2\alpha + e^{i\lambda}|} + \frac{2\cos \lambda}{3|2\alpha + e^{i\lambda}|}. \]

The functions \(g(z) = z(1 - z)^{-2\cos \lambda e^{-i\lambda}}\) and \(p(z) = \frac{1 + e^{-i\lambda}z}{1 - e^{-i\lambda}z}\) show that the results are sharp.

Using the second coefficient estimate for the class \(Y(\alpha, \lambda)\), we obtain the following result similar to Koebe ‘s covering theorem.

**Theorem 2.6.** Let \(f(z)\) be in the class \(Y(\alpha, \lambda)\) and let \(\omega\) be any complex number such that \(f(z) \neq \omega\) for \(z\) in \(E\). Then

\[ |\omega| \geq \frac{\alpha + 1}{3\alpha + 4}, \quad (\alpha \geq 0). \]

**Proof.** Let us write

\[ f_1(z) = \frac{\omega f(z)}{\omega - f(z)}. \]

Then \(f_1(z)\) belong to \(S\) and \(f_1(z) = z + (a_2 + \frac{1}{\omega})z^2 + \cdots\). Hence, \(|a_2 + \frac{1}{\omega}| \leq 2\). By Theorem 2.5, we obtain

\[ \left| \frac{1}{\omega} \right| \leq \frac{(2 + \cos \lambda)|\alpha + e^{i\lambda}| + \cos \lambda}{|\alpha + e^{i\lambda}|} \leq \frac{4 + 3\alpha}{1 + \alpha}. \]

Hence,

\[ |\omega| \geq \frac{\alpha + 1}{3\alpha + 4}. \]

\[ \square \]
A generalization of Silvia class of functions

References


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