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MULTIPLICITY RESULT FOR SEMILINEAR PARABOLIC EQUATIONS

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ABSTRACT. An Ambrosetti-Prodi type multiplicity result for periodic-Dirichlet problem to semilinear parabolic equation is treated.

1. Introduction

Let Z^+, Z, R^* and R be the set of all positive integers, integers, nonnegative reals and reals, respectively, and let $\Omega \subseteq R^n$, $n \geq 1$, be a bounded domain with smooth boundary $\partial\Omega$ which is assumed to be of class C^2.

Let $Q = (0, 2\pi) \times \Omega$ and $L^2(Q)$ be the space of measurable Lebesgue square integrable real-valued functions on Q with usual inner product $< \cdot, \cdot >$ and corresponding norm $\| \cdot \|_2$.

By $H_0^1(\Omega)$ we mean the completion of $C_0^1(\Omega)$ with respect to the norm $\| \cdot \|_1$ defined by

$$\|\phi\|_1^2 = \int_\Omega \sum_{|\alpha| \leq 1} |D^\alpha \phi(x)|^2 dx.$$

$H^2(\Omega)$ stands for the usual Sovolev space ; i.e., the completion of $C^2(\bar{\Omega})$ with respect to the norm $\| \cdot \|_2$ defined by

$$\|\phi\|_2^2 = \int_\Omega \sum_{|\alpha| \leq 2} |D^\alpha \phi(x)|^2 dx.$$

Let $g : R \to R$ be a continuous function. Moreover, we assume that there exist constants a_0 and b_0 such that

$$(H_1) \qquad |g(u)| \leq a_0|u| + b_0 \text{ for all } u \in R.$$

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The purpose of this work is to investigate a multiplicity result for weak solution of the nonlinear parabolic equations

$$(E) \qquad \frac{\partial u}{\partial t} - \Delta_x u - \lambda_1 u + g(u) = \frac{s\phi_1}{\sqrt{2\pi}} + h(t,x) \text{ in } Q$$

$$(B_1) \qquad u(t,x) = 0 \text{ on } (0,2\pi) \times \partial\Omega$$

$$(B_2) \qquad u(0,x) = u(2\pi,x) \text{ on } \Omega$$

where λ_1 denotes the first eigenvalue of $-\Delta$ with zero Dirichlet boundary condition and ϕ_1 is the correspanding positive normalized eigenfunction; i.e., $\phi_1(x) > 0$ on Ω and $\int_\Omega \phi_1^2(x)dx = 1$, and $h \in L^2(Q)$ with

$$\iint_Q h(t,x)\phi_1(x)dtdx = 0.$$

More precisely, the purpose is to find constants $s_0 < s_1$ such that the problem
$(E)(B_1)(B_2)$ has no solution, at least one solution, or at least two solutions according to $s < s_0$, $s = s_1$ or $s > s_1$.

This type of result, so-called an Ambrosetti-Prodi type result has been initiated by Ambrosetti-Prodi [1] in 1972 in the study of a Dirichlet problem to elliptic equations and developed in various directions by several authors to ordinary and partial differential equations. A notable discussion for AP type results for periodic and Dirichlet boundary value problem has been done by Fabry, Mawhin and Nkashama [4] and Chiappinelli, Mawhin and Nugari [2], respectively, for second order ordinary differential equations. For AP type result for periodic solutions of higher order ordinary differential equations, we refer the results of Ding and Mawhin in [3]. AP type results for Lienard systems have been done by Kim [8], and Hirano and Kim [7], and AP type results for dissipative hyperbolic equations have been done by Kim [9]. Lazer and Mckenna treated AP type multiplicity result for elliptic and parabolic equations in [10]. In our result, we assume the coercive growth condition on g

and make use of degree theory in our proof. Our result, in particular, is different from that of [10].

Here we assume the following

$$(H_2) \qquad \lim_{|u|\to\infty} \inf g(u) = +\infty,$$

$$(H_3) \qquad \lim_{u\to-\infty} \sup |\frac{g(u)}{u}| < \lambda_2 - \lambda_1.$$

Then we have that

THEOREM. *Assume (H_1), (H_2) and (H_3). Then there exist real numbers $s_0 \leq s_1$ such that*

(i) *$(E)(B_1)(B_2)$ has no solution for $s < s_0$.*

(ii) *$(E)(B_1)(B_2)$ has at least one solution for $s = s_1$.*

(iii) *$(E)(B_1)(B_2)$ has at least two solution for $s > s_1$.*

2. Preliminary results

Let's define the linear operator

$$L : DomL \subseteq L^2(Q) \to L^2(Q)$$

by

$$DomL = \{u \in L^2((0, 2\pi), H^2(\Omega) \cap H_0^1(\Omega)) | \frac{\partial u}{\partial t} \in L^2(Q), \\ u(0, x) = u(2\pi, x), x \in \Omega\}$$

and

$$Lu = \frac{\partial u}{\partial t} - \Delta u - \lambda_1 u$$

Using Fourier series and Parseval inequality, we get easily

$$< Lu, \frac{\partial u}{\partial t} >= \|\frac{\partial u}{\partial t}\|_{L^2}^2 \text{ for all } u \in DomL.$$

Hence $kerL = ker(\Delta + \lambda_1 I) = kerL^*$ since $\Delta + \lambda_1 I$ is self-adjoint and $ker(\Delta + \lambda_1 I)$ is one space dimension generated by the eigenfunction ϕ_1 . Therefore L is a closed, densely defined linear operator and $Im(L) = [kerL]^{\perp}$; i.e., $L^2(Q) = kerL \bigoplus ImL$. Let's consider a continous projection $P_1 : L^2(Q) \to L^2(Q)$ such that $kerP_1 = ImL$. Then $L^2(Q) = kerL \bigoplus kerP_1$. We consider another continuous projection $P_2 : L^2(Q) \to L^2(Q)$ defined by

$$(P_2 h)(t,x) = \frac{1}{2\pi} \iint_Q h(t,x)\phi_1(x)dtdx\phi_1(x).$$

Then we have $L^2(Q) = ImP_1 \bigoplus ImL$, $kerP_2 = ImL$, and $L^2(Q)/ImL$ is isomorphism to ImP_2.

Since $dim[L^2(Q)/ImL] = dim[ImP_2] = dim[kerL] = 1$, we have an isomorphism $J : ImP_2 \to kerL$.

By the closed graph theorm, the generalized right inverse of L defined by

$$K = [L|_{DomL \cap ImL}]^{-1} : ImL \to ImL$$

is continuous. If we equip the space $DomL$ with the norm

$$\|u\|_{DomL} = \iint_Q [u^2 + (\frac{\partial u}{\partial t})^2 + \sum_{|\beta| \le 2} (D_x^{\beta} u)^2] dtdx,$$

then there exist a constant $c > 0$ independently of $h \in ImL$, $u = Kh$ such that

$$\|Kh\|_{DomL} \le c\|h\|_{L^2}.$$

Therefore $K : ImL \to ImL$ is continuous and by the compact imbedding of $DomL$ in $L^2(Q)$, we have that $K : ImL \to ImL$ is compact

LEMMA 2.1. *L is closed , densely defined linear operator such that $kerL = [ImL]^{\perp}$ and such that the right inverse $K : ImL \to ImL$ is completely continuous.*

3. Multiplicity result

Let us consider the following

$$(E_s^{\mu}) \qquad \frac{\partial u}{\partial t} - \Delta_x u - \lambda_1 u + \mu g(u) = \mu s\phi + \mu h(t,x) \text{ in } Q$$

$$(B_1) \qquad u(t,x) = 0 \quad \text{on} \quad (0,2\pi) \times \partial\Omega$$

$$(B_2) \qquad u(0,x) = u(2\pi,x) \quad \text{on} \quad \Omega$$

where $\mu \in [0,1]$ and $\phi(x) = \frac{\phi_1(x)}{\sqrt{2\pi}}$.

Let $L : DomL \subseteq L^2(Q) \to L^2(Q)$ be defined as before. If we define a substitution operator $N_s^\mu : L^2(Q) \to L^2(Q)$ by

$$(N_s^\mu)(t,x) = \mu g(u) - \mu s\phi - \mu h(t,x)$$

for $u \in L^2(Q)$ and $(t,x) \in Q$, then N_s^μ maps continuously into itself and take bounded sets into bounded set. Let G be any open bounded subset of $L^2(Q)$. Then $P_2 N_s^\mu : \bar{G} \to L^2(Q)$ is bounded and $K(I - P_2) : \bar{G} \to L^2(Q)$ is compact and continuous. Thus N_s^μ is L-compact on $\bar{G}$.

The coincidence degree $D_L(L + N_s^\mu, G)$ is well defined and constant in μ if $Lu + N_s^\mu \neq 0$ for $\mu \in [0,1]$, $s \in R$ and $u \in DomL \cap \partial G$. It is easy to check that (u,μ) is a weak solution of (E_s^μ) if and only if $u \in DomL$ and

$$(3.1_s^\mu) \qquad Lu + N_s^\mu u = 0.$$

From (H_2) and (H_3), we may assume that

$$m = \inf_{u \in R} g(u) > -\infty$$

and there exist $a \in (0, \lambda_2 - \lambda_1)$ and $b \geq 0$ such that

$$|g(u)| \leq a|u| + b \text{ for all } u \leq 0.$$

Here we have the following lemma.

LEMMA 3.1. *If (H_1) (H_2) and (H_3) is satisfied, then for any $s^* \in R^+$, there exists $M(s^*) > 0$ such that*

$$\|u\|_{L^2} \leq M(s^*)$$

holds for each possible weak solution $u = \alpha\phi_1 + \tilde{u}$, with $\alpha \in R$ and $\tilde{u} \in ImL$, of (E_s^μ) with $\mu \in [0,1]$ and $|s| \leq s^*$.

PROOF. Suppose there exists a constant s with $|s| \leq s^*$ and corresponding solutions (u_n, μ_n) of $(3.1_s^{\mu_n})$ such that

$$\lim_{n \to \infty} \|u_n\|_{L^2} = \infty.$$

For each $n \geq 1$, we put $u_n(t,x) = \alpha_n \phi(x) + \tilde{u}_n(t,x)$.

By extracting subsequence, we may assume that

$$\lim_{n \to \infty} \frac{|\alpha_n|}{\|\tilde{u}_n\|_{L^2}} = c < \infty.$$

If it is not the case, then we have from the positivity of $\phi(x)$ that

$$\lim_{n \to \infty} |u_n(t,x)| = \infty \text{ a.e. on } Q.$$

By taking the inner product with ϕ on both sides of (3.1_s^{μ}), we have

$$\iint_Q g(u_n(t,x))\phi(x)dtdx = s \leq s^*.$$

On the other hand, by (H_2) and Fatou's lemma, we have

$$\lim_{n \to \infty} \inf \iint_Q g(u_n(t,x))\phi(x)dtdx = \infty$$

which leads to a contradiction. First, we assume that $0 < c < \infty$. Then there exist $n_0 \in N$ such that

$$(c/2)\|\tilde{u}_n\|_{L^2} \leq |\alpha_n| \leq (3c/2)\|\tilde{u}_n\|_{L^2} \text{ for all } n \geq n_0.$$

For given $\epsilon > 0$, we may choose $\delta > 0$ such that

$$\iint_A |\phi|^2 dtdx < \epsilon \|\phi\|_{L^2}^2$$

for any measurable set $A \subset \bar{Q}$ with $|A| \leq \delta$.

Let $0 < \beta < \|\phi\|_\infty$ and $\Omega_0 = \{x \in \Omega : \phi(x) \geq \beta\}$. Choose $M_0 > 0$ such that

$$\delta M_0 - |m| \iint_Q \phi dtdx > s^*.$$

Then, since $\lim_{|u|\to\infty} \inf g(u) = \infty$, we have that

$$m_0 = sup\{|u| : \beta g(u) < M_0\} < \infty.$$

We put

$$Q_n = \{(t,x) \in (0, 2\pi) \times \Omega_0 : |u_n(t,x)| \geq m_0\}.$$

Then we have $|Q_n| \leq \delta$. In fact if $|Q_n| > \delta$, then from the definition of m_0 we have

$$\begin{aligned}
&\iint_Q g(u_n(t,x))\phi(x)dtdx \\
&= \iint_{Q_n} g(u_n)\phi(x)dtdx + \iint_{Q\setminus Q_n} g(u_n)\phi(x)dtdx \\
&> \delta M_0 - |m| \iint_Q \phi(x)dtdx \\
&> s^*
\end{aligned}$$

and this lead a contradiction. Therefore, we have

$$\iint_{Q\setminus Q_n} |\alpha_n\phi|^2 \geq (1-\epsilon) \iint_Q |\alpha_n\phi|^2.$$

On the other hand,

$$\begin{aligned}
0 &= \iint_Q \alpha_n\phi\tilde{u}_n \\
&= \iint_{Q\setminus Q_n} \alpha_n\phi\tilde{u}_n + \iint_{Q_n} \alpha_n\phi\tilde{u}_n \\
&\leq (1/2) \iint_{Q\setminus Q_n} (|\alpha_n\phi + \tilde{u}_n|^2 - |\alpha_n\phi|^2 - |\tilde{u}_n|^2) + \iint_{Q_n} |\alpha_n\phi||\tilde{u}_n|.
\end{aligned}$$

From the definition of m_0 and the above facts, we have, for all $n \geq n_0$,

$$\begin{aligned} 0 &\leq (1/2)m_0^2 - (1/2)(1-\epsilon)(c/2)\|\tilde{u}_n\|_{L^2}^2 + \epsilon(3c/2)\|\tilde{u}_n\|_{L^2}^2 \\ &= (1/2)m_0^2 - (c/4)(1+5\epsilon c)\|\tilde{u}_n\|_{L^2}^2. \end{aligned}$$

Therefore, $\{\|\tilde{u}_n\|_{L^2}\}$ is bounded and hence $\{\|u_n\|_{L^2}\}$ is bounded which lead a contraction.

Next, we assume $c = 0$. Then $\lim_{n\to\infty} \frac{\|u_n\|}{\|\tilde{u}_n\|_{L^2}} = 1$.

Taking the inner product with u_n on both sides of (3.1_s^μ), we have

$$(\lambda_2 - \lambda_1)\|\tilde{u}_n\|_{L^2}^2 + < g(u_n), u_n > \leq s^*|\alpha_n| + \|h\|_{L^2}\|\tilde{u}_n\|_{L^2}$$

and hence

$$\lim_{n\to\infty} \sup(\lambda_2 - \lambda_1 - a)\|\tilde{u}_n\|_{L^2} \leq [\max\{|m|, b\}|Q|^{1/2} + \|h\|_{L^2}].$$

Thus $\{\|\tilde{u}_n\|_{L^2}\}$ is bounded and thus $\{\|u_n\|_{L^2}\}$ is bounded which leads to another contradiction. □

REMARK. By Lemma 3.1, we may have an a'priori bounds $M_1(s^*) > 0$ and $\gamma_1(s^*) > 0$ such that

$$|\alpha| \leq \gamma_1(s^*), \|\tilde{u}\|_{L^2} \leq M_1(s^*)$$

for each possible weak solution $u = \alpha\phi + \tilde{u}$ of (E_s^μ) with $|s| \leq s^*$ and $\mu \in [0,1]$.

LEMMA 3.2. *If (H_1), (H_2) and (H_3) are satisfied, then, for each $s^* \in R^+$, we can find an open bounded set $G(s^*)$ in $L^2(Q)$ such that, for each open bounded set G in $L^2(Q)$ such that $G \supseteq G(s^*)$, we have*

$$D_L(L + N_s^1, G) = 0 \quad \text{for all} \quad |s| \leq s^*.$$

PROOF. Suppose that $\alpha \in R$ and $|\alpha| \to \infty$, then $|\alpha\phi(x)| \to \infty$ for each $x \in \Omega_0$. Let M $= \min_{x\in\bar{\Omega}} g(\alpha\phi(x))\phi(x)$ and W$=(0,2\pi)\times\Omega_0$. Then, by Fatou's lemma and (H_2), we have

$$\begin{aligned}
&\lim_{|\alpha|\to\infty} \inf \iint_Q g(\alpha\phi(x))\phi(x)dtdx \\
&= \lim_{|\alpha|\to\infty} \inf \iint_Q [g(\alpha\phi(x))\phi(x) - M]dtdx + M|Q| \\
&\geq \iint_W \lim_{|\alpha|\to\infty} \inf[g(\alpha\phi(x))\phi(x) - M]dtdx + M|Q| \\
&= \infty.
\end{aligned}$$

Hence, there exists $r_2(s^*) > 0$ such that, for $|\alpha| > r_2(s^*)$, we have

$$\iint_Q g(\alpha\phi(x))\phi(x)dtdx > s^*.$$

Let

$$G(s^*) = \{u \in L^2(Q) | -\tilde{r}(s^*)\phi(x) < \alpha\phi(x) < \tilde{r}(s^*)\phi(x) \text{ for } x \in \Omega, \|\tilde{u}\|_{L^2} < M\}$$

where $u = \alpha\phi(x) + \tilde{u}$ with $\tilde{r}(s^*) > \max\{r_1(s^*), r_2(s^*)\}$ and $\tilde{M} > M$ which are given in Lemma 3.1 and Remark. Let

$$s_0 = d \min_{u\in R} g(u)$$

where $d = 2\pi \int_\Omega \phi(x)dx$. If $(3.1_{\bar{s}}^{\mu})$ has a solution u for some $\bar{s} \in R$ and $\mu \in [0,1]$, then by taking the inner product with ϕ on the both sides of the equation $(3.1_{\bar{s}}^{\mu})$, we have

$$s_0 \leq \iint_Q g(u(t,x))\phi(x)dtdx = \bar{s}.$$

Thus $(3.1_{\bar{s}}^{\mu})$ has no solution for $\bar{s} < s_0$.

Hence for each open bounded set $G \supseteq G(s^*)$, we have

$$D_L(L + N_{\bar{s}}^1, G) = 0 \quad \text{for} \quad \bar{s} < s_0.$$

Choose $\bar{s} < s_0$ and define

$$F : (D(L) \cap G) \times [0,1] \to L^2(\Omega)$$

by

$$F(u,\mu) = Lu + N_{(1-\mu)\bar{s}+\mu s}(u) \quad \text{for} \quad |s| \le s^*.$$

They by Lemma 3.1 and Remark, we have

$$0 \notin F(D(L) \cap \partial G) \times [0,1] \quad \text{for} \quad |s| \le s^*.$$

By the homotopy invariance of degree, we have, for all $|s| \le s^*$,

$$\begin{aligned} D_L(L + N_s^1, G) &= D_L(F(\cdot,1), G) \\ &= D_L(F(\cdot,0), G) \\ &= D_L(L + N_{\bar{s}}^1, G) \\ &= 0 \end{aligned}$$

and the proof is completed. □

LEMMA 3.3. *If (H_1), (H_2) and (H_3) are satisfied, then there exists $s_1 > s_0$ such that, for each $s^* > s_1$, we can find an open bounded set $\Delta(G(s^*))$ in $L^2(Q)$ on which*

$$|D_L(L + N_s^1, \Delta(G(s^*)))| = 1$$

for all $s_1 < s \le s^*$.

PROOF. Let

$$g(\alpha_0\phi(x_0) + \tilde{u}_0) = \min_{\substack{x \in \bar{\Omega} \\ |\alpha| \le \tilde{\gamma}(s^*) \\ |\tilde{u}| \le \tilde{M}}} g(\alpha\phi(x) + \tilde{u})$$

and $s_1 = |d \max_{\substack{x \in \bar{\Omega} \\ u \in [\alpha_0\phi(x) - \tilde{M}, \alpha_0\phi(x) + \tilde{M}]}} g(u)|$.

Define

$$\Delta(G(s^*)) = \{u \in L^2(Q) | \alpha_0\phi(x) < \alpha\phi(x) < \tilde{\gamma}(s^*)\phi(x) \text{ for } x \in \Omega, \|\tilde{u}\|_{L^2} < \tilde{M}\}$$

where $\tilde{\gamma}(s^*)$ and $\tilde{M}$ are given in Lemma 3.2.

If $s > s_1$, $\mu \in [0,1]$ and (u,μ) is a possible solution of (3.1_s^μ) such that $u \in \partial\Delta(G(s^*))$, then by (B_1), Lemma 3.1 and Remark, we have necessarily $\alpha = \alpha_0$ and

$$\alpha_0\phi(x) - \tilde{M} < \alpha\phi(x) + \tilde{u}(t,x) < \alpha_0\phi(x) + \tilde{M}, \quad (t,x) \in (0,2\pi) \times \bar{\Omega}.$$

By taking the inner product with ϕ on the both sides of (3.1_s^μ), we have

$$\iint_Q g(u(t,x))\phi(x)dtdx = s.$$

But

$$s_1 \geq \iint_Q g(u(t,x))\phi(x)dtdx = s$$

which is impossible, thus for $s \geq s_1$, and $\mu \in [0,1]$, $D_L(L+N_s^\mu, \Delta(G(s^*)))$ is well defined and

$$D_L(L + N_s^\mu, \Delta(G(s^*))) = D_B(JP_2N_s^\mu, \Delta(G(s^*)) \cap kerL, 0)$$

where $P_2N_s^\mu : L^2(\Omega) \to kerL$ is defined by

$$(P_2N_s^\mu u)(t,x) = [\mu \iint_Q g(u(t,x))\phi(x)dtdx - s]\phi(x).$$

Now let $T : kerL \to R$ be defined by

$$T(\alpha\phi(x)) = \alpha.$$

Then, for $\mu = 1$,

$$\begin{aligned} D_L(L + N_s^1, \Delta(G(s^*))) &= D_B(JP_2N_s^1, \Delta(G(s^*)) \cap kerL, 0) \\ &= D_B(T(JP_2N_s^1)T^{-1}, T(\Delta(G(s^*)) \cap kerL), 0). \end{aligned}$$

If we let $J : ImP_2 \to kerL$ be the identity map, then the operator $\Phi = T(JP_2N_s^1)T^{-1}$ will be defined by

$$\Phi(\alpha) = \iint_Q g(\alpha\phi(x))\phi(x)dtdx - s.$$

Thus, for $s_1 < s \le s^*$, we have

$$\Phi(\alpha_0) = \iint_Q g(\alpha_0\phi(x))\phi(x)dtdx - s < s_1 - s < 0$$

and by the choice of $\tilde{\gamma}(s^*)$, we have

$$\begin{aligned}\Phi(\tilde{\gamma}(s^*) &= \iint_Q [g(\tilde{\gamma}(s^*)\phi(x))\phi(x)]dtdx - s \\ &> s^* - s \\ &\ge 0.\end{aligned}$$

Therefore $|D_L(L + N_s^1, \Delta(G(s^*))| = 1$ and the proof is completed. □

PROOF OF THEOREM. Let s_0 and s_1 be constants defined in Lemma 3.2 and Remark. Part (i) has been proved in Lemma 3.3. For part (iii), if $s > s_1$ then we can choose $G \supsetneq \Delta(G(s))$, where G and $\Delta(G(s))$ are defined in Lemma 3.2 and Lemma 3.3, respectively.

By the additivity of degree, we have

$$0 = (D_L(L+N_s^1, G) = D_L(L+N_s^1, \Delta(G(s))) + D_L(L+N_s^1, G - \overline{\Delta(G(s))})$$

and hence, by Lemma 3.3,

$$|D_L(L + N_s^1, G - \overline{\Delta(G(s))})| = 1.$$

Therefore (3.1_s^1) has one solution in $\Delta(G(s))$ and one in $G - \overline{\Delta(G(s))}$. For part (ii), let $\{s_{(n)}\}$ be a sequence in R with $s_{(1)} > s_{(2)} > \cdots > s_1$ such that $s_{(n)} \to s_1$ and let $\{u_n\}$ be the corresponding sequence of solutions of (3.1_s^1). Then $u_n = \alpha_n\phi(x) + \tilde{u}_n$ with $\alpha_n \in R$ and $\tilde{u}_n \in ImL$. By Lemma 3.1, we have a subsequence $\{\alpha_{n_k}\}$ of $\{\alpha_n\}$ which converges to some α in R.

On the other hand, by (H_1), Lemma 3.1 and Remark, we can see that $\{Lu_{n_k}\}$ is a bounded sequence in $ImL \subseteq L^2(Q)$. Since $K : ImL \to ImL$ is a compact operator, and $\tilde{u}_{n_k} = K(Lu_{n_k})$, we have a subsequence, say again, $\{\tilde{u}_{n_k}\}$ converging to $\tilde{u}$ in $DomL \cap ImL$.

Therefore, we have a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ which converges to $u = \alpha\phi + \tilde{u}$ with $\alpha \in R$ and $\tilde{u} \in ImL$. Since L is a closed operator, $u \in DomL$ and u is a solution of (3.1_s^1) for $s = s_1$. This completes our proof. □

REMARK. It is another question whether we can find a constant s_0 such that the problem $(E)(B_1)(B_2)$ has no solution, at least one solution, or at least two solutions according to $s < s_0$, $s = s_0$, or $s > s_0$. The author would like to refer to [5] containing the multiplicity results for doubly-periodic boundary value problem to semilinear dissipative hyperbolic in one dimentional space.

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