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A NON-STANDARD CLASS OF SOBOLEV ORTHOGONAL POLYNOMIALS

S. S. HAN, I. H. JUNG, K. H. KWON, AND J. K. LEE

ABSTRACT. When τ is a quasi-definite moment functional on $\mathcal{P}$, the vector space of all real polynomials, we consider a symmetric bilinear form $\phi(\cdot,\cdot)$ on $\mathcal{P}\times\mathcal{P}$ defined by

$$\phi(p,q)=\lambda p(a)q(a)+\mu p(b)q(b)+\langle\tau,p'q'\rangle,$$

where λ,μ,a, and b are real numbers. We first find a necessary and sufficient condition for $\phi(\cdot,\cdot)$ to be quasi-definite. When τ is a semi-classical moment functional, we discuss algebraic properties of the orthogonal polynomials relative to $\phi(\cdot,\cdot)$ and show that such orthogonal polynomials satisfy a fifth order differential equation with polynomial coefficients.

1. Introduction

Recently, there have been many works([1, 3, 5-10, 12, 13]) on polynomials orthogonal relative to Sobolev pseudo-inner products of the form

$$\int_{-\infty}^{\infty}p(x)q(x)d\mu_0(x)+\lambda\int_{-\infty}^{\infty}p'(x)q'(x)d\mu_1(x), \tag{1.1}$$

where $d\mu_0(x)$ and $d\mu_1(x)$ are positive or signed Borel measures on the real line and λ is a real constant. When $\lambda=0$, we have ordinary orthogonal polynomials, of which the general theory is rather well developed ([2]). When $\lambda\neq 0$, we have the so-called Sobolev orthogonal polynomials or Sobolev-type orthogonal polynomials in case $d\mu_1(x)$ is discrete.

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In most of works, it is assumed that either both $d\mu_0(x)$ and $d\mu_1(x)$ are continuous ([6, 12]) or $d\mu_0(x)$ is continuous and $d\mu_1(x)$ is discrete ([13]). See [1] and references therein for an excellent survey on Sobolev orthogonal polynomials.

On the other hand, Kwon and Littlejohn [9] (see also [3, 5]) found several non-standard Sobolev orthogonal polynomials, which are orthogonal relative to a Sobolev pseudo-inner product (1.1), where $d\mu_0(x)$ is a discrete measure with one or two mass points and $d\mu_1(x)$ is a classical measure. Kwon and Littlejohn obtained such examples in classifying all polynomials orthogonal relative to a symmetric bilinear form on the space of polynomials

$$\phi(p,q) := \langle \sigma, pq \rangle + \langle \tau, p'q' \rangle, \tag{1.2}$$

where σ and τ are moment functionals, which also satisfy a second order differential equation of hypergeometric type

$$\alpha(x)y''(x) + \beta(x)y'(x) = \lambda_n y(x).$$

Generalizing examples found in [9], we now consider discrete Sobolev pseudo- inner products of the form

$$\phi(p,q) := \lambda p(a)q(a) + \mu p(b)q(b) + \langle \tau, p'q' \rangle, \tag{1.3}$$

where $\lambda(\neq 0)$, μ, and $a, b\,(a \neq b)$ are real constants and τ is an arbitrary quasi-definite moment functional. Inner products such as in (1.3) with $\mu = 0$ was first appeared in [3] and studied in general in [7].

We first find necessary and sufficient conditions for $\phi(\cdot,\cdot)$ in (1.3) to be quasi-definite and then express Sobolev orthogonal polynomials $\{R_n(x)\}_{n=0}^{\infty}$ relative to $\phi(\cdot,\cdot)$ in terms of orthogonal polynomials $\{P_n(x)\}_{n=0}^{\infty}$ relative to τ. When τ is semi-classical, we find a differential operator, which is symmetric relative to $\phi(\cdot,\cdot)$ and then investigate various difference-differential relations between $\{P_n(x)\}_{n=0}^{\infty}$ and $\{R_n(x)\}_{n=0}^{\infty}$ by such differential operator and also show that such orthogonal polynomials $\{R_n(x)\}_{n=0}^{\infty}$ satisfy a fifth order differential equation with polynomial coefficients.

2. Quasi-definiteness of $\phi(\cdot,\cdot)$

Let $\mathcal{P}$ be the space of all real polynomials in one variable and use deg(p) to denote the degree of a polynomial $p(x)$ with the convention that deg$(0)= -1$. By a polynomial system(PS), we mean a sequence $\{P_n(x)\}_{n=0}^{\infty}$ of polynomials with deg$(P_n) = n$ for $n \geq 0$.

For any moment functional τ (i.e., a linear functional on $\mathcal{P}$), we call $\{\tau_n := \langle \tau, x^n\rangle\}_{n=0}^{\infty}$ the moments of τ and say that τ is quasi-definite(respectively, positive-definite) (see [2]) if

$$\Delta_n(\tau) := \det[\tau_{i+j}]_{i,j=0}^{n} \neq 0 \text{ (respectively,} \, \Delta_n(\tau) > 0), \; n \geq 0.$$

More generally, for a symmetric bilinear form $\phi(\cdot,\cdot)$ as in (1.2) or (1.3), we call $\{\phi_{mn} := \phi(x^m, x^n)\}_{m,n=0}^{\infty}$ the moments of $\phi(\cdot,\cdot)$ and say that $\phi(\cdot,\cdot)$ is quasi-definite(respectively, positive-definite) if

$$\Delta_n(\phi) := \det[\phi_{ij}]_{i,j=0}^{n} \neq 0 \text{ (respectively,} \, \Delta_n(\phi) > 0), \; n \geq 0.$$

It is then easy to see that $\phi(\cdot,\cdot)$ is quasi-definite or positive-definite if and only if there is a PS $\{R_n(x)\}_{n=0}^{\infty}$ such that

$$\phi(R_m, R_n) = K_n \delta_{mn}, \quad m \text{ and } n \geq 0,$$

where $K_n, n \geq 0$, is a nonzero or positive constant, respectively. In this case, we call $\{R_n(x)\}_{n=0}^{\infty}$ a Sobolev orthogonal polynomial system (SOPS) relative to $\phi(\cdot,\cdot)$ (or simply, an orthogonal polynomial system (OPS) relative to σ, when $\tau = 0$ in (1.2)). We note that when $\phi(\cdot,\cdot)$ is quasi-definite, each $R_n(x)$, $n \geq 0$, is uniquely determined up to a nonzero constant multiple.

In this work, we consider only $\phi(\cdot,\cdot)$ as in (1.3), where $\lambda \neq 0$, $a \neq b$, and τ is always assumed to be quasi-definite. We let $\{P_n(x)\}_{n=0}^{\infty}$ be the OPS relative to τ such that the leading coefficient of $P_n(x)$, $n \geq 0$, is $n+1$.

We now set

$$Q_0(x) = 1 \text{ and } Q_n(x) = \int_a^x P_{n-1}(t)dt, \; n \geq 1.$$

Then, $\{Q_n(x)\}_{n=0}^{\infty}$ is a monic PS such that

$$Q_n(a) = 0 \text{ and } Q_n'(x) = P_{n-1}(x), \; n \geq 1. \tag{2.1}$$

When is $\phi(\cdot,\cdot)$ quasi-definite ?

THEOREM 2.1. *Let $\phi(\cdot,\cdot)$ be a symmetric bilinear form on $\mathcal{P}\times\mathcal{P}$ as in (1.3). Then $\phi(\cdot,\cdot)$ is quasi-definite if and only if*

$$\lambda+\mu+\lambda\mu G_n(b,b)\neq 0,\ \ n\geq 0, \tag{2.2}$$

where

$$G_0(x,b)=0 \text{ and } G_n(x,b):=\sum_{j=1}^{n}\frac{Q_j(b)Q_j(x)}{\langle\tau,P_{j-1}^2\rangle},\ \ n\geq 1.$$

Furthermore if $\phi(\cdot,\cdot)$ is quasi-definite, then the monic SOPS $\{R_n(x)\}_{n=0}^{\infty}$ relative to $\phi(\cdot,\cdot)$ is given by

(2.3)

$$R_n(x)=\begin{cases}1, & n=0\\ Q_n(x)-\dfrac{\lambda\mu Q_n(b)}{\lambda+\mu+\lambda\mu G_{n-1}(b,b)}(G_{n-1}(x,b)+\dfrac{1}{\lambda}), & n\geq 1,\end{cases}$$

$$\phi(R_n,R_n)=\begin{cases}\lambda+\mu, & n=0\\ \dfrac{\lambda+\mu+\lambda\mu G_n(b,b)}{\lambda+\mu+\lambda\mu G_{n-1}(b,b)}\langle\tau,P_{n-1}^2\rangle, & n\geq 1,\end{cases} \tag{2.4}$$

and

$$R_n(a)=\frac{-\mu Q_n(b)}{\lambda+\mu+\lambda\mu G_{n-1}(b,b)},\ \ n\geq 1. \tag{2.5}$$

PROOF. Assume that $\phi(\cdot,\cdot)$ is quasi-definite and let $\{R_n(x)\}_{n=0}^{\infty}$ be the corresponding monic SOPS. Then

$$\begin{aligned}\phi(R_m,R_n)&=\lambda R_m(a)R_n(a)+\mu R_m(b)R_n(b)+\langle\tau,R_m'R_n'\rangle\\ &=K_n\delta_{mn}\ \ (m \text{ and } n\geq 0),\end{aligned}$$

where K_n, $n\geq 0$, is a nonzero constant. In particular, we have

$$\lambda+\mu\neq 0 \tag{2.6}$$

$$\lambda R_n(a)+\mu R_n(b)=0,\ n\geq 1. \tag{2.7}$$

For $n \geq 1$, we can write $R'_n(x)$ as

$$R'_n(x) = \sum_{j=0}^{n-1} C_{nj} P_j(x), \ \ n \geq 1.$$

From the orthogonality of $\{P_n(x)\}_{n=0}^{\infty}$ relative to τ and (2.1), we obtain:

$$(2.8) \qquad C_{nj} = \begin{cases} \dfrac{-\mu R_n(b) Q_{j+1}(b)}{\langle \tau, P_j^2 \rangle}, & 1 \leq j \leq n-2 \\ \dfrac{\phi(R_n, R_n) - \mu R_n(b) Q_n(b)}{\langle \tau, P_{n-1}^2 \rangle}, & j = n-1 \end{cases}$$

and so

$$(2.9) \quad R'_n(x) = \frac{\phi(R_n, R_n)}{\langle \tau, P_{n-1}^2 \rangle} P_{n-1}(x) - \mu R_n(b) \sum_{j=0}^{n-1} \frac{P_j(x) Q_{j+1}(b)}{\langle \tau, P_j^2 \rangle}, \ \ n \geq 1.$$

Integrating (2.9) from a to x, we obtain by (2.7)

$$(2.10) \qquad R_n(x) = \frac{\phi(R_n, R_n)}{\langle \tau, P_{n-1}^2 \rangle} Q_n(x) - \mu R_n(b)(G_n(x,b) + \frac{1}{\lambda}).$$

Evaluating (2.10) at $x = b$, we have

$$(2.11) \qquad [\lambda + \mu + \lambda\mu G_n(b,b)] R_n(b) = \frac{\lambda \phi(R_n, R_n)}{\langle \tau, P_{n-1}^2 \rangle} Q_n(b), \ \ n \geq 1.$$

Now we shall show that $\lambda + \mu + \lambda\mu G_n(b,b) \neq 0$ for $n \geq 0$. For $n = 0$, $\lambda + \mu \neq 0$ by (2.6). Assume $\lambda + \mu + \lambda\mu G_n(b,b) = 0$ for some $n \geq 1$. Then by (2.11), we have $Q_n(b) = 0$ and so $\lambda + \mu + \lambda\mu G_{n-1}(b,b) = 0$ and $Q_{n-1}(b) = 0$. Continuing this process, we obtain $Q_1(b) = b - a = 0$, which contradicts the assumption $a \neq b$. Thus, by (2.11), we obtain

$$(2.12) \qquad R_n(b) = \frac{\lambda \phi(R_n, R_n) Q_n(b)}{(\lambda + \mu + \lambda\mu G_n(b,b)) \langle \tau, P_{n-1}^2 \rangle}, \ \ n \geq 1.$$

On the other hand, we have from (2.8)

$$\phi(R_n, R_n) = \mu R_n(b)Q_n(b) + \langle \tau, P_{n-1}^2\rangle, \quad n \geq 1 \tag{2.13}$$

since $C_{n\,n-1} = 1$. Hence, by (2.12) and (2.13), we obtain

$$R_n(b) = \frac{\lambda Q_n(b)}{\lambda + \mu + \lambda\mu G_{n-1}(b,b)}, \quad n \geq 1. \tag{2.14}$$

From (2.10), (2.13), and (2.14), we obtain (2.3) and (2.4). Finally, (2.5) follows immediately from (2.1) and (2.3) since $G_n(a,b) = 0$, $n \geq 0$.

Conversely, assume that (2.2) holds. Define $R_n(x)$ by (2.3). Then $\{R_n(x)\}_{n=0}^{\infty}$ is a monic PS. By using $Q_n(a) = 0, n \geq 1$ and $G_n(a,b) = 0, n \geq 0$, we can easily show that (2.7) and (2.14) hold. For $0 \leq k \leq n$, we have

$$\phi(R_n, Q_k) = \lambda R_n(a)Q_k(a) + \mu R_n(b)Q_k(b) + \langle \tau, R_n' P_{k-1}\rangle,$$

where

$$R_n'(x) = \begin{cases} P_0(x) = 1, & n = 1 \\ P_{n-1}(x) - \dfrac{\lambda\mu Q_n(b)}{\lambda + \mu + \lambda\mu G_{n-1}(b,b)} \displaystyle\sum_{j=1}^{n-1} \frac{Q_j(b)P_{j-1}(x)}{\langle \tau, P_{j-1}^2\rangle}, & n \geq 2. \end{cases}$$

Hence, we have by (2.7) and (2.14)

$$\phi(R_n, Q_0) = \lambda R_n(a) + \mu R_n(b) = 0, \quad n \geq 1;$$

$$\phi(R_n, Q_k) = \mu R_n(b)Q_k(b) - \frac{\lambda\mu Q_n(b)Q_k(b)}{\lambda + \mu + \lambda\mu G_{n-1}(b,b)} = 0, \quad 1 \leq k < n.$$

Also for $k = n$, we have

$$\begin{aligned} \phi(R_n, R_n) &= \phi(R_n, Q_n) = \lambda R_n(a)Q_n(a) + \mu R_n(b)Q_n(b) + \langle \tau, P_{n-1}^2\rangle, \\ &= \begin{cases} \lambda + \mu, & n = 0 \\ \dfrac{\lambda + \mu + \lambda\mu G_n(b,b)}{\lambda + \mu + \lambda\mu G_{n-1}(b,b)} \langle \tau, P_{n-1}^2\rangle, & n \geq 1, \end{cases} \end{aligned}$$

which are nonzero by (2.2). Hence, $\{R_n(x)\}_{n=0}^{\infty}$ is a monic SOPS relative to $\phi(\cdot,\cdot)$ and so $\phi(\cdot,\cdot)$ is quasi-definite. □

COROLLARY 2.2. (i) (cf. Theorem 2.1 in [7]) *If $\mu = 0$, then $\phi(\cdot,\cdot)$ is quasi-definite (respectively, positive-definite) if and only if $\lambda \neq 0$ (respectively, $\lambda > 0$ and τ is positive-definite).*
(ii) *If $\lambda + \mu + \lambda\mu G_n(b,b) > 0$ for all $n \geq 0$ and τ is positive-definite, then $\phi(\cdot,\cdot)$ is positive-definite.*

3. Difference-differential relations and differential equations

For a moment functional σ and a polynomial $f(x)$, we let σ' and $f\sigma$ be the moment functionals defined by

$$\langle \sigma', g\rangle = -\langle \sigma, g\rangle \text{ and } \langle f\sigma, g\rangle = \langle \sigma, fg\rangle, \quad g \in \mathcal{P}.$$

Then, we have the Leibniz rule :

$$(f\sigma)' = f'\sigma + f\sigma'.$$

We now assume that τ is a semi-classical moment functional (cf.[14]) satisfying

$$\alpha\tau' = \beta\tau, \tag{3.1}$$

where $\alpha(x)$ and $\beta(x)$ are polynomials with $\deg(\alpha) \geq 0$ and $\deg(\alpha' + \beta) \geq 1$.

LEMMA 3.1. (cf.Theorem 3.8 in [11]) *For any polynomial $\gamma(x)$, we have*

$$\langle \tau, (\gamma\,[\alpha D^2 + \beta D][p])'q'\rangle = \langle \tau, p'(\gamma\,[\alpha D^2 + \beta D][q])'\rangle,\ p, q \in \mathcal{P}, \tag{3.2}$$

where $D = d/dx$ and $D^2 = \frac{d^2}{dx^2}$.

PROOF. We have, by (3.1) and the Leibniz rule for τ

$$\begin{aligned}
&\langle \tau, (\gamma\,[\alpha D^2 + \beta D][p])'q'\rangle \\
&\qquad = \langle \tau, (\gamma\,[\alpha D^2 + \beta D][p]q')' - \gamma\,[\alpha D^2 + \beta D][p]q''\rangle \\
&\qquad = -\langle \gamma\alpha\tau', p''q'\rangle - \langle \tau', \gamma\beta p'q'\rangle - \langle \tau, \gamma\,[\alpha D^2 + \beta D][p]q''\rangle \\
&\qquad = \langle \tau, (\gamma\beta)'p'q'\rangle + \langle (\gamma\alpha\tau)', p'q''\rangle + \langle \tau, \gamma\alpha p'q'''\rangle \\
&\qquad = \langle \tau, p'(\gamma\,[\alpha D^2 + \beta D][q])'\rangle
\end{aligned}$$

since $\gamma\alpha\tau' = \gamma\beta\tau$ for any polynomial $\gamma(x)$. □

We now define a linear operator $\mathcal{F}$ on $\mathcal{P}$ by

$$\mathcal{F} := f(x)[\alpha(x)D^2 + \beta(x)D], \tag{3.3}$$

where $f(x) = (x-a)(x-b)$.

Then we obtain :

PROPOSITION 3.2. (cf. Theorem 3.2 in [12]) *The linear operator $\mathcal{F}$ is symmetric relative to $\phi(\cdot,\cdot)$, that is,*

$$\phi(\mathcal{F}[p], q) = \phi(p, \mathcal{F}[q]) \ (p, q \in \mathcal{P}). \tag{3.4}$$

PROOF. Since $f(a) = f(b) = 0$, $\phi(\mathcal{F}[p], q) = \langle \tau, \mathcal{F}[p]'q' \rangle$. Hence, by (3.2) we obtain (3.4). □

From now on we always assume that τ is a semi-classical moment functional satisfying (3.1) and $\phi(\cdot,\cdot)$ is quasi-definite. The PS's $\{P_n(x)\}_{n=0}^{\infty}$, $\{Q_n(x)\}_{n=0}^{\infty}$, and $\{R_n(x)\}_{n=0}^{\infty}$ are the same as in Section 2.

Note that the linear operator $\mathcal{F}$ maps a polynomial of degree n into a polynomial of degree at most $n+t$, where

$$t := \max\{\deg(\alpha), \deg(\beta)+1\}. \tag{3.5}$$

From the symmetrical character of the linear differential operator $\mathcal{F}$ relative to $\phi(\cdot,\cdot)$, we can obtain various difference-differential relations among $\{R_n(x)\}_{n=0}^{\infty}$, $\{Q_n(x)\}_{n=0}^{\infty}$, and $\{P_n(x)\}_{n=0}^{\infty}$.

THEOREM 3.3. *We have the following difference-differential relation:*

$$\mathcal{F}[R_n]'(x) = \sum_{i=n-t-1}^{n+t-1} \alpha_{ni} P_i(x), \ n \geq t+1, \tag{3.6}$$

where $\alpha_{ni} = \frac{\phi(R_n, \mathcal{F}[Q_{i+1}])}{\langle \tau, P_i^2 \rangle}$, $n-t-1 \leq i \leq n+t-1$,

$$\mathcal{F}[Q_n](x) = \sum_{i=n-t}^{n+t} \beta_{ni} R_i(x), \ n \geq t, \tag{3.7}$$

where $\beta_{ni} = \frac{\langle\tau, P_{n-1}\mathcal{F}[R_i]'\rangle}{\phi(R_i,R_i)}$, $n-t \le i \le n+t$, and

$$\mathcal{F}[R_n](x) = \sum_{i=n-t}^{n+t} \gamma_{ni} R_i(x),\ n \ge t, \tag{3.8}$$

where $\gamma_{ni} = \frac{\phi(R_n,\mathcal{F}[R_i])}{\phi(R_i,R_i)}$, $n-t \le i \le n+t$.

PROOF. Since degree of the polynomial $\mathcal{F}[R_n](x)$ is at most $n+t$, we can express $\mathcal{F}[R_n]'(x)$ as

$$\mathcal{F}[R_n]'(x) = \sum_{i=0}^{n+t-1} \alpha_{ni} P_i(x).$$

For $0 \le k \le n+t-1$, we have by (3.4)

$$\begin{aligned}\alpha_{nk}\langle\tau, P_k^2\rangle &= \langle\tau, \mathcal{F}[R_n]'P_k\rangle = \langle\tau, \mathcal{F}[R_n]'Q'_{k+1}\rangle \\ &= \phi(\mathcal{F}[R_n], Q_{k+1}) = \phi(R_n, \mathcal{F}[Q_{k+1}]).\end{aligned}$$

Hence, $\alpha_{nk} = 0$ for $k < n-t-1$ and

$$\alpha_{nk} = \frac{\phi(R_n, \mathcal{F}[Q_{k+1}])}{\langle\tau, P_k^2\rangle},\ n-t-1 \le k \le n+t-1.$$

Similarly, we can write $\mathcal{F}[Q_n](x)$ as

$$\mathcal{F}[Q_n](x) = \sum_{i=0}^{n+t} \beta_{ni} R_i(x).$$

For $0 \le k \le n+t$, we have by (3.4)

$$\begin{aligned}\beta_{nk}\phi(R_k, R_k) &= \phi(\mathcal{F}[Q_n], P_k) = \phi(Q_n, \mathcal{F}[R_k]) \\ &= \langle\tau, P_{n-1}\mathcal{F}[R_k]'\rangle.\end{aligned}$$

Hence we have $\beta_{nk} = 0$ for $k < n-t$ and

$$\beta_{nk} = \frac{\langle\tau, P_{n-1}\mathcal{F}[R_k]'\rangle}{\phi(R_k, R_k)},\ n-t \le k \le n+t.$$

We also have

$$\mathcal{F}[R_n](x) = \sum_{i=0}^{n+t} \gamma_{ni} R_i(x).$$

By using (3.4) and the orthogonality of $\{R_n(x)\}_{n=0}^{\infty}$, we have $\gamma_{nk} = 0$ for $k < n - t$ and

$$\gamma_{nk} = \frac{\phi(R_n, \mathcal{F}[R_k])}{\phi(R_k, R_k)}, \ n - t \le k \le n + t. \qquad \square$$

As a semi-classical OPS, $\{P_n(x)\}_{n=0}^{\infty}$ satisfies the three term recurrence relation (see [2]) ;

$$\text{(3.9)} \qquad \begin{aligned} P_{n+1}(x) = (\alpha_n x - \beta_n) P_n(x) - \gamma_n P_{n-1}(x), \ n \ge 0 \\ (P_{-1}(x) = 0 \text{ and } \alpha_n \gamma_{n+1} \ne 0, n \ge 0) \end{aligned}$$

and the structure relation (see [14]) ;

$$\text{(3.10)} \qquad \alpha(x) P_n'(x) = \sum_{i=n-s-1}^{n+\tilde{s}-1} \alpha_{ni} P_i(x), \ n \ge s + 1,$$

where $s = \max\{\deg(\alpha) - 2, \deg(\beta) - 1\}$ and $\tilde{s} = \deg(\alpha)$.

In the following, we will denote by $\pi_k(x, n)$ a polynomial of degree at most k (k is independent of n) such that its coefficients may depend on n. Also the polynomial $\pi_k(x, n)$ may not be the same in different formulas even though we use the same notation $\pi_k(x, n)$.

By (3.9), (3.6) and (3.10) can be written as

$$\text{(3.11)} \qquad \mathcal{F}[R_n]'(x) = \pi_{t-1}(x, n) P_n(x) + \pi_t(x, n) P_{n-1}(x)$$

and

$$\text{(3.12)} \qquad \alpha(x) P_n'(x) = \pi_{\tilde{s}-1}(x, n) P_n(x) + \pi_s(x, n) P_{n-1}(x).$$

Differentiating (3.11) and then multiplying by $\alpha(x)$, and using (3.12), we obtain

$$\text{(3.13)} \qquad \alpha(x) \mathcal{F}[R_n]''(x) = \pi_{t+s}(x, n) P_n(x) + \pi_{t+s+1}(x, n) P_{n-1}(x).$$

Again differentiating (3.13) and multiplying by $\alpha(x)$, and then using (3.12), we have

(3.14)
$$\alpha(x)(\alpha(x)\mathcal{F}[R_n]'')'(x) = \pi_{t+2s+1}(x,n)P_n(x) + \pi_{t+2s+2}(x,n)P_{n-1}(x).$$

By (3.11) and (3.13), we also have

(3.15)
$$\pi_{2t+s}(x,n)P_n(x) = \pi_t(x,n)\alpha(x)\mathcal{F}[R_n]''(x) + \pi_{t+s+1}(x,n)\mathcal{F}[R_n]'(x)$$

and

(3.16)
$$\tilde{\pi}_{2t+s}(x,n)P_{n-1}(x) = \pi_{t-1}(x,n)\alpha(x)\mathcal{F}[R_n]''(x) + \pi_{t+s}(x,n)\mathcal{F}[R_n]'(x).$$

Now from (3.14), (3.15) and (3.16), we obtain the following :

THEOREM 3.4. *When τ is a semi-classical moment functional satisfying (3.1), the monic SOPS $\{R_n(x)\}_{n=0}^{\infty}$ relative to $\phi(\cdot,\cdot)$ satisfies a fifth order differential equation with polynomial coefficients :*

$$A(x,n)\mathcal{F}[R_n]'''(x) + B(x,n)\mathcal{F}[R_n]''(x) + C(x,n)\mathcal{F}[R_n]'(x) = 0,$$

where $\deg(A) \leq 4t + 2s + 2\tilde{s}$, $\deg(B) \leq 4t + 3s + \tilde{s} + 1$, *and* $\deg(C) \leq 4t + 4s + 2$.

4. Examples

In this section, we shall consider the case when τ is a classical moment functional. It is well known that if an OPS $\{P_n(x)\}_{n=0}^{\infty}$ relative to τ is classical, then, by Sonine-Hahn characterization (see [4, 15]), $\{P_n'(x)\}_{n=1}^{\infty}$ is also classical.

EXAMPLE 4.1. **The Laguerre case**

Let τ be the moment functional defined by the weight function(or distribution) $w(x) = x_+^{\alpha+1}e^{-x}$ on $[0,\infty)$, $\alpha \neq -2, -3, \cdots$. In this case, the corresponding orthogonal polynomials are the Laguerre polynomials given by

$$L_{n-1}^{(\alpha+1)}(x) = (-1)^{n-1}n!\sum_{k=0}^{n-1}\binom{n+\alpha}{n-k-1}\frac{(-x)^k}{k!},\ n \geq 1$$

and $\langle \tau, (L_{n-1}^{(\alpha+1)})^2 \rangle = n(n)!(n+\alpha)!$, $n \geq 1$ (see [2]).

Hence,

$$\phi_1(p,q) = \lambda p(a)q(a) + \mu p(b)q(b) + \langle x_+^{\alpha+1} e^{-x}, p'q' \rangle$$

is quasi-definite if and only if $\lambda + \mu + \lambda\mu G_n(b,b) \neq 0$, $n \geq 0$, where

$$G_n(x,b) = \sum_{j=1}^{n} \frac{(L_j^{(\alpha)}(x) - L_j^{(\alpha)}(a))(L_j^{(\alpha)}(b) - L_j^{(\alpha)}(a))}{j(j)!(j+\alpha)!}, \quad n \geq 1.$$

When $\phi_1(\cdot,\cdot)$ is quasi-definite, by Theorem 2.1, the monic SOPS $\{R_n(x)\}_{n=0}^{\infty}$ relative to $\phi_1(\cdot,\cdot)$ is given by

$$R_n(x) = \begin{cases} 1, & n = 0 \\ (L_n^{(\alpha)}(x) - L_n^{(\alpha)}(a)) - \dfrac{\lambda\mu(L_n^{(\alpha)}(b) - L_n^{(\alpha)}(a))}{\lambda + \mu + \lambda\mu G_{n-1}(b,b)} \\ \qquad \times (G_{n-1}(x,b) + \dfrac{1}{\lambda}), & n \geq 1 \end{cases}$$

and

$$\phi_1(R_n, R_n) = \begin{cases} \lambda + \mu, & n = 0 \\ \dfrac{\lambda + \mu + \lambda\mu G_n(b,b)}{\lambda + \mu + \lambda\mu G_{n-1}(b,b)} n(n)!(n+\alpha)!, & n \geq 1. \end{cases}$$

It is well known that the moment functional τ satisfies the following functional equation

$$x\tau' = (1 + \alpha - x)\tau.$$

Now let $\mathcal{F}$ be the linear operator on $\mathcal{P}$ defined by

$$\mathcal{F} := (x-a)(x-b)[xD^2 + (1+\alpha-x)D].$$

Then, by Theorem 3.3, we obtain

$$\mathcal{F}[R_n]'(x) = \sum_{i=n-3}^{n+1} \alpha_{ni} L_i^{(\alpha+1)}(x), \ n \geq 3,$$

where

$$\alpha_{n\,n-3} = -\frac{\lambda+\mu+\lambda\mu G_n(b,b)}{\lambda+\mu+\lambda\mu G_{n-1}(b,b)} n^2(n-1)(n+\alpha)(n+\alpha-1)$$

$$\alpha_{ni} = \frac{\phi_1(R_n, \mathcal{F}[L_{i+1}^{(\alpha)}])}{(i+1)(i+1)!(i+\alpha+1)!},\ n-2 \le i \le n+1.$$

Moreover, $\{R_n(x)\}_{n=0}^{\infty}$ satisfies a fifth order differential equation with polynomial coefficients :

$$A(x,n)\mathcal{F}[R_n]'''(x) + B(x,n)\mathcal{F}[R_n]''(x) + C(x,n)\mathcal{F}[R_n]'(x) = 0,$$

where $\deg(A) \le 10, \deg(B) \le 10$, and $\deg(C) \le 10$. When $\alpha = -1, \mu = 0$, and $a = 0$, $R_n(x) = \frac{1}{n+1}L_n^{(-1)}(x)$, where $R_n(0) = 0$, $n \ge 1$(see [5, 9]).

EXAMPLE 4.2. **The Jacobi case**

Let τ be the moment functional defined by the weight function (or distribution) $w(x) = (1-x)_+^{\alpha+1}(1+x)_+^{\beta+1}$ on $[-1,1]$, where $\alpha \neq -2, -3, \cdots$, $\beta \neq -2, -3, \cdots$, and $\alpha+\beta \neq -4, -5, \cdots$. In this case, the orthogonal polynomials are the Jacobi polynomials given by

$$J_{n-1}^{(\alpha+1,\beta+1)}(x) = n\sum_{k=0}^{n-1} \frac{\binom{n+\alpha}{n-k-1}\binom{n+\beta}{k}}{\binom{2n+\alpha+\beta}{n-1}}(x-1)^k(x+1)^{n-k-1}, n \ge 1$$

and $\langle \tau, (J_{n-1}^{(\alpha+1,\beta+1)})^2\rangle = 2^{2n+\alpha+\beta+1}n^2(n+\alpha+\beta+1)\cdot B(n+\alpha+\beta+1, n)\cdot B(n+\alpha+1, n+\beta+1)$, $n \ge 1$ (see [2]), where $B(\cdot,\cdot)$ denotes the beta function defined by

$$B(x,y) := \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Then, by Theorem 2.1,

$$\phi_2(p,q) = \lambda p(a)q(a) + \mu p(b)q(b) + \langle (1-x)_+^{\alpha+1}(1+x)_+^{\beta+1}, p'q'\rangle$$

is quasi-definite if and only if $\lambda+\mu+\lambda\mu G_n(b,b) \neq 0$, $n \ge 0$, where

$$G_n(x,b) = \sum_{j=1}^{n} \frac{(J_j^{(\alpha,\beta)}(x) - J_j^{(\alpha,\beta)}(a))(J_j^{(\alpha,\beta)}(b) - J_j^{(\alpha,\beta)}(a))}{2^{2j+\alpha+\beta+1}(j+\alpha+\beta+1)B(j+\alpha+\beta+1, j)} \times \frac{1}{B(j+\alpha+1, j+\beta+1)j^2},\ n \ge 1.$$

When $\phi_2(\cdot,\cdot)$ is quasi-definite, by Theorem 2.1, the monic SOPS $\{R_n(x)\}_{n=0}^{\infty}$ relative to $\phi_2(\cdot,\cdot)$ is given by

$$R_n(x) = \begin{cases} 1, & n=0 \\ (J_n^{(\alpha,\beta)}(x) - J_n^{(\alpha,\beta)}(a)) - \dfrac{\lambda\mu(J_n^{(\alpha,\beta)}(b) - J_n^{(\alpha,\beta)}(a))}{\lambda+\mu+\lambda\mu G_{n-1}(b,b)} \\ \qquad \times (G_{n-1}(x,b) + \dfrac{1}{\lambda}), \ n \geq 1 \end{cases}$$

and

$$\phi_2(R_n,R_n) = \begin{cases} \lambda+\mu, & n=0 \\ \dfrac{\lambda+\mu+\lambda\mu G_n(b,b)}{\lambda+\mu+\lambda\mu G_{n-1}(b,b)} n^2 2^{2n+\alpha+\beta+1}(n+\alpha+\beta+1) \\ \times B(n+\alpha+\beta+1,n)\cdot B(n+\alpha+1,n+\beta+1), \ n\geq 1. \end{cases}$$

In particular, when $a=1$, $b=-1$, and $\alpha=\beta=-1$, $\{R_n(x)\}_{n=0}^{\infty}$ is $\{J_n^{(-1,-1)}(x)\}_{n=0}^{\infty}$, where we note that $J_n^{(-1,-1)}(\pm 1)=0$ for all $n\geq 2$ (see [5,9]). In this case, $G_n(-1,-1)=2$, $n\geq 1$ and so $\phi_2(\cdot,\cdot)$ is quasi-definite (respectively, positive-definite) if and only if $\lambda+\mu\neq 0$ and $\lambda+\mu+2\lambda\mu\neq 0$ (respectively, $\lambda+\mu>0$ and $\lambda+\mu+2\lambda\mu>0$), which exactly agree with the result by Kwon and Littlejohn ([9]).

It is well known that the moment functional τ satisfies the following functional equation

$$(1-x^2)\tau' = [\alpha+\beta+2+(\alpha-\beta)x]\tau.$$

If $\mathcal{F}$ is the linear operator on $\mathcal{P}$ defined by

$$\mathcal{F} := (x-a)(x-b)[(1-x^2)D^2 + (\alpha+\beta+2+(\alpha-\beta)x)D],$$

Then, by Theorem 3.3, we have

$$\mathcal{F}[R_n]'(x) = \sum_{i=n-3}^{n+1} \alpha_{ni} J_i^{(\alpha+1,\beta+1)}(x), \ n\geq 3,$$

where

$$\alpha_{n\,n-3} = \frac{(n-2)(3-n+\alpha-\beta)\phi_2(R_n, R_n)}{\langle \tau, (J_{n-3}^{(\alpha+1,\beta+1)})^2 \rangle}$$

$$\alpha_{ni} = \frac{\phi_2(R_n, \mathcal{F}[J_{j+1}^{(\alpha,\beta)}])}{\langle \tau, (J_j^{(\alpha+1,\beta+1)})^2 \rangle}, \; n-2 \le i \le n+1.$$

Moreover, $\{R_n(x)\}_{n=0}^{\infty}$ satisfies a fifth order differential equation with polynomial coefficients :

$$A(x,n)\mathcal{F}[R_n]'''(x) + B(x,n)\mathcal{F}[R_n]''(x) + C(x,n)\mathcal{F}[R_n]'(x) = 0,$$

where $\deg(A) \le 12$, $\deg(B) \le 11$, and $\deg(C) \le 10$. When $\alpha = \beta = -1, \lambda = \mu \neq 0$, $a = 1$, and $b = -1$, $R_n(x) = \frac{1}{n+1} J_n^{(-1,-1)}(x)$.

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S. S. Han
Department of Liberal Arts
Myongji university
Yongin 449-728, Korea
E-mail: hahn@wh.myongji.ac.kr

I. H. Jung and K. H. Kwon
Department of Mathematics
KAIST
Taejon 305-701, Korea
E-mail: khkwon@jacobi.kaist.ac.kr

J. K. Lee
Department of Mathematics
Sunmoon university
Cheonan 336-840, Korea
E-mail: jklee@omega.sunmoon.ac.kr