A NON-STANDARD CLASS OF SOBOLEV ORTHOGONAL POLYNOMIALS

S. S. HAN, I. H. JUNG, K. H. KWON, AND J. K. LEE

ABSTRACT. When $\tau$ is a quasi-definite moment functional on $\mathcal{P}$, the vector space of all real polynomials, we consider a symmetric bilinear form $\phi(\cdot, \cdot)$ on $\mathcal{P} \times \mathcal{P}$ defined by

$$\phi(p, q) = \lambda p(a)q(a) + \mu p(b)q(b) + \langle \tau, p'q' \rangle,$$

where $\lambda, \mu, a,$ and $b$ are real numbers. We first find a necessary and sufficient condition for $\phi(\cdot, \cdot)$ to be quasi-definite. When $\tau$ is a semi-classical moment functional, we discuss algebraic properties of the orthogonal polynomials relative to $\phi(\cdot, \cdot)$ and show that such orthogonal polynomials satisfy a fifth order differential equation with polynomial coefficients.

1. Introduction

Recently, there have been many works([1, 3, 5-10, 12, 13]) on polynomials orthogonal relative to Sobolev pseudo-inner products of the form

$$\int_{-\infty}^{\infty} p(x)q(x)d\mu_0(x) + \lambda \int_{-\infty}^{\infty} p'(x)q'(x)d\mu_1(x), \tag{1.1}$$

where $d\mu_0(x)$ and $d\mu_1(x)$ are positive or signed Borel measures on the real line and $\lambda$ is a real constant. When $\lambda = 0$, we have ordinary orthogonal polynomials, of which the general theory is rather well developed ([2]). When $\lambda \neq 0$, we have the so-called Sobolev orthogonal polynomials or Sobolev-type orthogonal polynomials in case $d\mu_1(x)$ is discrete.

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In most of works, it is assumed that either both $d\mu_0(x)$ and $d\mu_1(x)$ are continuous ([6, 12]) or $d\mu_0(x)$ is continuous and $d\mu_1(x)$ is discrete ([13]). See [1] and references therein for an excellent survey on Sobolev orthogonal polynomials.

On the other hand, Kwon and Littlejohn [9] (see also [3, 5]) found several non-standard Sobolev orthogonal polynomials, which are orthogonal relative to a Sobolev pseudo-inner product (1.1), where $d\mu_0(x)$ is a discrete measure with one or two mass points and $d\mu_1(x)$ is a classical measure. Kwon and Littlejohn obtained such examples in classifying all polynomials orthogonal relative to a symmetric bilinear form on the space of polynomials

\begin{equation}
\phi(p, q) := \langle \sigma, pq \rangle + \langle \tau, p'q' \rangle,
\end{equation}

where $\sigma$ and $\tau$ are moment functionals, which also satisfy a second order differential equation of hypergeometric type

\[ \alpha(x)y''(x) + \beta(x)y'(x) = \lambda_n y(x). \]

Generalizing examples found in [9], we now consider discrete Sobolev pseudo-inner products of the form

\begin{equation}
\phi(p, q) := \lambda p(a)q(a) + \mu p(b)q(b) + \langle \tau, p'q' \rangle,
\end{equation}

where $\lambda(\neq 0)$, $\mu$, and $a, b (a \neq b)$ are real constants and $\tau$ is an arbitrary quasi-definite moment functional. Inner products such as in (1.3) with $\mu = 0$ was first appeared in [3] and studied in general in [7].

We first find necessary and sufficient conditions for $\phi(\cdot, \cdot)$ in (1.3) to be quasi-definite and then express Sobolev orthogonal polynomials \( \{R_n(x)\}_{n=0}^{\infty} \) relative to $\phi(\cdot, \cdot)$ in terms of orthogonal polynomials \( \{P_n(x)\}_{n=0}^{\infty} \) relative to $\tau$. When $\tau$ is semi-classical, we find a differential operator, which is symmetric relative to $\phi(\cdot, \cdot)$ and then investigate various difference-differential relations between \( \{P_n(x)\}_{n=0}^{\infty} \) and \( \{R_n(x)\}_{n=0}^{\infty} \) by such differential operator and also show that such orthogonal polynomials \( \{R_n(x)\}_{n=0}^{\infty} \) satisfy a fifth order differential equation with polynomial coefficients.
2. Quasi-definiteness of $\phi(\cdot, \cdot)$

Let $\mathcal{P}$ be the space of all real polynomials in one variable and use $\text{deg}(p)$ to denote the degree of a polynomial $p(x)$ with the convention that $\text{deg}(0) = -1$. By a polynomial system (PS), we mean a sequence $\{P_n(x)\}_{n=0}^{\infty}$ of polynomials with $\text{deg}(P_n) = n$ for $n \geq 0$.

For any moment functional $\tau$ (i.e., a linear functional on $\mathcal{P}$), we call $\{\tau_n := \langle \tau, x^n \rangle\}_{n=0}^{\infty}$ the moments of $\tau$ and say that $\tau$ is quasi-definite (respectively, positive-definite) (see [2]) if
\[
\Delta_n(\tau) := \det[\tau_{i+j}]_{i,j=0}^{n} \neq 0 \quad \text{(respectively, } \Delta_n(\tau) > 0), \ n \geq 0.
\]

More generally, for a symmetric bilinear form $\phi(\cdot, \cdot)$ as in (1.2) or (1.3), we call $\{\phi_{mn} := \phi(x^m, x^n)\}_{m,n=0}^{\infty}$ the moments of $\phi(\cdot, \cdot)$ and say that $\phi(\cdot, \cdot)$ is quasi-definite (respectively, positive-definite) if
\[
\Delta_n(\phi) := \det[\phi_{ij}]_{i,j=0}^{n} \neq 0 \quad \text{(respectively, } \Delta_n(\phi) > 0), \ n \geq 0.
\]

It is then easy to see that $\phi(\cdot, \cdot)$ is quasi-definite or positive-definite if and only if there is a PS $\{R_n(x)\}_{n=0}^{\infty}$ such that
\[
\phi(R_m, R_n) = K_n \delta_{mn}, \quad m \text{ and } n \geq 0,
\]
where $K_n, n \geq 0$, is a nonzero or positive constant, respectively. In this case, we call $\{R_n(x)\}_{n=0}^{\infty}$ a Sobolev orthogonal polynomial system (SOPS) relative to $\phi(\cdot, \cdot)$ (or simply, an orthogonal polynomial system (OPS) relative to $\sigma$, when $\tau = 0$ in (1.2)). We note that when $\phi(\cdot, \cdot)$ is quasi-definite, each $R_n(x), n \geq 0$, is uniquely determined up to a nonzero constant multiple.

In this work, we consider only $\phi(\cdot, \cdot)$ as in (1.3), where $\lambda \neq 0$, $a \neq b$, and $\tau$ is always assumed to be quasi-definite. We let $\{P_n(x)\}_{n=0}^{\infty}$ be the OPS relative to $\tau$ such that the leading coefficient of $P_n(x), n \geq 0$, is $n + 1$.

We now set
\[
Q_0(x) = 1 \text{ and } Q_n(x) = \int_{a}^{x} P_{n-1}(t)dt, \ n \geq 1.
\]
Then, $\{Q_n(x)\}_{n=0}^{\infty}$ is a monic PS such that
\[
(2.1) \quad Q_n(a) = 0 \text{ and } Q'_n(x) = P_{n-1}(x), \ n \geq 1.
\]
When is $\phi(\cdot, \cdot)$ quasi-definite?
THEOREM 2.1. Let $\phi(\cdot, \cdot)$ be a symmetric bilinear form on $\mathcal{P} \times \mathcal{P}$ as in (1.3). Then $\phi(\cdot, \cdot)$ is quasi-definite if and only if

$$\lambda + \mu + \lambda \mu G_n(b, b) \neq 0, \quad n \geq 0,$$

where

$$G_0(x, b) = 0 \text{ and } G_n(x, b) := \sum_{j=1}^{n} \frac{Q_j(b)Q_j(x)}{\langle \tau, P_{j-1}^2 \rangle}, \quad n \geq 1.$$

Furthermore if $\phi(\cdot, \cdot)$ is quasi-definite, then the monic SOPS $\{R_n(x)\}_{n=0}^{\infty}$ relative to $\phi(\cdot, \cdot)$ is given by

$$R_n(x) = \begin{cases} 1, & n = 0 \\ Q_n(x) - \frac{\lambda \mu Q_n(b)}{\lambda + \mu + \lambda \mu G_{n-1}(b, b)}(G_{n-1}(x, b) + \frac{1}{\lambda}), & n \geq 1, \end{cases}$$

and

$$\phi(R_n, R_n) = \begin{cases} \lambda + \mu, & n = 0 \\ \frac{\lambda + \mu + \lambda \mu G_n(b, b)}{\lambda + \mu + \lambda \mu G_{n-1}(b, b)} \langle \tau, P_{n-1}^2 \rangle, & n \geq 1, \end{cases}$$

and

$$R_n(a) = \frac{-\mu Q_n(b)}{\lambda + \mu + \lambda \mu G_{n-1}(b, b)}, \quad n \geq 1.$$

PROOF. Assume that $\phi(\cdot, \cdot)$ is quasi-definite and let $\{R_n(x)\}_{n=0}^{\infty}$ be the corresponding monic SOPS. Then

$$\phi(R_m, R_n) = \lambda R_m(a)R_n(a) + \mu R_m(b)R_n(b) + \langle \tau, R'_mR'_n \rangle$$

$$= K_n \delta_{mn} \quad (m \text{ and } n \geq 0),$$

where $K_n, \ n \geq 0,$ is a nonzero constant. In particular, we have

$$\lambda + \mu \neq 0$$

$$\lambda R_n(a) + \mu R_n(b) = 0, \quad n \geq 1.$$
For $n \geq 1$, we can write $R'_n(x)$ as

$$R'_n(x) = \sum_{j=0}^{n-1} C_{nj} P_j(x), \ n \geq 1.$$ 

From the orthogonality of \( \{P_n(x)\}_{n=0}^{\infty} \) relative to $\tau$ and (2.1), we obtain:

$$C_{nj} = \begin{cases} 
-\mu R_n(b) Q_{j+1}(b) \\ \frac{\langle \tau, P_j^2 \rangle}{\phi(R_n, R_n) - \mu R_n(b) Q_n(b)} 
\end{cases}, \ 1 \leq j \leq n - 2$$

$$C_{nj} = \frac{\phi(R_n, R_n) - \mu R_n(b) Q_n(b)}{\langle \tau, P_{n-1}^2 \rangle}, \ j = n - 1$$

and so

$$R'_n(x) = \frac{\phi(R_n, R_n)}{\langle \tau, P_{n-1}^2 \rangle} P_{n-1}(x) - \mu R_n(b) \sum_{j=0}^{n-1} \frac{P_j(x) Q_{j+1}(b)}{\langle \tau, P_j^2 \rangle}, \ n \geq 1.$$ 

Integrating (2.9) from $x$ to $a$, we obtain by (2.7)

$$R_n(x) = \frac{\phi(R_n, R_n)}{\langle \tau, P_{n-1}^2 \rangle} Q_n(x) - \mu R_n(b)(C_n(x, b) + \frac{1}{\lambda}).$$

Evaluating (2.10) at $x = b$, we have

$$[\lambda + \mu + \lambda \mu G_n(b, b)] R_n(b) = \frac{\lambda \phi(R_n, R_n)}{\langle \tau, P_{n-1}^2 \rangle} Q_n(b), \ n \geq 1.$$ 

Now we shall show that $\lambda + \mu + \lambda \mu G_n(b, b) \neq 0$ for $n \geq 0$. For $n = 0$, $\lambda + \mu \neq 0$ by (2.6). Assume $\lambda + \mu + \lambda \mu G_n(b, b) = 0$ for some $n \geq 1$. Then by (2.11), we have $Q_n(b) = 0$ and so $\lambda + \mu + \lambda \mu G_{n-1}(b, b) = 0$ and $Q_{n-1}(b) = 0$. Continuing this process, we obtain $Q_1(b) = b - a = 0$, which contradicts the assumption $a \neq b$. Thus, by (2.11), we obtain

$$R_n(b) = \frac{\lambda \phi(R_n, R_n) Q_n(b)}{(\lambda + \mu + \lambda \mu G_n(b, b)) \langle \tau, P_{n-1}^2 \rangle}, \ n \geq 1.$$
On the other hand, we have from (2.8)

\begin{equation}
(2.13) \quad \phi(R_n, R_n) = \mu R_n(b)Q_n(b) + \langle \tau, P^2_{n-1} \rangle, \ n \geq 1
\end{equation}

since \( C_{n,n-1} = 1 \). Hence, by (2.12) and (2.13), we obtain

\begin{equation}
(2.14) \quad R_n(b) = \frac{\lambda Q_n(b)}{\lambda + \mu + \lambda \mu G_{n-1}(b,b)}, \ n \geq 1.
\end{equation}

From (2.10), (2.13), and (2.14), we obtain (2.3) and (2.4). Finally, (2.5) follows immediately from (2.1) and (2.3) since \( G_n(a,b) = 0, \ n \geq 0 \).

Conversely, assume that (2.2) holds. Define \( R_n(x) \) by (2.3). Then \( \{R_n(x)\}_{n=0}^{\infty} \) is a monic PS. By using \( Q_n(a) = 0, n \geq 1 \) and \( G_n(a,b) = 0, n \geq 0 \), we can easily show that (2.7) and (2.14) hold. For \( 0 \leq k \leq n \), we have

\[ \phi(R_n, Q_k) = \lambda R_n(a)Q_k(a) + \mu R_n(b)Q_k(b) + \langle \tau, R'_n P_{k-1} \rangle, \]

where

\[ R'_n(x) = \begin{cases} 
P_0(x) = 1, & n = 1 \\
R_{n-1}(x) - \frac{\lambda \mu Q_n(b)}{\lambda + \mu + \lambda \mu G_{n-1}(b,b)} \sum_{j=1}^{n-1} \frac{Q_j(b)P_{j-1}(x)}{\langle \tau, P_{j-1}^2 \rangle}, & n \geq 2.
\end{cases} \]

Hence, we have by (2.7) and (2.14)

\[ \phi(R_n, Q_0) = \lambda R_n(a) + \mu R_n(b) = 0, \ n \geq 1; \]

\[ \phi(R_n, Q_k) = \mu R_n(b)Q_k(b) - \frac{\lambda \mu Q_n(b)Q_k(b)}{\lambda + \mu + \lambda \mu G_{n-1}(b,b)} = 0, \ 1 \leq k < n. \]

Also for \( k = n \), we have

\[ \phi(R_n, R_n) = \phi(R_n, Q_n) = \lambda R_n(a)Q_n(a) + \mu R_n(b)Q_n(b) + \langle \tau, P^2_{n-1} \rangle, \]

\[ = \begin{cases} 
\lambda + \mu, & n = 0 \\
\lambda + \mu + \lambda \mu G_n(b,b) \langle \tau, P^2_{n-1} \rangle, & n \geq 1,
\end{cases} \]

which are nonzero by (2.2). Hence, \( \{R_n(x)\}_{n=0}^{\infty} \) is a monic SOPS relative to \( \phi(\cdot, \cdot) \) and so \( \phi(\cdot, \cdot) \) is quasi-definite. □
COROLLARY 2.2. (i) (cf. Theorem 2.1 in [7]) If $\mu = 0$, then $\phi(\cdot, \cdot)$ is quasi-definite (respectively, positive-definite) if and only if $\lambda \neq 0$ (respectively, $\lambda > 0$ and $\tau$ is positive-definite).
(ii) If $\lambda + \mu + \lambda \mu G_\infty(b, b) > 0$ for all $n \geq 0$ and $\tau$ is positive-definite, then $\phi(\cdot, \cdot)$ is positive-definite.

3. Difference-differential relations and differential equations

For a moment functional $\sigma$ and a polynomial $f(x)$, we let $\sigma'$ and $f \sigma$ be the moment functionals defined by

$$\langle \sigma', g \rangle = -\langle \sigma, g \rangle \text{ and } \langle f \sigma, g \rangle = \langle \sigma, fg \rangle, \ g \in \mathcal{P}.$$  

Then, we have the Leibniz rule:

$$(f \sigma)' = f' \sigma + f \sigma'.$$

We now assume that $\tau$ is a semi-classical moment functional (cf.[14]) satisfying

$$(3.1) \quad \alpha \tau' = \beta \tau,$$

where $\alpha(x)$ and $\beta(x)$ are polynomials with $\deg(\alpha) \geq 0$ and $\deg(\alpha' + \beta) \geq 1$.

LEMMA 3.1. (cf.Theorem 3.8 in [11]) For any polynomial $\gamma(x)$, we have

$$(3.2) \quad \langle \tau, (\gamma [\alpha D^2 + \beta D][p])' q' \rangle = \langle \tau, p' (\gamma [\alpha D^2 + \beta D][q])' \rangle, \ p, q \in \mathcal{P},$$

where $D = d/dx$ and $D^2 = \frac{d^2}{dx^2}$.

PROOF. We have, by (3.1) and the Leibniz rule for $\tau$

$$\langle \tau, (\gamma [\alpha D^2 + \beta D][p])' q' \rangle$$

$$= \langle \tau, (\gamma [\alpha D^2 + \beta D][p]q')' - \gamma [\alpha D^2 + \beta D][p]q'' \rangle$$

$$= -\langle \gamma \alpha \tau', p'' q' \rangle - \langle \tau', \gamma \beta p' q' \rangle - \langle \tau, \gamma [\alpha D^2 + \beta D][p]q'' \rangle$$

$$= \langle \tau, (\gamma \beta)' p' q' \rangle + \langle (\gamma \alpha \tau)', p'' q'' \rangle + \langle \tau, \gamma \alpha p' q''' \rangle$$

$$= \langle \tau, p' (\gamma [\alpha D^2 + \beta D][q])' \rangle.$$
since $\gamma \alpha' = \gamma \beta$ for any polynomial $\gamma(x)$. \hfill \Box

We now define a linear operator $\mathcal{F}$ on $\mathcal{P}$ by

$$\mathcal{F} := f(x)[\alpha(x)D^2 + \beta(x)D],$$

where $f(x) = (x - a)(x - b)$.

Then we obtain:

**Proposition 3.2.** (cf. Theorem 3.2 in [12]) The linear operator $\mathcal{F}$ is symmetric relative to $\phi(\cdot, \cdot)$, that is,

$$\phi(\mathcal{F}[p], q) = \phi(p, \mathcal{F}[q]) \quad (p, q \in \mathcal{P}).$$

**Proof.** Since $f(a) = f(b) = 0$, $\phi(\mathcal{F}[p], q) = \langle \tau, \mathcal{F}[p]'q' \rangle$. Hence, by (3.2) we obtain (3.4). \hfill \Box

From now on we always assume that $\tau$ is a semi-classical moment functional satisfying (3.1) and $\phi(\cdot, \cdot)$ is quasi-definite. The PS's $\{P_n(x)\}_{n=0}^{\infty}$, $\{Q_n(x)\}_{n=0}^{\infty}$, and $\{R_n(x)\}_{n=0}^{\infty}$ are the same as in Section 2.

Note that the linear operator $\mathcal{F}$ maps a polynomial of degree $n$ into a polynomial of degree at most $n + t$, where

$$t := \max\{\deg(\alpha), \deg(\beta) + 1\}.$$

From the symmetrical character of the linear differential operator $\mathcal{F}$ relative to $\phi(\cdot, \cdot)$, we can obtain various difference-differential relations among $\{R_n(x)\}_{n=0}^{\infty}$, $\{Q_n(x)\}_{n=0}^{\infty}$, and $\{P_n(x)\}_{n=0}^{\infty}$.

**Theorem 3.3.** We have the following difference-differential relation:

$$\mathcal{F}[R_n]'(x) = \sum_{i=n-t}^{n+t-1} \alpha_{ni} P_i(x), \quad n \geq t + 1,$$

where $\alpha_{ni} = \frac{\phi(R_n, \mathcal{F}[Q_{i+1}])}{\langle \tau, P_i^2 \rangle}$, $n - t - 1 \leq i \leq n + t - 1$,

$$\mathcal{F}[Q_n](x) = \sum_{i=n-t}^{n+t} \beta_{ni} R_i(x), \quad n \geq t,$$
where $\beta_{ni} = \frac{\langle \tau, P_{n-1}^t F[R_n] \rangle}{\phi(R_i, R_i)}$, $n - t \leq i \leq n + t$, and

$$\mathcal{F}[R_n](x) = \sum_{i=n-t}^{n+t} \gamma_{ni} R_i(x), \quad n \geq t,$$

where $\gamma_{ni} = \frac{\phi(R_n, F[R_i])}{\phi(R_i, R_i)}$, $n - t \leq i \leq n + t$.

**Proof.** Since degree of the polynomial $\mathcal{F}[R_n](x)$ is at most $n + t$, we can express $\mathcal{F}[R_n]'(x)$ as

$$\mathcal{F}[R_n]'(x) = \sum_{i=0}^{n+t-1} \alpha_{ni} P_i(x).$$

For $0 \leq k \leq n + t - 1$, we have by (3.4)

$$\alpha_{nk} \langle \tau, P_k^2 \rangle = \langle \tau, \mathcal{F}[R_n]' P_k \rangle = \langle \tau, \mathcal{F}[R_n]' Q_{k+1} \rangle = \phi(\mathcal{F}[R_n], Q_{k+1}) = \phi(R_n, \mathcal{F}[Q_{k+1}]).$$

Hence, $\alpha_{nk} = 0$ for $k < n - t - 1$ and

$$\alpha_{nk} = \frac{\phi(R_n, \mathcal{F}[Q_{k+1}])}{\langle \tau, P_k^2 \rangle}, \quad n - t - 1 \leq k \leq n + t - 1.$$

Similarly, we can write $\mathcal{F}[Q_n](x)$ as

$$\mathcal{F}[Q_n](x) = \sum_{i=0}^{n+t} \beta_{ni} R_i(x).$$

For $0 \leq k \leq n + t$, we have by (3.4)

$$\beta_{nk} \phi(R_n, R_n) = \phi(\mathcal{F}[Q_n], P_k) = \phi(Q_n, \mathcal{F}[R_k]) = \langle \tau, P_{n-1} \mathcal{F}[R_k]' \rangle.$$

Hence we have $\beta_{nk} = 0$ for $k < n - t$ and

$$\beta_{nk} = \frac{\langle \tau, P_{n-1} \mathcal{F}[R_k]' \rangle}{\phi(R_k, R_k)}, \quad n - t \leq k \leq n + t.$$
We also have
\[ \mathcal{F}[R_n](x) = \sum_{i=0}^{n+t} \gamma_{ni} R_i(x). \]
By using (3.4) and the orthogonality of \( \{R_n(x)\}_{n=0}^{\infty} \), we have \( \gamma_{nk} = 0 \)
for \( k < n - t \) and
\[ \gamma_{nk} = \frac{\phi(R_n, \mathcal{F}[R_k])}{\phi(R_k, R_k)}, \ n - t \leq k \leq n + t. \]

As a semi-classical OPS, \( \{P_n(x)\}_{n=0}^{\infty} \) satisfies the three term recurrence relation (see [2]);
\[ P_{n+1}(x) = (\alpha_n x - \beta_n) P_n(x) - \gamma_n P_{n-1}(x), \ n \geq 0 \]
(3.9)
\( (P_{-1}(x) = 0 \text{ and } \alpha_n \gamma_{n+1} \neq 0, \ n \geq 0) \)
and the structure relation (see [14]);
\[ \alpha(x) P_n'(x) = \sum_{i=n-s-1}^{n+s-1} \alpha_{ni} P_i(x), \ n \geq s + 1, \]
(3.10)
where \( s = \max\{\deg(\alpha) - 2, \deg(\beta) - 1\} \) and \( \tilde{s} = \deg(\alpha) \).

In the following, we will denote by \( \pi_k(x, n) \) a polynomial of degree
at most \( k \) (\( k \) is independent of \( n \)) such that its coefficients may depend
on \( n \). Also the polynomial \( \pi_k(x, n) \) may not be the same in different
formulas even though we use the same notation \( \pi_k(x, n) \).

By (3.9), (3.6) and (3.10) can be written as
\[ \mathcal{F}[R_n]'(x) = \pi_{t-1}(x, n) P_n(x) + \pi_t(x, n) P_{n-1}(x) \]
(3.11)
and
\[ \alpha(x) P_n'(x) = \pi_{\tilde{s}-1}(x, n) P_n(x) + \pi_s(x, n) P_{n-1}(x). \]
(3.12)
Differentiating (3.11) and then multiplying by \( \alpha(x) \), and using (3.12),
we obtain
\[ \alpha(x) \mathcal{F}[R_n]''(x) = \pi_{t+s}(x, n) P_n(x) + \pi_{t+s+1}(x, n) P_{n-1}(x). \]
(3.13)
Again differentiating (3.13) and multiplying by \( \alpha(x) \), and then using (3.12), we have
\[
\alpha(x)(\alpha(x)\mathcal{F}[R_n])''(x) = \pi_{t+2s+1}(x,n)P_n(x) + \pi_{t+2s+2}(x,n)P_{n-1}(x).
\]
By (3.11) and (3.13), we also have
\[
\pi_{2t+s}(x,n)P_n(x) = \pi_t(x,n)\alpha(x)\mathcal{F}[R_n]'(x) + \pi_{t+s+1}(x,n)\mathcal{F}[R_n]'(x)
\]
and
\[
\tilde{\pi}_{2t+s}(x,n)P_{n-1}(x) = \pi_{t-1}(x,n)\alpha(x)\mathcal{F}[R_n]''(x) + \pi_{t+s}(x,n)\mathcal{F}[R_n]'(x).
\]
Now from (3.14), (3.15) and (3.16), we obtain the following:

**Theorem 3.4.** When \( \tau \) is a semi-classical moment functional satisfying (3.1), the monic SOPS \( \{R_n(x)\}_{n=0}^{\infty} \) relative to \( \phi(\cdot, \cdot) \) satisfies a fifth order differential equation with polynomial coefficients:

\[
A(x,n)\mathcal{F}[R_n]'''(x) + B(x,n)\mathcal{F}[R_n]''(x) + C(x,n)\mathcal{F}[R_n]'(x) = 0,
\]
where \( \deg(A) \leq 4t + 2s + 2\tilde{s}, \deg(B) \leq 4t + 3s + \tilde{s} + 1 \), and \( \deg(C) \leq 4t + 4s + 2 \).

**4. Examples**

In this section, we shall consider the case when \( \tau \) is a classical moment functional. It is well known that if an OPS \( \{P_n(x)\}_{n=0}^{\infty} \) relative to \( \tau \) is classical, then, by Sonine-Hahn characterization (see [4, 15]), \( \{P_n'(x)\}_{n=1}^{\infty} \) is also classical.

**Example 4.1. The Laguerre case**

Let \( \tau \) be the moment functional defined by the weight function (or distribution) \( w(x) = x_+^{\alpha+1}e^{-x} \) on \([0, \infty)\), \( \alpha \neq -2, -3, \ldots \). In this case, the corresponding orthogonal polynomials are the Laguerre polynomials given by

\[
L_n^{(\alpha+1)}(x) = (-1)^{n-1}n! \sum_{k=0}^{n-1} \binom{n + \alpha}{n - k - 1} \frac{(-x)^k}{k!}, \quad n \geq 1
\]
and \( \langle \tau, (L^{(\alpha+1)}_{n-1})^2 \rangle = n(n)! (n+\alpha)!, \ n \geq 1 \) (see [2]).

Hence,

\[
\phi_1(p, q) = \lambda p(a)q(a) + \mu p(b)q(b) + \langle x_+^{\alpha+1}e^{-x}, p'q' \rangle
\]
is quasi-definite if and only if \( \lambda + \mu + \lambda \mu G_n(b, b) \neq 0, \ n \geq 0, \) where

\[
G_n(x, b) = \sum_{j=1}^{n} \frac{(L^{(\alpha)}_j(x) - L^{(\alpha)}_j(a))(L^{(\alpha)}_j(b) - L^{(\alpha)}_j(a))}{j!(j + \alpha)!}, \ n \geq 1.
\]

When \( \phi_1(\cdot, \cdot) \) is quasi-definite, by Theorem 2.1, the monic SOPS \( \{R_n(x)\}_{n=0}^{\infty} \) relative to \( \phi_1(\cdot, \cdot) \) is given by

\[
R_n(x) = \begin{cases} 
1, & n = 0 \\
(L^{(\alpha)}_n(x) - L^{(\alpha)}_n(a)) - \frac{\lambda \mu (L^{(\alpha)}_n(b) - L^{(\alpha)}_n(a))}{\lambda + \mu + \lambda \mu G_{n-1}(b, b)} \\
\times (G_{n-1}(x, b) + \frac{1}{\lambda}), & n \geq 1
\end{cases}
\]

and

\[
\phi_1(R_n, R_n) = \begin{cases} 
\lambda + \mu, & n = 0 \\
\frac{\lambda + \mu + \lambda \mu G_n(b, b)}{\lambda + \mu + \lambda \mu G_{n-1}(b, b)} n(n)! (n+\alpha)!, & n \geq 1.
\end{cases}
\]

It is well known that the moment functional \( \tau \) satisfies the following functional equation

\[
x \tau' = (1 + \alpha - x) \tau.
\]

Now let \( F \) be the linear operator on \( P \) defined by

\[
F := (x - a)(x - b)[xD^2 + (1 + \alpha - x)D].
\]

Then, by Theorem 3.3, we obtain

\[
F[R_n]'(x) = \sum_{i=n-3}^{n+1} \alpha_i L^{(\alpha+1)}_i(x), \ n \geq 3,
\]
where
\[ \alpha_{n,n-3} = \frac{\lambda + \mu + \lambda \mu G_n(b, b)}{\lambda + \mu + \lambda \mu G_{n-1}(b, b)} n^2(n - 1)(n + \alpha)(n + \alpha - 1) \]
and
\[ \alpha_{ni} = \frac{\phi_1(R_n, \mathcal{F}[L_i^{(\alpha)}])}{(i + 1)(i + 1)!(i + \alpha + 1)!}, \quad n - 2 \leq i \leq n + 1. \]

Moreover, \( \{R_n(x)\}_{n=0}^{\infty} \) satisfies a fifth order differential equation with polynomial coefficients:
\[ A(x, n)\mathcal{F}[R_n]'''(x) + B(x, n)\mathcal{F}[R_n]''(x) + C(x, n)\mathcal{F}[R_n]'(x) = 0, \]
where \( \deg(A) \leq 10, \deg(B) \leq 10, \) and \( \deg(C) \leq 10. \) When \( \alpha = -1, \mu = 0, \) and \( a = 0, \) \( R_n(x) = \frac{1}{n+1} L_n^{(-1)}(x), \) where \( R_n(0) = 0, n \geq 1(\text{see [5, 9]}). \)

**Example 4.2. The Jacobi case**

Let \( \tau \) be the moment functional defined by the weight function (or distribution) \( w(x) = (1-x)^{\alpha+1}(1+x)^{\beta+1} \) on \( [-1, 1], \) where \( \alpha \neq -2, -3, \ldots, \beta \neq -2, -3, \ldots, \) and \( \alpha + \beta \neq -4, -5, \ldots. \) In this case, the orthogonal polynomials are the Jacobi polynomials given by
\[ J_{n-1}^{(\alpha+1, \beta+1)}(x) = \frac{n \sum_{k=0}^{n-1} \left( \begin{array}{c} n+\alpha \\ n-k-1 \end{array} \right) \left( \begin{array}{c} n+\beta \\ k \end{array} \right)}{\left( \begin{array}{c} 2n+\alpha+\beta \\ n-1 \end{array} \right)} (x - 1)^k (x + 1)^{n-k-1}, n \geq 1 \]
and \( \langle \tau, (J_{n-1}^{(\alpha+1, \beta+1)})^2 \rangle = 2^{2n+\alpha+\beta+1} n^2 (n + \alpha + \beta + 1) \cdot B(n + \alpha + \beta + 1, n) \cdot B(n + \alpha + 1, n + \beta + 1), n \geq 1 (\text{see [2]}), \) where \( B(\cdot, \cdot) \) denotes the beta function defined by
\[ B(x, y) := \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)}. \]

Then, by Theorem 2.1,
\[ \phi_2(p, q) = \lambda p(a)q(a) + \mu p(b)q(b) + \langle (1-x)^{\alpha+1}(1+x)^{\beta+1}, p'q' \rangle \]
is quasi-definite if and only if \( \lambda + \mu + \lambda \mu G_n(b, b) \neq 0, n \geq 0, \) where
\[ G_n(x, b) = \sum_{j=1}^{n} \frac{(J_j^{(\alpha, \beta)}(x) - J_j^{(\alpha, \beta)}(a))(J_j^{(\alpha, \beta)}(b) - J_j^{(\alpha, \beta)}(a))}{2^{2j+\alpha+\beta+1}(j + \alpha + \beta + 1)B(j + \alpha + \beta + 1, j)} \]
\[ \times \frac{1}{B(j + \alpha + 1, j + \beta + 1)j^2}, n \geq 1. \]
When $\phi_2(\cdot, \cdot)$ is quasi-definite, by Theorem 2.1, the monic SOPS \( \{R_n(x)\}_{n=0}^{\infty} \) relative to $\phi_2(\cdot, \cdot)$ is given by
\[
R_n(x) = \begin{cases} 
1, & n = 0 \\
(J_n^{(\alpha,\beta)}(x) - J_n^{(\alpha,\beta)}(a)) - \frac{\lambda\mu(J_n^{(\alpha,\beta)}(b) - J_n^{(\alpha,\beta)}(a))}{\lambda + \mu + \lambda\mu G_{n-1}(b,b)} \\
\times (G_{n-1}(x,b) + \frac{1}{\lambda}), & n \geq 1 
\end{cases}
\]
and
\[
\phi_2(R_n, R_n) = \begin{cases} 
\lambda + \mu, & n = 0 \\
\frac{\lambda + \mu + \lambda\mu G_n(b,b)}{\lambda + \mu + \lambda\mu G_{n-1}(b,b)} n^{2n+\alpha+\beta+1}(n + \alpha + \beta + 1) \\
\times B(n + \alpha + \beta + 1, n) \cdot B(n + \alpha + 1, n + \beta + 1), & n \geq 1 
\end{cases}
\]

In particular, when $a = 1$, $b = -1$, and $\alpha = \beta = -1$, \( \{R_n(x)\}_{n=0}^{\infty} \) is \( \{J_n^{(-1,-1)}(x)\}_{n=0}^{\infty} \), where we note that $J_n^{(-1,-1)}(\pm 1) = 0$ for all $n \geq 2$ (see [5,9]). In this case, $G_n(-1,-1) = 2$, $n \geq 1$ and so $\phi_2(\cdot, \cdot)$ is quasi-definite (respectively, positive-definite) if and only if $\lambda + \mu \neq 0$ and $\lambda + \mu + 2\lambda\mu \neq 0$ (respectively, $\lambda + \mu > 0$ and $\lambda + \mu + 2\lambda\mu > 0$), which exactly agree with the result by Kwon and Littlejohn ([9]).

It is well known that the moment functional $\tau$ satisfies the following functional equation
\[
(1 - x^2)\tau' = [\alpha + \beta + 2 + (\alpha - \beta)x]\tau.
\]

If $\mathcal{F}$ is the linear operator on $\mathcal{P}$ defined by
\[
\mathcal{F} := (x - a)(x - b)[(1 - x^2)D^2 + (\alpha + \beta + 2 + (\alpha - \beta)x)D],
\]
Then, by Theorem 3.3, we have
\[
\mathcal{F}[R_n]'(x) = \sum_{i=n-3}^{n+1} \alpha_{ni} J_i^{(\alpha+1,\beta+1)}(x), \quad n \geq 3,
\]
where
\[
\alpha_{n,n-3} = \frac{(n-2)(3-n+\alpha-\beta)\phi_2(R_n, R_n)}{\langle \tau, (J_{n-3}^{(\alpha+1,\beta+1)})^2 \rangle},
\]
\[
\alpha_{ni} = \frac{\phi_2(R_n, F[J_{j+1}^{(\alpha,\beta)}])}{\langle \tau, (J_{j}^{(\alpha+1,\beta+1)})^2 \rangle}, \quad n-2 \leq i \leq n+1.
\]
Moreover, \(\{R_n(x)\}_{n=0}^{\infty}\) satisfies a fifth order differential equation with polynomial coefficients:
\[
A(x, n)F[R_n]'''(x) + B(x, n)F[R_n]''(x) + C(x, n)F[R_n]'(x) = 0,
\]
where \(\deg(A) \leq 12, \deg(B) \leq 11, \) and \(\deg(C) \leq 10.\) When \(\alpha = \beta = -1, \lambda = \mu \neq 0, a = 1, \) and \(b = -1, \) \(R_n(x) = \frac{1}{n+1}J_n^{(-1,-1)}(x).\)

References

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S. S. Han  
Department of Liberal Arts  
Myongji university  
Yongin 449-728, Korea  
*E-mail*: hahn@wh.myongji.ac.kr

I. H. Jung and K. H. Kwon  
Department of Mathematics  
KAIST  
Taejon 305-701, Korea  
*E-mail*: khhkwon@jacobi.kaist.ac.kr

J. K. Lee  
Department of Mathematics  
Sunmoon university  
Cheonan 336-840, Korea  
*E-mail*: jklee@omega.sunmoon.ac.kr