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CONFORMAL TRANSFORMATIONS OF DIFFERENCE TENSORS OF FINSLER SPACE WITH AN (α, β)-METRIC

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ABSTRACT. In the Finsler space with an (α, β)-metric, we can consider the difference tensors of the Finsler connection. The properties of the conformal transformation of these difference tensors are investigated in the present paper. Some conformal invariant tensors are formed in the Finsler space with an (α, β)-metric related with the difference tensors.

1. Introduction

Let $F^n = (M^n, L(\alpha, \beta))$ be an n-dimensional Finsler space with an (α, β)-metric $L(\alpha, \beta)$. The fundamental function $L(\alpha, \beta)$ is a positive homogeneous of degree one in α and β, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a differential 1-form in M^n. In F^n, the Riemannian space $R^n = (M^n, \alpha)$ is called an associated Riemannian space with F^n and the Riemannian connection constructed by α is called the associated Riemannian connection with F^n, which is denoted by the Christoffel symbol $\{{}_j{}^i{}_k\}$ of R^n. In F^n, the difference tensors of the Finsler connection are given by the differences of the h-connection coefficients of the Finsler connection and the associated Riemannian connection. The fundamental Finsler connections are the Cartan connection $C\Gamma = (\Gamma^*{}_j{}^i{}_k, G^i{}_j, C_j{}^i{}_k)$ and the Berwald connection $B\Gamma = (G_j{}^i{}_k, G^i{}_j, 0)$. We denote the difference tensors of $C\Gamma$ and $B\Gamma$ by

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$D_j{}^i{}_k, 'D_j{}^i{}_k$, that is, $D_j{}^i{}_k = \Gamma^*{}_j{}^i{}_k - \{_j{}^i{}_k\}$, $'D_j{}^i{}_k = G_j{}^i{}_k - \{_j{}^i{}_k\}$ respectively. It is well known [7] that if the covariant vector b_i is parallel with respect to the Riemannian connection, then $D_j{}^i{}_k = 0$ and the space becomes a Berwald space. In the present paper, we consider the conformal transformation of $D_j{}^i{}_k$, $'D_j{}^i{}_k$ and some conformal invariant tensors in the Finsler space with an (α, β)-metric. Throughout the present paper we shall use the terminologies and notations in Matsumoto's monograph [5].

2. Preliminaries

We shall consider an n-dimensional Finsler space $F^n = (M^n, L(\alpha, \beta))$ with (α, β)-metric $L(\alpha, \beta)$. It is well known [7] that the fundamental tensor $g_{ij}(x,y) = \dot{\partial}_j \dot{\partial}_i L^2(x,y)/2$, the angular metric tensor $h_{ij} = L\dot{\partial}_i \dot{\partial}_j L$ and the Cartan C-tensor $C_{ijk}(x,y) = \dot{\partial}_k g_{ij}(x,y)/2$ are given by

$$(2.1)\qquad \begin{aligned} g_{ij} &= pa_{ij} + p_0 b_i b_j + p_{-1}(b_i Y_j + b_j Y_i) + p_{-2} Y_i Y_j, \\ h_{ij} &= pa_{ij} + q_0 b_i b_j + q_{-1}(b_i Y_j + b_j Y_i) + q_{-2} Y_i Y_j, \\ 2pC_{ijk} &= p_{-1}(h_{ij} m_k + h_{jk} m_i + h_{ki} m_j) + r_{-1} m_i m_j m_k, \end{aligned}$$

respectively, where we put

$$(2.2)\qquad \begin{aligned} &Y_i = a_{ij} y^j, \quad p = LL_\alpha \alpha^{-1}, \quad p_0 = q_0 + L_\beta^2, \\ &p_{-1} = q_{-1} + L^{-1} p L_\beta, \quad p_{-2} = q_{-2} + p^2 L^{-2}, \\ &q_{-1} = LL_{\alpha\beta} \alpha^{-1}, \quad q_{-2} = L\alpha^{-2}(L_{\alpha\alpha} - L_\alpha \alpha^{-1}), \\ &r_{-1} = pp_{0\beta} - 3p_{-1} q_0, \quad m_i = b_i - \alpha^{-2} \beta Y_i, \\ &L_\alpha = \partial_\alpha L(\alpha, \beta), \quad L_\beta = \partial_\beta L(\alpha, \beta), \quad q_0 = LL_{\beta\beta}. \end{aligned}$$

In the following, we shall take the symbols used in [7]:

$$(2.3)\qquad \begin{aligned} b_{jk} &= \partial b_j / \partial x^k - b_r \{_j{}^r{}_k\}, \\ E_{jk} &= (b_{jk} + b_{kj})/2 = b_{(jk)}, \\ F_{jk} &= (b_{jk} - b_{kj})/2 = b_{[jk]}. \end{aligned}$$

A direct calculuation leads us to

$$\begin{aligned}\gamma_i{}^k{}_j = \{{}_i{}^k{}_j\} + N^k E_{ij} + N_i F^k{}_j + N_j F^k{}_i + \{{}_0{}^s{}_j\} C_i{}^k{}_s \\ + \{{}_0{}^s{}_i\} C_j{}^k{}_s - \{{}_0{}^s{}_m\} g^{mk} C_{ijs} + b_{0j} K^k{}_i + b_{0i} K^k{}_j \\ - b_{0m} g^{mk} K_{ij},\end{aligned} \tag{2.4}$$

where we put

$$\begin{aligned}&N_k = p_0 b_k + p_{-1} Y_k, \quad N^i = g^{mi} N_m, \quad F^k{}_i = g^{kr} F_{ri}, \\ &K_{ik} = \{p_{-1}(a_{ik} - \alpha^{-2} Y_i Y_k) + p_{0\beta} m_i m_k\}/2, \quad K^k{}_i = g^{rk} K_{ri}.\end{aligned} \tag{2.5}$$

For the symmetric tensor K_{ik} and covariant vector N_k, we get

$$K_{i0} = 0, \quad N_0 = p_0\beta + p_{-1}\alpha^2 = q, \quad \dot{\partial}_i N_k = 2K_{ik}. \tag{2.6}$$

where the suffix "0" means the contraction by y^i.

Putting $2G^i = \gamma_0{}^i{}_0$, we have from (2.4)

$$G^i = (\{{}_0{}^i{}_0\} + N^i E_{00} + 2qF^i{}_0)/2. \tag{2.7}$$

The non-linear connection $G^i{}_j = \dot{\partial}_j G^i$ is obtained as follows:

$$\begin{aligned}G^i{}_j = \{{}_j{}^i{}_0\} + N^i E_{j0} + N_j F^i{}_0 + (K^i{}_j \\ - N^m C_m{}^i{}_j) E_{00} + q(F^i{}_j - 2F^m{}_0 C_j{}^i{}_m).\end{aligned} \tag{2.8}$$

The Cartan h-connection $\Gamma^*{}_j{}^i{}_k$ of the Finsler space with an (α,β)-metric is well-known [7] as follows:

$$\begin{aligned}\Gamma^*{}_j{}^i{}_k &= \gamma_j{}^i{}_k + g^{im} C_{jkr} G^r{}_m - C_k{}^i{}_r G^r{}_j - C_j{}^i{}_r G^r{}_k \\ &= \{{}_j{}^i{}_k\} + N^i E_{jk} + N_j F^i{}_k + N_k F^i{}_j + b_{0j} K^i{}_k + b_{0k} K^i{}_j \\ &\quad - b_{0m} g^{mi} K_{kj} - (C_j{}^i{}_m A^m{}_k + C_m{}^i{}_k A^m{}_j - C_{mjk} A^m{}_s g^{is}) \\ &\quad + \lambda^s (C_j{}^i{}_m C_s{}^m{}_k + C_k{}^i{}_m C_s{}^m{}_j - C_k{}^m{}_j C_m{}^i{}_s),\end{aligned} \tag{2.9}$$

where we put

$$\begin{aligned}&A^m{}_k = K^m{}_k E_{00} + N^m E_{k0} + N_k F^m{}_0 + qF^m{}_k, \\ &\lambda^s = N^s E_{00} + 2qF^s{}_0.\end{aligned} \tag{2.10}$$

From (2.3) and (2.6), we have $A^m{}_k y^k = \lambda^m$. In the Berwald h-connection $G_j{}^i{}_k = \dot\partial_k G^i{}_j$ of the Finsler space with an (α,β)-metric, we get

$$\begin{aligned} G_j{}^i{}_k = \{{}_j{}^i{}_k\} &+ N^i E_{jk} + N_k F^i{}_j + N_j F^i{}_k + g^{im} K_{mj\,(k)} E_{00} \\ &+ 2(K^i{}_j E_{k0} + K^i{}_k E_{j0} + K_{jk} F^i{}_0) - 2S_{(jk)}\{C_m{}^i{}_j A^m{}_k\} \\ &+ \lambda^s (2C_j{}^i{}_m C_s{}^m{}_k - C_s{}^i{}_{j(k)}), \end{aligned} \tag{2.11}$$

where we put

$$C_s{}^i{}_{j\,(k)} = \dot\partial_k C_s{}^i{}_j, \quad K_{mj\,(k)} = \dot\partial_k K_{mj},$$

$$S_{(jk)}\{C_m{}^i{}_j A^m{}_k\} = C_m{}^i{}_j A^m{}_k + C_m{}^i{}_k A^m{}_j.$$

From(2.9), the difference tensor of the Cartan connection $C\Gamma$ is given [7] as follows:

$$\begin{aligned} D_j{}^i{}_k = N^i E_{jk} &+ N_j F^i{}_k + N_k F^i{}_j + b_{0j} K^i{}_k + b_{0k} K^i{}_j \\ &- b_{0m} g^{im} K_{kj} - (C_m{}^i{}_j A^m{}_k + C_m{}^i{}_k A^m{}_j - C_{mjk} A^m{}_s g^{is}) \\ &+ \lambda^s (C_j{}^i{}_m C_s{}^m{}_k + C_k{}^i{}_m C_j{}^m{}_s - C_k{}^m{}_j C_m{}^i{}_s). \end{aligned} \tag{2.12}$$

Next, from (2.11), the difference tensor of Berwald connection $B\Gamma$ is given by

$$\begin{aligned} {}'D_j{}^i{}_k = N^i E_{jk} &+ N_k F^i{}_j + N_j F^i{}_k + g^{im} K_{mj(k)} E_{00} \\ &+ 2(K^i{}_j E_{k0} + K^i{}_k E_{j0} + K_{jk} F^i{}_0) - 2S_{(jk)}\{C_m{}^i{}_j A^m{}_k\} \\ &- \lambda^s (C_s{}^i{}_j \,|_k - C_j{}^i{}_r C_s{}^r{}_k - C_s{}^r{}_j C_r{}^i{}_k + C_s{}^i{}_r C_j{}^r{}_k), \end{aligned} \tag{2.13}$$

where $C_s{}^i{}_j \,|_k$ is v-covariant derivatives of $C_s{}^i{}_j$.

The $(v)hv$-torsion tensor $P_j{}^i{}_k$ is given by the difference of (2.9) and (2.11) as follows:

$$\begin{aligned} P_j{}^i{}_k &= G_j{}^i{}_k - \Gamma^*{}_j{}^i{}_k \\ &= S_{(jk)}\{K^i{}_j b_{0k} - C_m{}^i{}_j A^m{}_k\} - \lambda^s C_s{}^i{}_j \,|_k \\ &+ g^{il}(K_{jk} b_{0l} + C_{mjk} A^m{}_l + K_{lj(k)} E_{00}). \end{aligned} \tag{2.14}$$

The tensors above are used later.

3. Conformal Transformations of the Cartan and the Berwald connections of the Finsler space with an (α, β)-metric

A transformation of a Finsler space $F^n = (M^n, L(\alpha,\beta))$ to another Finsler space $\overline{F}^n = (M^n, \overline{L}(\alpha,\beta))$ satisfying

$$\overline{L}(\alpha,\beta) = e^{\sigma} L(\alpha,\beta), \quad (\sigma = \sigma(x)) \tag{3.1}$$

is called a conformal transformation of a Finsler space with an (α,β)-metric. A conformal transformation is a homothetic transformation if $\sigma_j = 0$, where $\sigma_j = \partial\sigma/\partial x_j$. Under the conformal transformation of the Finsler space $F^n = (M^n, L(\alpha,\beta))$, we have the following relations

$$\overline{\{{}_j{}^i{}_k\}} = \{{}_j{}^i{}_k\} + \sigma_j \delta_k{}^i + \sigma_k \delta_j{}^i - \sigma^i a_{jk}, \tag{3.2}$$

$$\overline{b}_{ij} = e^{\sigma}(b_{ij} - \sigma_i b_j + a_{ij}\sigma_m b^m), \tag{3.3}$$

$$\begin{aligned} \overline{E}_{ij} &= e^{\sigma}(E_{ij} - \sigma_{(i}b_{j)} + a_{ij}\sigma_m b^m), \quad \overline{F}_{ij} = e^{\sigma}(F_{ij} - \sigma_{[i}b_{j]}), \\ \overline{E}_{00} &= e^{\sigma}(E_{00} - \sigma_0 b_0 + \alpha^2 \sigma_m b^m), \quad \overline{F}^k{}_i = e^{-\sigma}(F^k{}_i - g^{kr}\sigma_{[r}b_{i]}), \end{aligned} \tag{3.4}$$

$$\begin{aligned} &\overline{N}_k = e^{\sigma} N_k, \quad \overline{q} = e^{\sigma} q, \quad \overline{q}_{-2} = e^{-2\sigma} q_{-2}, \quad \overline{p}_0 = p_0, \\ &\overline{N}^i = e^{-\sigma} N^i, \quad \overline{K}_{ik} = e^{\sigma} K_{ik}, \quad \overline{K}^i{}_k = e^{-\sigma} K^i{}_k. \end{aligned} \tag{3.5}$$

From the relation (2.10), (3.4) and (3.5), we get the transformation formula

$$\overline{A}^m{}_k = A^m{}_k + R^{mr}{}_k \sigma_r, \quad \overline{\lambda}^m = \lambda^m + R^{mr}\sigma_r, \tag{3.6}$$

where we put

$$\begin{aligned} R^{mr}{}_k &= K^m{}_k(\alpha^2 b^r - y^r b_0) + N^m\{Y_k b^r - (y^r b_k + \delta^r_k b_0)/2\} \\ &\quad - \{N_k g^{ml}(\delta^r_l b_0 - y^r b_l) + q g^{ml}(\delta^r_l b_k - \delta^r_k b_l)\}/2, \end{aligned} \tag{3.7}$$

$$R^{mr} = N^m(\alpha^2 b^r - y^r b_0) - 2qg^{ml}(\delta_l^r b_0 - y^r b_l)/2. \tag{3.8}$$

We shall investigate the connection coefficients of the Cartan and the Berwald connections of a conformal transformation in the Finsler space with an (α, β)-metric. By (3.2), (3.4) and (3.5), the conformal transformation of (2.7) is reduced to the following forms

$$\overline{G}^i = (\overline{\{{}_0{}^i{}_0\}} + \overline{N}^i \overline{E}_{00} + 2\overline{q}\overline{F}^i{}_0)/2 = G^i - B^{ir}\sigma_r, \tag{3.9}$$

where we put

$$\begin{aligned} B^{ir} = \{&\alpha^2 a^{ir} - 2y^i y^r + N^i(y^r b_0 - \alpha^2 b^r) \\ &+ qg^{il}(\delta_l^r b_0 - y^r b_l)\}/2. \end{aligned} \tag{3.10}$$

Next, differentiating (3.9) with respect to y^i, we have

$$\overline{G}^i{}_j = \dot\partial_j G^i - \dot\partial_j(B^{ir}\sigma_r) = G^i{}_j - B^{ir}{}_j\sigma_r, \tag{3.11}$$

where we put

$$\begin{aligned} B^{ir}{}_j = &Y_j a^{ir} - y^i\delta_j^r - y^r\delta_j^i + N^i\{(\delta_j^r b_0 + y^r b_j)/2 - Y_j b^r\} \\ &+ (N_j g^{il} - 2qC_j^{il})(\delta_l^r b_0 - y^r b_l)/2 + qg^{il}\delta_{[l}^r b_{j]} \\ &+ (K^i{}_j - N_l C_j^{il})(y^r b_0 - \alpha^2 b^r). \end{aligned} \tag{3.12}$$

Furthermore, differentiating (3.11) with respect to y^k, we obtain

$$\overline{G}_k{}^i{}_j = \dot\partial_k G^i{}_j - \dot\partial_k(B^{ir}{}_j\sigma_r) = G_j{}^i{}_k - B_{jk}^{ir}\sigma_r, \tag{3.13}$$

where we put

$$\begin{aligned} B_{jk}^{ir} = &Q_{jk}^{ir} + N^i(\delta_{[j}^r b_{k]} - a_{jk}b^r) + 2S_{(jk)}\{(K^i{}_k \\ &- N_l C_k^{il})(\delta_j^r b_0 + y^r b_j)/2 - Y_j b^r\} + \{K_{jk}g^{il} - S_{(jk)}(N_j C_k^{il}) \\ &- qC_{j(k)}^{il}\}(\delta_l^r b_0 - y^r b_l) + S_{(jk)}\{(N_j g^{il} - 2qC^{il}{}_j)\delta_{[l}^r b_{k]}\} \\ &+ \{g^{il}K_{lj(k)} - N_l C_{j(k)}^{il} - 2S_{(jk)}(K_{lj}C_k^{il})\}(y^r b_0 - \alpha^2 b^r), \end{aligned} \tag{3.14}$$

$$Q_{jk}^{ir} = a_{jk}a^{ir} - \delta_k^i\delta_j^r - \delta_j^i\delta_k^r.$$

Therefore, we have the following

THEOREM 3.1. *Under the conformal transformation of the Finsler space with an (α,β)-metric, the connection coefficients of $G_j{}^i{}_k$, $G^i{}_j$ of a Berwald connection $B\Gamma$ are transformed as (3.11), (3.13) respectively.*

From the relations (3.3), (3.4), (3.5), we can prove easily the following.

THEOREM 3.2. *The tensors $B^{ir}{}_j$, B^{ir}_{jk} are invariant under the conformal transformation of the Finsler space with an (α,β)-metric .*

Next, we shall calculate the transformed quantity $\overline{\Gamma}^*{}_j{}^i{}_k$ of the Cartan connection coefficient $\Gamma^*{}_j{}^i{}_k$ under the conformal transformation. Using (3.2), (3.3), (3.4), (3.5) and (3.6), we can see that (2.9) is transformed to the following forms

$$\begin{aligned}\overline{\Gamma}^*{}_j{}^i{}_k &= \overline{\{{}_j{}^i{}_k\}} + \overline{N}^i\overline{E}_{jk} + \overline{N}_j\overline{F}^i{}_k + \overline{N}_k\overline{F}^i{}_j + \bar{b}_{0j}\overline{K}^i{}_k \\ &\quad + \bar{b}_{0k}\overline{K}^i{}_j - \bar{b}_{0m}\overline{g}^{im}\overline{K}_{kj} - (C_m{}^i{}_j\overline{A}^m{}_k + C_m{}^i{}_k\overline{A}^m{}_j \\ &\quad - \overline{C}_{mjk}\overline{A}^m{}_s\overline{g}^{is}) + \overline{\lambda}^s(C_j{}^i{}_mC_s{}^m{}_k + C_k{}^i{}_mC_s{}^m{}_j \\ &\quad - C_k{}^m{}_jC_m{}^i{}_s) \\ &= \Gamma^*{}_j{}^i{}_k - U^{ir}_{jk}\sigma_r,\end{aligned} \tag{3.15}$$

where we put

$$\begin{aligned}U^{ir}_{jk} &= Q^{ir}_{jk} + N^i(\delta^r_{(j}b_{k)} - a_{jk}b^r) + S_{(jk)}[(N_jg^{il})\delta^r_{[l}b_{k]} \\ &\quad + K^i{}_j(y^rb_k - Y_kb^r) - N^mC_j{}^i{}_m\{(y^rb_k + \delta^r_kb_0)/2 - Y_kb_r\}] \\ &\quad - K_{jk}g^{im}(y^rb_m - Y_mb^r) - (C_j{}^i{}_mK^m{}_k + C_k{}^i{}_mK^m{}_j \\ &\quad - C_{jkm}g^{is}K^m{}_s)(y^rb_0 - \alpha^2b^r) + N^mC_{jkm}g^{is}\{(y^rb_s \\ &\quad + \delta^r_sb_0)/2 - Y_sb^r\} - g^{ml}(C_j{}^i{}_mN_k + C_k{}^i{}_mN_j \\ &\quad - C_{jkm}N^i)(\delta^r_lb_0 - y^rb_l)/2 - qg^{ml}(C_j{}^i{}_m\delta^r_{[l}b_{k]} \\ &\quad + C_k{}^i{}_m\delta^r_{[l}b_{j]} - C_{jkm}g^{is}\delta^r_{[l}b_{s]}) + (C_j{}^i{}_mC_s{}^m{}_k + C_k{}^i{}_mC_s{}^m{}_j \\ &\quad - C_k{}^m{}_jC_m{}^i{}_s)\{N^s(y^rb_0 - \alpha^2b^r) + qg^{sl}(\delta^r_lb_0 - y^rb_l)\}.\end{aligned} \tag{3.16}$$

From the properties of conformal transformation formula (3.2), (3.3), (3.4), (3.5) and (3.6), we find

THEOREM 3.3. *Under the conformal transformation of the Finsler space with an (α,β)-metric, U^{ir}_{jk} is invariant and symmetric in j, k.*

4. Conformal transformation of difference tensors in the Finsler space with an (α, β)-metric

From the difference of (3.2) and (3.13), we have

$$\overline{'D}_j{}^i{}_k = \overline{G}_j{}^i{}_k - \overline{\{{}_j{}^i{}_k\}} = {}'D_j{}^i{}_k - {}'T^{ir}_{jk}\sigma_r, \tag{4.1}$$

where ${}'T^{ir}_{jk} = B^{ir}_{jk} - Q^{ir}_{jk}$. From (3.14), we have

$$\begin{aligned} {}'T^{ir}_{jk} &= N^i(\delta^r_{[j}b_{k]} - a_{jk}b^r) \\ &\quad + 2S_{(jk)}\{(K^i{}_k - N_lC^{il}{}_k)(\delta^r_j b_0 + y^r b_j)/2 - Y_j b^r\} \\ &\quad + (K_{jk}g^{il} - S_{(jk)}\{N_jC^{il}{}_k\} - qC^{il}_{j(k)})(\delta^r_l b_0 - y^r b_l) \\ &\quad + S_{(jk)}\{(N_jg^{il} - 2qC^{il}_j)\delta^r_{[l}b_{k]}\} + (g^{il}K_{lj(k)} - N_lC^{il}{}_{j(k)} \\ &\quad - 2S_{(jk)}\{K_{lj}C^{il}_k\})(y^r b_0 - \alpha^2 b^r). \end{aligned} \tag{4.2}$$

Thus we have

THEOREM 4.1. *A difference tensor ${}'D_j{}^i{}_k$ of the Berwald connection of the Finsler space with an (α, β)-metric is invariant under the conformal transformation if and only if ${}'T^{ir}_{jk}\sigma_r = 0$.*

Secondly, we shall calculate the conformal transformation of difference tensor $D_j{}^i{}_k$ of $C\Gamma$. From (3.2), (3.15), we have

$$\overline{D}_j{}^i{}_k = \overline{\Gamma}^*{}_j{}^i{}_k - \overline{\{{}_j{}^i{}_k\}} = D_j{}^i{}_k - T^{ir}_{jk}\sigma_r, \tag{4.3}$$

where $T^{ir}_{jk} = U^{ir}_{jk} - Q^{ir}_{jk}$. From (3.16), we find

$$\begin{aligned} T^{ir}_{jk} &= N^i(\delta^r_{[j}b_{k]} - a_{jk}b^r) + S_{(jk)}[(N_jg^{il})\delta^r_{[l}b_{k]} \\ &\quad + K^i{}_j(y^r b_k - Y_k b^r) - N^m C_j{}^i{}_m\{(y^r b_k + \delta^r_k b_0)/2 - Y_k b^r\}] \\ &\quad - K_{jk}g^{im}(y^r b_m - Y_m b^r) - (C_j{}^i{}_m K^m{}_k + C_k{}^i{}_m K^m{}_j \\ &\quad - C_{jkm}g^{is}K^m{}_s)(y^r b_0 - \alpha^2 b^r) + N^m C_{jkm}g^{is}\{(y^r b_s \\ &\quad + \delta^r_s b_0)/2 - Y_s b^r\} - g^{ml}(C_j{}^i{}_m N_k + C_k{}^i{}_m N_j \\ &\quad - C_{jkm}N^i)(\delta^r_l b_0 - y^r b_l)/2 - qg^{ml}(C_j{}^i{}_m \delta^r_{[l}b_{k]} \\ &\quad + C_k{}^i{}_m \delta^r_{[l}b_{j]} - C_{jkm}g^{is}\delta^r_{[l}b_{s]}) + (C_j{}^i{}_m C_s{}^m{}_k + C_k{}^i{}_m C_s{}^m{}_j \\ &\quad - C_k{}^m{}_j C_m{}^i{}_s)\{N^s(y^r b_0 - \alpha^2 b^r) + qg^{sl}(\delta^r_l b_0 - y^r b_l)\}. \end{aligned} \tag{4.4}$$

Thus we have

THEOREM 4.2. *The difference tensor* $D_j{}^i{}_k$ *of the Cartan connection is invariant under the conformal transformation of the Finsler space with an* (α, β)*-metric if and only if* $T^{ir}_{jk}\sigma_r = 0$.

Thirdly, we shall consider that the $(v)hv$-torsion tensor $P_j{}^i{}_k$ under the conformal transformation of the Finsler space with an (α, β)-metric. From (3.13), (3.15), we get the following equation

$$\overline{P}_j{}^i{}_k = \overline{G}_j{}^i{}_k - \overline{\Gamma}^*{}_j{}^i{}_k = P_j{}^i{}_k - V^{ir}_{jk}\sigma_r, \tag{4.5}$$

where $V^{ir}_{jk} = B^{ir}_{jk} - U^{ir}_{jk}$. From (3.14), (3.16), we get the following

$$\begin{aligned} V^{ir}_{jk} = {} & S_{(jk)}[K^i{}_j(\delta^r_k b_0 - Y_k b^r) - N^m C_m{}^i{}_j\{(\delta^r_k b_0 + y^r b_k)/2 \\ & - Y_k b^r\}] + \{(C^{il}{}_j N_k + C^{il}{}_k N_j - C^l{}_{jk} N^i)/2 \\ & + (K_{jk} g^{il} - q C^{il}_{j(k)})\}(\delta^r_l b_0 - y^r b_l) + K_{jk} g^{is}(y^r b_s - Y_s b^r) \\ & - N^l C_{jkl} g^{is}\{(y^r b_s + \delta^r_s b_0)/2 - Y_s b^r\} - (K_{lj} C^{il}{}_k + K_{lk} C^{il}{}_j \\ & - K^i{}_l C_j{}^l{}_k - g^{il} K_{lj(k)} + N_l C^{il}{}_{j(k)})(y^r b_0 - \alpha^2 b^r) \\ & - q\{C_{jk}{}^l g^{is}(\delta^r_{[l} b_{s]}) + C^{il}{}_j(\delta^r_{[l} b_{k]}) + C^{il}{}_k(\delta^r_{[l} b_{j]})\} \\ & - (C_j{}^i{}_m C_s{}^m{}_k + C_k{}^i{}_m C_s{}^m{}_j - C_k{}^m{}_j C_m{}^i{}_s)\{N^s(y^r b_0 - \alpha^2 b^r) \\ & + q g^{sl}(\delta^r_l b_0 - y^r b_l)\}. \end{aligned} \tag{4.6}$$

Thus we have, from (4.5)

THEOREM 4.3. *A Landsberg space remains to be a Landsberg space by a conformal transfor mation of the Finsler space with an* (α, β)*-metric if and only if* $V^{ir}_{jk}\sigma_r = 0$.

From (3.4), (3.5), (4.2), (4.4) and (4.6), we find

THEOREM 4.4. *The tensors* $'T^{ir}_{jk}$, T^{ir}_{jk}, V^{ir}_{jk} *are conformally invariant.*

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