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INTEGRAL FORMULAS FOR STRIPS

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ABSTRACT. For n random strips chosen so as to meet a fixed bounded convex set K of the plane we let ν be the number of intersection regions that meet K. We develop the integral formula for the mean value of ν and ν^2 involving the area and the perimeter of K and the breadths of the strips. We get some geometric inequalities in way of studying integral geometry.

1. Introduction

The study on integral geometry is originated with Buffon's needle problem which is stated and solved by Buffon in his Essai d'Arithmétique Morale. And the title "Integral Geometry" was initiated by W. Blaschke and his school in the mathematics seminar of the University of Hamburg. Many problems treated in integral geometry had their roots in geometric probability theory and it was main purpose to investigate whether probabilistic ideas could be applied to get interesting results in the geometry of convex bodies. Convexity is closely related to integral geometry. P. M. Gruber and J. M. Wills([3], [4]) and L. A. Santaló [9] are good references in this line.

Let K be a bounded convex set of the plane with interior points and denote the area of K by F and the perimeter of K by L. Assume that n random lines intersect K. Then these lines determine a number ν of intersection points of pairs of lines that are interior to K. By definition we have $E(\nu) = \frac{1}{L^n} \int_{G_i \cap K \neq \emptyset} \nu dG_1 \wedge dG_2 \wedge \cdots \wedge dG_n$. Then the mean number of intersection points that are inside K is

$$E(\nu) = \frac{n(n-1)\pi F}{L^2}.$$

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We can also find $E(\nu^2)$. Using the theory of Poisson line process([1], [2], [5], [6], [7], [8]) we get the mean area, the mean number of the sides, and the mean perimeter of the regions into which the plane is partitioned by the random lines([9]).

In this paper, we get the mean value of the number of intersections that meet a fixed convex set K for n random strips chosen so as to meet K and we get some geometric inequalities in way of studying integral geometry.

2. Preliminaries and Notations

Throughout this paper the underlying plane is the Euclidean plane. A line on the plane may be determined by its distance p from the origin of the plane and the angle ϕ of the normal of the line with the x axis. We denote it by $G(p,\phi)$. The function $p(\phi)$ is the support function of the envelope K of the lines $G(p,\phi)$. We have the formula for the perimeter L of K (See [9]):

$$L = \int_0^{2\pi} p d\phi \tag{1}$$

The measure of a set of lines $G(p,\phi)$ is defined by the integral, over the set, of the differential form

$$dG = dp \wedge d\phi, \tag{2}$$

called the density for sets of lines. Using (1) and (2) we have the formula for the measure of the set of lines that meet a fixed bounded convex set K:

$$m(G; G \cap K \neq \emptyset) = L, \tag{3}$$

where L is the perimeter of K. For the proof see [9]. Now we define a strip $B(p,\phi)$ as the closed part of the plane consisting of all points that lie between two parallel lines at a distance $\frac{a}{2}$ from the midparallel $G(p,\phi)$. We call the set $B(p,\phi)$ a strip of breadth a. The position of $B(p,\phi)$ can be determined by the position of its midparallel $G(p,\phi)$. Thus the density for sets of strips of fixed breadth is $dB = dp \wedge d\phi$. For more informations see [9]. Assume that K is a bounded convex set of perimeter L in the plane and $K_{\frac{a}{2}}$ is the parallel set of K in the distance $\frac{a}{2}$. If $B(p,\phi) \cap K \neq \emptyset$, the medparallel $G(p,\phi)$ of $B(p,\phi)$ intersects $K_{\frac{a}{2}}$. Conversely, if the midparallel $G(p,\phi)$ of $B(p,\phi)$ intersects $K_{\frac{a}{2}}$, then $B(p,\phi)$ intersects K. And the perimeter $L_{\frac{a}{2}}$ of the parallel set $K_{\frac{a}{2}}$ of K in the distance $\frac{a}{2}$ is $L+\pi a$([9]). Therefore using (3) the measure of the set of strips B of breadth a that intersect a convex set K is

$$m(B; B\cap K\neq\emptyset)=\int_{B\cap K\neq\emptyset} dB = L+\pi a.$$

Now we have the following lemma.

LEMMA 1. *Let K be a plane convex set of area F and let B be a strip of breadth a. Then*

$$\int_{B\cap K\neq\emptyset} F_{B\cap K}dB=\pi a F,$$

where $F_{B\cap K}$ is the area of $B\cap K$.

PROOF. Let G_1, G_2 be any lines. Consider the measure of the set of pairs of lines G_1, G_2 and one strip B such that $G_1\cap G_2\cap B\cap K\neq\emptyset$. The measure of this set is the integral of the form $dB\wedge dG_1\wedge dG_2$ over the set $G_1\cap G_2\cap B\cap K\neq\emptyset$. The first way, first fixing the lines G_1, G_2 and then integrating over the strips B, gives

$$\begin{aligned} & m(G_1, G_2, B; G_1\cap G_2\cap B\cap K\neq\emptyset)\\ = & \int_{G_1\cap K\neq\emptyset}\int_{G_2\cap(K\cap G_1)\neq\emptyset}\int_{B\cap(G_1\cap G_2\cap K)\neq\emptyset} dB dG_2 dG_1\\ = & \pi a\int_{G_1\cap K\neq\emptyset}\int_{G_2\cap(K\cap G_1)\neq\emptyset} dG_2 dG_1\\ = & 2\pi a\int_{G_1\cap K\neq\emptyset}\sigma_1 dG_1\\ = & 2\pi^2 a F.\end{aligned}$$

And the second way, first fixing the strip B and then integrating over the lines G_1 and G_2, gives

$$\begin{aligned} & m(G_1, G_2, B; G_1 \cap G_2 \cap B \cap K \neq \emptyset) \\ = & \int_{B\cap K\neq\emptyset} \int_{G_1\cap(B\cap K)\neq\emptyset} \int_{G_2\cap(G_1\cap B\cap K)\neq\emptyset} dG_2 dG_1 dB \\ = & \int_{B\cap K\neq\emptyset} \int_{G_1\cap(B\cap K)\neq\emptyset} 2\sigma_{G_1\cap B\cap K} dG_1 dB \\ = & \int_{B\cap K\neq\emptyset} 2\pi F_{B\cap K} dB \\ = & 2\pi \int_{B\cap K\neq\emptyset} F_{B\cap K} dB, \end{aligned}$$

where $\sigma_{G_1\cap B\cap K}$ is the length of the segment $G_1 \cap B \cap K$. This completes the proof. □

3. Main Results

In this section we investigate the measure of the set of lines and strips that meet a fixed bounded convex set K and study some geometric inequalities and the geometry of the number of intersections that meet K for n random strips that meet K.

THEOREM 1. *Let K be a plane convex set of area F and perimeter L and let G be a line; and let B_1, B_2 be strips of breadths a_1, a_2, respectively. Then*

$$m(B_1, B_2, G; B_1 \cap B_2 \cap G \cap K \neq \emptyset) = \pi^2\{2F(a_1 + a_2) + a_1 a_2 L\}.$$

PROOF. Let $F_{G\cap K}$ denote the area of $G \cap K$ and $\sigma_{B_1\cap G\cap K}$ the length of $B_1 \cap G \cap K$. Denoting u_1 the perimeter of $B_1 \cap K$ and $F_{B_1\cap K}$ the area of $B_1 \cap K$,

$$\begin{aligned} & m(B_1, B_2, G; B_1 \cap B_2 \cap G \cap K \neq \emptyset) \\ = & \int_{B_1\cap K\neq\emptyset} \int_{G\cap(B_1\cap K)\neq\emptyset} \int_{B_2\cap(B_1\cap G\cap K)\neq\emptyset} dB_2 dG dB_1 \\ = & \int_{B_1\cap K\neq\emptyset} \int_{G\cap(B_1\cap K)\neq\emptyset} \{2\sigma_{G\cap B_1\cap K} + \pi a_2\} dG dB_1 \end{aligned}$$

$$= \int_{B_1 \cap K \neq \emptyset} \{2\pi F_{B_1 \cap K} + \pi a_2 u_1\} dB_1$$
$$= 2\pi \int_{B_1 \cap K \neq \emptyset} F_{B_1 \cap K} dB_1 + \pi a_2 (2\pi F + \pi a_1 L).$$

By Lemma 1, this completes the proof. □

REMARK 1. Theorem 1 says that if B_1 and B_2 are independent strips that meet a fixed convex set K and the line G meets K then the probability that $B_1 \cap B_2 \cap G \cap K \neq \emptyset$ is

$$p = \frac{\pi^2 \{2F(a_1 + a_2) + a_1 a_2 L\}}{L(L + \pi a_1)(L + \pi a_2)}$$

COROLLARY 1. *Let K be a plane convex set of area F and perimeter L and let G be a line; and let B_1, B_2 be strips of breadth a. Then*

$$m(B_1, B_2, G; B_1 \cap B_2 \cap G \cap K \neq \emptyset) = \pi^2 a \{4F + aL\}.$$

PROOF. We have the result from Theorem 1 immediately. □

THEOREM 2. *Let K be a plane convex set of area F and perimeter L. If B is a strip of breadth a, then*

$$\int_{B \cap K \neq \emptyset} u^2 dB \leq L\{2\pi F + \pi a L\}, \tag{4}$$

where u is the perimeter of $B \cap K$.

PROOF. Let G_1 and G_2 be the lines in the plane and compute the measure by first fixing the strip B and then integrating over the lines G_1 and G_2. Then we have

$$m(G_1, G_2, B; G_1 \cap B \cap K \neq \emptyset, G_2 \cap B \cap K \neq \emptyset)$$
$$= \int_{G_1 \cap B \cap K \neq \emptyset, G_2 \cap B \cap K \neq \emptyset} dG_1 dG_2 dB$$
$$= \int_{B \cap K \neq \emptyset} dB \int_{G_1 \cap B \cap K \neq \emptyset} dG_1 \int_{G_2 \cap B \cap K \neq \emptyset} dG_2$$
$$= \int_{B \cap K \neq \emptyset} dB \int_{G_1 \cap B \cap K \neq \emptyset} u dG_1$$
$$= \int_{B \cap K \neq \emptyset} u^2 dB.$$

On the other hand, if we compute this measure by first fixing the line G_2 and the strip B and then integrating over the line G_1, then we have

$$\begin{aligned} & m(G_1, G_2, B; G_1 \cap B \cap K \neq \emptyset, G_2 \cap B \cap K \neq \emptyset) \\ = & \int_{G_1 \cap B \cap K \neq \emptyset, G_2 \cap B \cap K \neq \emptyset} dG_1 dG_2 dB \\ = & \int_{G_2 \cap B \cap K \neq \emptyset} \{ \int_{G_1 \cap B \cap K \neq \emptyset} dG_1 \} dB dG_2 \\ = & \int_{G_2 \cap K \neq \emptyset} \{ \int_{B \cap (G_2 \cap K) \neq \emptyset} u dB \} dG_2. \end{aligned}$$

Because u has the positive value and there is the strip B such that $B \cap K \neq \emptyset$ and $B \cap (K \cap G_2) = \emptyset$, we have

$$\int_{B \cap (G_2 \cap K) \neq \emptyset} u dB \leq \int_{B \cap K \neq \emptyset} u dB.$$

Thus we have

$$\begin{aligned} \int_{G_2 \cap K \neq \emptyset} \{ \int_{B \cap (G_2 \cap K) \neq \emptyset} u dB \} dG_2 & \leq \int_{G_2 \cap K \neq \emptyset} \{ \int_{B \cap K \neq \emptyset} u dB \} dG_2 \\ & = L \{ 2\pi F + \pi a L \}. \end{aligned}$$

This completes the proof. □

Let K be a plane convex set with the origin in its interior. For the direction $0 \leq \theta \leq 2\pi$ the distance $Br(\theta)$ between two parallel lines perpendicular to the direction θ is called the breadth of K in the direction θ. Because the distance function $Br(\theta)$ is continuous on $[0, 2\pi]$ $Br(\theta)$ has the minimal value on $[0, 2\pi]$. From now on *we assume that the breadth a_i of the strip B_i is always less than the minimal value of $Br(\theta)$.*

THEOREM 3. *Let K be a plane convex set of perimeter L and area F, and let $B_i, 1 \leq i \leq n$, be a strip of breadth a_i. Let ν be the number of intersections of B_i and B_j that meet K. Then the mean value $E(\nu)$ of ν is*

$$E(\nu) = \sum_{h<k} \frac{\{2\pi F + \pi(a_h + a_k)L + \pi^2 a_h a_k\}}{(L + \pi a_h)(L + \pi a_k)}. \tag{5}$$

PROOF. Let ν_{hk} $(h \neq k)$ denote the function of B_h, B_k such that

$$\nu_{hk} = \begin{cases} 1, & \text{if} \quad B_h \cap B_k \cap K \neq \emptyset \\ 0, & \text{if} \quad B_h \cap B_k \cap K = \emptyset. \end{cases} \tag{6}$$

Then it is trivial that $\nu = \sum_{h<k} \nu_{hk}$. Denote u_h the perimeter of $B_h \cap K$, $1 \leq h \leq n$. Then

$$\begin{aligned} \int \nu_{hk} dB_h \wedge dB_k &= \int_{B_h \cap B_k \cap K \neq \emptyset} dB_h \wedge dB_k \\ &= \int_{B_k \cap K \neq \emptyset} \{\int_{B_h \cap (B_k \cap K) \neq \emptyset} dB_h\} dB_k \\ &= \int_{B_k \cap K \neq \emptyset} \{u_k + \pi a_h\} dB_k \\ &= \int_{B_k \cap K \neq \emptyset} u_k dB_k + \pi a_h (L + \pi a_k) \\ &= 2\pi F + \pi a_k L + \pi a_h L + \pi^2 a_h a_k. \end{aligned}$$

Thus

$$\begin{aligned} &\int \nu dB_1 \wedge \cdots \wedge dB_n \\ = &\int \sum_{h<k} \nu_{hk} dB_1 \wedge \cdots \wedge dB_n \\ = &\sum_{h<k} \{2\pi F + \pi(a_k + a_h)L + \pi^2 a_h a_k\} \int dB_1 \wedge \cdots \wedge \overbrace{dB_h}^{\text{omitted}} \wedge \cdots \wedge \overbrace{dB_k}^{\text{omitted}} \wedge \cdots \wedge dB_n \\ = &\sum_{h<k} \{2\pi F + \pi(a_k + a_h)L + \pi^2 a_h a_k\} (L + \pi a_1) \cdots \overbrace{(L + \pi a_h)}^{\text{omitted}} \cdots \overbrace{(L + \pi a_k)}^{\text{omitted}} \cdots (L + \pi a_n). \end{aligned}$$

Thus we have now easily the result from the definition of the mean value

$$E(\nu) = \frac{\int \nu dB_1 \wedge \cdots \wedge dB_n}{\int dB_1 \wedge \cdots \wedge dB_n}.$$

□

COROLLARY 2. *Let K be a plane convex set of perimeter L and area F. Then the mean value $E(\nu)$ of the number ν of intersections of the strips B_i and B_j of breadth a, $1 \leq i \leq n$, that meet K is*

$$E(\nu) = \frac{n(n-1)\pi\{F + aL + \frac{\pi a^2}{2}\}}{(L + \pi a)^2}.$$

PROOF. Because in the proof of Theorem 3 $a_i = a$ for all $i \in \{1, 2, \cdots, n\}$ and the summation $\sum_{h<k}$ has $\binom{n}{2}$ terms we have the result □

From Theorem 3 we can get the well-known result:

COROLLARY 3. *For n lines $G_i, i = 1, 2, \cdots, n$, chosen at random so as to meet a bounded convex set K of area F and perimeter L, the mean number of intersection points that are inside K is*

$$E(\nu) = \frac{n(n-1)\pi F}{L^2}.$$

PROOF. In the equation (5) $a_h = a_k = 0$ we have

$$E(\nu) = \sum_{h<k} \frac{2\pi F}{L^2}.$$

And since the summation $\sum_{h<k}$ has $\binom{n}{2}$ terms

$$E(\nu) = \frac{n(n-1)\pi F}{L^2}.$$ □

THEOREM 4. *Let K be a plane convex set of perimeter L and area F, and let B_i, $1 \leq i \leq n$, be a strip of breadth a_i. Let ν be the number of intersections of B_i and B_j that meets K. Then*

$$\begin{aligned} &E(\nu^2) - E(\nu) \\ &= 6 \sum_{i<j<h<k} \frac{\{2\pi F + \pi(a_i + a_j)L + \pi^2 a_i a_j\}\{2\pi F + \pi(a_h + a_k)L + \pi^2 a_h a_k\}}{(L + \pi a_i)(L + \pi a_j)(L + \pi a_h)(L + \pi a_k)} \\ &+ 6 \sum_{i<j<h} \frac{\pi^2 a_j a_h (L + \pi a_i) + \pi(a_j + a_h)\{2\pi F + \pi a_i L\} + (I_2)_i}{(L + \pi a_i)(L + \pi a_j)(L + \pi a_h)}, \end{aligned} \tag{7}$$

where $(I_2)_i$ denotes the integral $\int_{B_i \cap K \neq \emptyset} u_i^2 dB_i$.

PROOF. Let ν_{hk} be the function in (6). Note that

$$\begin{aligned} &\int \nu^2 dB_1 \wedge \cdots \wedge dB_n \\ &= \int \{\sum_{h<k} {\nu_{hk}}^2 + 2 \sum_{(i,j)\neq(k,h)} \nu_{ij}\nu_{kh}\} dB_1 \wedge \cdots \wedge dB_n \\ &= \sum_{h<k} \int {\nu_{hk}}^2 dB_1 \wedge \cdots \wedge dB_n + 2 \sum_{(i,j)\neq(k,h)} \int \nu_{ij}\nu_{kh} dB_1 \wedge \cdots \wedge dB_n. \end{aligned}$$

First $\sum_{h<k} \int \nu_{hk}{}^2 dB_1 \wedge \cdots \wedge dB_n$ is equal to

$$\sum_{h<k} \int_{B_i \cap K \neq \emptyset} \{ \int_{B_h \cap B_k \cap K \neq \emptyset} dB_h \wedge dB_k \} dB_1 \wedge \cdots \wedge \overbrace{dB_h}^{\text{omitted}} \wedge \cdots \wedge \overbrace{dB_k}^{\text{omitted}} \wedge \cdots \wedge dB_n.$$

And this equals

$$\sum_{h<k} \{2\pi F + \pi(a_h + a_k)L + \pi^2 a_h a_k\}(L + \pi a_1) \cdots \overbrace{(L + \pi a_h)}^{\text{omitted}} \cdots \overbrace{(L + \pi a_k)}^{\text{omitted}} \cdots (L + \pi a_n).$$

Now

$$\begin{aligned} & 2 \sum_{(i,j) \neq (h,k)} \int \nu_{ij} \nu_{kh} dB_1 \wedge \cdots \wedge dB_n \\ = & \; 2 \sum_{k \neq i, j \neq h, j \neq k} \int \nu_{ij} \nu_{kh} dB_1 \wedge \cdots \wedge dB_n + 2 \sum_{i<j<h} \int \nu_{ij} \nu_{ih} dB_1 \wedge \cdots \wedge dB_n \\ = & \; 6 \sum_{i<j<h<k} \int \nu_{ij} \nu_{kh} dB_1 \wedge \cdots \wedge dB_n + 6 \sum_{i<j<h} \int \nu_{ij} \nu_{ih} dB_1 \wedge \cdots \wedge dB_n. \end{aligned} \tag{8}$$

And for $i < j < h$, we have

$$\begin{aligned} & \int \nu_{ij} \nu_{ih} dB_i \wedge dB_j \wedge dB_h \\ = & \int_{B_i \cap B_h \cap K \neq \emptyset} \int_{B_i \cap B_j \cap K \neq \emptyset} dB_i dB_j dB_h \\ = & \int_{B_i \cap B_h \cap K \neq \emptyset} \int_{B_i \cap K \neq \emptyset} \int_{B_i \cap B_j \cap K \neq \emptyset} dB_j dB_i dB_h \\ = & \int_{B_i \cap B_h \cap K \neq \emptyset} \int_{B_i \cap K \neq \emptyset} (u_i + \pi a_j) dB_i dB_h \\ = & \int_{B_i \cap K \neq \emptyset} \int_{B_i \cap B_h \cap K \neq \emptyset} (u_i + \pi a_j) dB_i dB_h \\ = & \int_{B_i \cap K \neq \emptyset} \{u_i + \pi a_j\} \{ \int_{B_i \cap B_h \cap K \neq \emptyset} dB_h \} dB_i \\ = & \int_{B_i \cap K \neq \emptyset} \{u_i + \pi a_j\} \{u_i + \pi a_h\} dB_i \end{aligned}$$

$$= \int_{B_i \cap K \neq \emptyset} {u_i}^2 dB_i + \pi(a_j + a_h) \int_{B_i \cap K \neq \emptyset} u_i dB_i - \pi^2 a_j a_h \int_{B_i \cap K \neq \emptyset} dB_i$$
$$= \pi^2 a_j a_h (L + \pi a_i) + \pi(a_j + a_h)(2\pi F + \pi a_i L) + \int_{B_i \cap K \neq \emptyset} {u_i}^2 dB_i.$$

Now as a matter of convenience put

$$A_{ijh} = \int \nu_{ij}\nu_{ih} dB_i \wedge dB_j \wedge dB_h$$

and for $i < j < h < k$, put

$$A_{ij} = \int_{B_i \cap B_j \cap K \neq \emptyset} \nu_{ij} dB_i dB_j$$

and

$$B_{ijhk} = \int dB_1 \wedge \cdots \wedge \overbrace{dB_i}^{\text{omitted}} \wedge \cdots \wedge \overbrace{dB_j}^{\text{omitted}} \wedge \cdots \wedge \overbrace{dB_h}^{\text{omitted}} \wedge \cdots \wedge \overbrace{dB_k}^{\text{omitted}} \wedge \cdots \wedge dB_n.$$

Then

$$\sum_{i<j<h} \int \nu_{ij}\nu_{ih} dB_1 \wedge \cdots \wedge dB_n$$
$$= \sum_{i<j<h} A_{ijh} \int dB_1 \wedge \cdots \wedge \overbrace{dB_i}^{\text{omitted}} \wedge \cdots \wedge \overbrace{dB_j}^{\text{omitted}} \wedge \cdots \wedge \overbrace{dB_h}^{\text{omitted}} \wedge \cdots \wedge dB_n$$
$$= \sum_{i<j<h} (L + \pi a_1) \cdots \overbrace{(L + \pi a_i)}^{\text{omitted}} \cdots \overbrace{(L + \pi a_j)}^{\text{omitted}} \cdots \overbrace{(L + \pi a_h)}^{\text{omitted}} \cdots (L + \pi a_n) A_{ijh}.$$

And we have

$$\sum_{i<j<h<k} \int \nu_{ij}\nu_{kh} dB_1 \wedge \cdots \wedge dB_n = \sum_{i<j<h<k} A_{ij} A_{hk} B_{ijhk}.$$

And

$$A_{ij} = 2\pi F + \pi(a_i + a_j)L + \pi^2 a_i a_j$$

and

$$B_{ijhk} = (L + \pi a_1) \cdots \overbrace{(L + \pi a_i)}^{\text{omitted}} \cdots \overbrace{(L + \pi a_j)}^{\text{omitted}} \cdots \overbrace{(L + \pi a_h)}^{\text{omitted}} \cdots \overbrace{(L + \pi a_k)}^{\text{omitted}} \cdots (L + \pi a_n).$$

Thus (8) is equal to

$$6 \sum_{i<j<h<k} A_{ij} A_{hk} B_{ijhk}$$

$$+6 \sum_{i<j<h} (L+\pi a_1)\cdots \overbrace{(L+\pi a_i)}^{\text{omitted}} \cdots \overbrace{(L+\pi a_j)}^{\text{omitted}} \cdots \overbrace{(L+\pi a_h)}^{\text{omitted}} \cdots (L+\pi a_n) A_{ijh}$$

$$= 6 \sum_{i<j<h<k} \{2\pi F + \pi(a_i+a_j)L + \pi^2 a_i a_j\}\{2\pi F + \pi(a_h+a_k)L + \pi^2 a_h a_k\}$$

$$\{(L+\pi a_1)\cdots \overbrace{(L+\pi a_i)}^{\text{omitted}} \cdots \overbrace{(L+\pi a_j)}^{\text{omitted}} \cdots \overbrace{(L+\pi a_h)}^{\text{omitted}} \cdots \overbrace{(L+\pi a_k)}^{\text{omitted}} \cdots (L+\pi a_n)\}$$

$$+6 \sum_{i<j<h} (L+\pi a_1)\cdots \overbrace{(L+\pi a_i)}^{\text{omitted}} \cdots \overbrace{(L+\pi a_j)}^{\text{omitted}} \cdots \overbrace{(L+\pi a_h)}^{\text{omitted}} \cdots (L+\pi a_n)$$

$$\{\pi^2 a_j a_h (L+\pi a_i) + \pi(a_j+a_h)(2\pi F + \pi a_i L) + \int_{B_i \cap K \neq \emptyset} u_i{}^2 dB_i\}.$$

Thus $\int \nu^2 dB_1 \wedge \cdots \wedge dB_n$ has the value

$$\sum_{h<k} \{2\pi F + \pi(a_h+a_k)L + \pi^2 a_h a_k\}(L+\pi a_1)\cdots \overbrace{(L+\pi a_h)}^{\text{omitted}} \cdots \overbrace{(L+\pi a_k)}^{\text{omitted}} \cdots (L+\pi a_n)$$

$$+6 \sum_{i<j<h<k} \{2\pi F + \pi(a_i+a_j)L + \pi^2 a_i a_j\}\{2\pi F + \pi(a_h+a_k)L + \pi^2 a_h a_k\}$$

$$\{(L+\pi a_1)\cdots \overbrace{(L+\pi a_i)}^{\text{omitted}} \cdots \overbrace{(L+\pi a_j)}^{\text{omitted}} \cdots \overbrace{(L+\pi a_h)}^{\text{omitted}} \cdots \overbrace{(L+\pi a_k)}^{\text{omitted}} \cdots (L+\pi a_n)\}$$

$$+6 \sum_{i<j<h} (L+\pi a_1)\cdots \overbrace{(L+\pi a_i)}^{\text{omitted}} \cdots \overbrace{(L+\pi a_j)}^{\text{omitted}} \cdots \overbrace{(L+\pi a_h)}^{\text{omitted}} \cdots (L+\pi a_n)$$

$$\{\pi^2 a_j a_h (L+\pi a_i) + \pi(a_j+a_h)(2\pi F + \pi a_i L) + \int_{B_i \cap K \neq \emptyset} u_i{}^2 dB_i\}.$$

Using the definition of the mean value of $E(\nu^2)$ and the inequality (4) and Theorem 3 we have the result. □

COROLLARY 4. *Let K be a plane convex set of perimeter L and area F, and let B_i, $1 \le i \le n$, be a strip of breadth a. Let ν be the number of intersections of B_i and B_j that meet K. Then*

$$E(\nu^2) - E(\nu)$$
$$= 6\binom{n}{4}\frac{\pi^2\{2F + 2aL + \pi a^2\}^2}{(L+\pi a)^4} + 6\binom{n}{3}\frac{\pi^2 a^2 (L+\pi a) + 2\pi a\{2\pi F + \pi a L\} + I_2}{(L+\pi a)^3},$$

where I_2 denotes the integral $\int_{B \cap K \neq \emptyset} u^2 dB$.

PROOF. Because in Theorem 4 the summation $\sum_{i<j<h<k}$ has $\binom{n}{4}$ terms and the summation $\sum_{i<j<h}$ has $\binom{n}{3}$ terms we have the results. □

Corollary 4 gives us well-known fact:

COROLLARY 5. *Let K be a bounded plane convex set of area F and perimeter L. For n random lines chosen so as to meet K let ν be the number of intersection points inside K. Then the mean value of ν^2 is*

$$E(\nu^2) = 2\pi\binom{n}{2}\frac{F}{L^2} + 24\binom{n}{4}\frac{\pi^2F^2}{L^4} + 24\binom{n}{3}\frac{\int_{G\cap K\neq\emptyset}\sigma^2 dG}{L^3},$$

where σ denotes the length of $K\cap G$.

PROOF. Since $a_i = a_j = a_h = a_k = 0$ in the inequality (7) and $u = 2\sigma$ and the summation $\sum_{i<j<h<k}$ has $\binom{n}{4}$ terms and the summation $\sum_{i<j<h}$ has $\binom{n}{3}$ terms we have the result. □

REMARK 2. From Theorem 3 and Theorem 4 we have also the result for the variance σ^2 of the number ν as follows:

$$\begin{aligned}\sigma^2(\nu) &= 6\sum_{i<j<h<k}\frac{\{2\pi F + \pi(a_i+a_j)L + \pi^2 a_i a_j\}\{2\pi F + \pi(a_h+a_k)L + \pi^2 a_h a_k\}}{(L+\pi a_i)(L+\pi a_j)(L+\pi a_h)(L+\pi a_k)}\\ &+6\sum_{i<j<h}\frac{\pi^2 a_j a_h(L+\pi a_i) + \pi(a_j+a_h)\{2\pi F + \pi a_i L\} + (I_2)_i}{(L+\pi a_i)(L+\pi a_j)(L+\pi a_h)}\\ &+\sum_{h<k}\frac{\{2\pi F + \pi(a_h+a_k)L + \pi^2 a_h a_k\}}{(L+\pi a_h)(L+\pi a_k)}\left(1-\sum_{h<k}\frac{\{2\pi F + \pi(a_h+a_k)L + \pi^2 a_h a_k\}}{(L+\pi a_h)(L+\pi a_k)}\right),\end{aligned}$$

where $(I_2)_i$ denotes the integral $\int_{B_i\cap K\neq\emptyset} u_i^2 dB_i$.

Corollary 4 gives also the formula for the variance σ^2 of the number ν in the case $a_i = a$ for all i:

$$\begin{aligned}\sigma^2(\nu) = \;& 6\binom{n}{4}\frac{\pi^2\{2F + 2aL + \pi a^2\}^2}{(L+\pi a)^4} + 6\binom{n}{3}\frac{\pi^2 a^2(L+\pi a) + 2\pi a\{2\pi F + \pi a L\} + I_2}{(L+\pi a)^3}\\ &+\frac{n(n-1)\pi\{F + aL + \frac{\pi a^2}{2}\}}{(L+\pi a)^2}\left(1-\frac{n(n-1)\pi\{F + aL + \frac{\pi a^2}{2}\}}{(L+\pi a)^2}\right),\end{aligned}$$

where I_2 denotes the integral $\int_{B\cap K\neq\emptyset} u^2 dB$.

Let $K(t)$ be a class of plane convex sets of area $F(t)$ and perimeter $L(t)$ depending on the parameter t. Assume that for any point $p \in E^2$

there exists a value t_p of t such that $p \in K(t)$ for all $t > t_p$. This means that $K(t)$ expands over the whole plane E^2 as $t \to \infty$. Independently of the shape of $K(t)$ $\lim_{t\to\infty} \frac{L(t)}{F(t)} = 0$. Let B_0 be a strip of breadth a. Let $K_0 = B_0 \cap K(t)$ and denote by u the perimeter of K_0. If n random strips of breadth a intersect $K(t)$, then the probability that exactly m of them meet K_0 is

$$P_m = \binom{n}{m}\left(\frac{u+\pi a}{L+\pi a}\right)^m \left(1 - \frac{u+\pi a}{L+\pi a}\right)^{n-m}$$

Assume that $K(t)$ expands to E^2 and that $n = n(t) \to \infty$ is such a way that $\lim_{t\to\infty} \frac{n(t)}{L(t)} = \frac{u}{u+\pi a}\lambda$, where $\lambda > 0$. Then $\lim_{t\to\infty} P_m = \frac{(u\lambda)^m}{m!}e^{-u\lambda}$ and the mean value of m is

$$\begin{aligned} E(m) &= \sum_{m=0}^{\infty} m(\lim_{t\to\infty} P_m) = \sum_{m=0}^{\infty} m\frac{(u\lambda)^m}{m!}e^{-u\lambda} \\ &= e^{-u\lambda}\left(u\lambda + (u\lambda)^2 + \frac{(u\lambda)^3}{2!} + \frac{(u\lambda)^4}{3!} + \cdots\right) = u\lambda. \end{aligned}$$

Thus λ equals the mean value of the number of strips meeting any convex set of unit perimeter. And Theorem 3 says that $\lim_{t\to\infty} E(\frac{\nu}{F}) = \pi(\frac{u}{u+\pi a})^2\lambda^2$ and Theorem 4 says that $\lim_{t\to\infty} E(\frac{\nu^2}{F^2}) = \pi^2(\frac{u}{u+\pi a})^4\lambda^4$. Thus $\sigma^2(\frac{\nu}{F}) = E(\frac{\nu^2}{F^2}) - (E(\frac{\nu}{F}))^2 \to 0$. So we have $\lim_{t\to\infty} \frac{\nu}{F} = \lim_{t\to\infty} E(\frac{\nu}{F}) = \pi(\frac{u}{u+\pi a})^2\lambda^2$. This is the mean value of the number of intersections per unit area.

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