CONFORMALLY FLAT COSYMPLECTIC MANIFOLDS

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ABSTRACT. We proved that if a fibred Riemannian space $\tilde{M}$ with cosymplectic structure is conformally flat, then $\tilde{M}$ is the locally product manifold of locally Euclidean spaces, that is locally Euclidean. Moreover, we investigated the fibred Riemannian space with cosymplectic structure when the Riemannian metric $\tilde{g}$ on $\tilde{M}$ is Einstein.

1. Introduction

From the Theorem 3.1 of D. E. Blair [3], it is well known that the locally product of a Käehler manifold with a circle or line admits cosymplectic structure. Moreover, D. Chinea, M. De Leon and J. C. Marreor [4] constructed an example of compact cosymplectic manifolds which is not a global product of a compact Kaehler manifold with the circle. On the other hand, one of the present authors [6] studied various cosymplectic structures on the fibred Riemannian space with invariant fibre. In this paper, we study conformally flat cosymplectic manifold on the fibred Riemannian space. It is also proved that conformally flat cosymplectic manifold $\tilde{M}$ is the locally product of Euclidean spaces, eventually $\tilde{M}$ is locally Euclidean.

2. Cosymplectic manifold

Let $\tilde{M}$ be an m-dimensional C^∞ - manifold and $\tilde{\phi}$ a tensor field of type (1,1) on $\tilde{M}$ such that

$$(2.1) \qquad \tilde{\phi}^2 = -I + \tilde{\xi} \otimes \tilde{\eta},$$

where I is the identity transformation, $\tilde{\xi}$ a vector field, and $\tilde{\eta}$ a 1-form on $\tilde{M}$ satisfying

$$(2.2) \qquad \tilde{\phi}\tilde{\xi} = \tilde{\eta} \circ \tilde{\phi} = 0 \quad \text{and} \quad \tilde{\eta}(\tilde{\xi}) = 1.$$

Then $\tilde{M}$ is said to have an almost cntact structure $(\tilde{\phi},\ \tilde{\xi},\ \tilde{\eta})$. It is known that there is a positive definite Riemannian metric $\tilde{g}$ on $\tilde{M}$ such that

$$(2.3) \qquad \tilde{g}(\tilde{\phi}\tilde{X}\ ,\ \tilde{Y}\ ) \ = \ -\tilde{g}(\tilde{X}\ ,\ \tilde{\phi}\tilde{Y})$$

$$(2.4) \qquad \tilde{g}(\tilde{\phi}\tilde{X}\ ,\ \tilde{\phi}\tilde{Y}\ ) \ = \ \tilde{g}(\tilde{X},\ \tilde{Y}) \ - \ \tilde{\eta}(\tilde{X})\ \tilde{\eta}(\tilde{Y})$$

and

$$(2.5) \qquad \tilde{g}(\tilde{\xi},\ \tilde{\xi}) \ = \ 1\ ,$$

where $\tilde{X}$ and $\tilde{Y}$ are vector fields on $\tilde{M}$. $(\tilde{\phi},\ \tilde{\xi},\ \tilde{\eta})$ is said to be normal [2] if

$$(2.6) \qquad [\tilde{\phi},\ \tilde{\phi}] \ + \ \xi \otimes d\eta \ = \ 0\ ,$$

where $[\tilde{\phi},\ \tilde{\phi}]$ is the Nijenhuis torsion of $\tilde{\phi}$. The fundamental 2-form $\tilde{\Phi}$ is defined by $\tilde{\Phi}(\tilde{X},\tilde{Y}) = \tilde{g}\ (\tilde{\phi}\ \tilde{X},\ \tilde{Y}\ )$. An almost contact metric structure $(\tilde{\phi},\ \tilde{\xi},\ \tilde{\eta},\ \tilde{g})$ is said to be cosymplectic if it is normal and both $\tilde{\Phi}$ and $\tilde{\eta}$ are closed. It can be shown [2] that the cosymplectic structure is characterized by

$$(2.7) \qquad \tilde{\nabla}_{\tilde{X}}\tilde{\phi} \ = \ 0 \quad \text{and} \quad \tilde{\nabla}_{\tilde{X}}\tilde{\eta} \ = \ 0,$$

where $\tilde{\nabla}$ is the Riemannian connection of $\tilde{g}$.

3. Fibred Riemannian space

Let $(\tilde{M}, B, \tilde{g}, \pi)$ be a fibred Riemannian space, that is, $\tilde{M}$ an m-dimensional total space with projectable Riemannian metric $\tilde{g}$, B an n-dimensional base space, and $\pi : \tilde{M} \to$ B a projection with maximal rank n. The fibre passing through a point $\tilde{p} \in \tilde{M}$ is denoted by $F(\tilde{p})$ or generally F, which is a p-dimensional submanifold of $\tilde{M}$, where p = m - n . Throughout this paper, the range of indices are as follows ;

$$h, i, j, k, \dots : 1, 2, \dots, m$$
$$a, b, c, d, \dots : 1, 2, \dots, n$$
$$\alpha, \beta, \gamma, \delta, \dots : n+1, \dots, n+p = m.$$

Let $h = (h_{\gamma\beta}{}^a)$ and $L = (L_{cb}{}^\beta)$ be the components of the second fundamental tensor and normal connection of each fibre respectively. The geometric objects $\{\bar{R}, \bar{S}, \bar{K}\}$ are the Riemannian curvature, Ricci curvature and scalar curvature tensors of $\tilde{M}$ respectively. $\{R, S, K\}$ and $\{\bar{R}, \bar{S}, \bar{K}\}$ are the corresponding objects of B and F. Then the structure equations are given by [2, 5, 6, 8]

$$(3.1) \qquad \tilde{R}_{dcb}{}^a = R_{dcb}{}^a - L_d{}^a{}_\epsilon L_{cb}{}^\epsilon + L_c{}^a{}_\epsilon L_{db}{}^\epsilon + 2L_{dc}{}^\epsilon L_b{}^a{}_\epsilon,$$

$$(3.2) \qquad \begin{aligned} \tilde{R}_{dc\beta}{}^\alpha = & {}^*\nabla_c h_\beta{}^\alpha{}_d - {}^*\nabla_d h_\beta{}^\alpha{}_c + 2^{**}\nabla_\beta L_{dc}{}^\alpha \\ & + L_{de}{}^\alpha L_c{}^e{}_\beta - L_{ce}{}^\alpha L_d{}^e{}_\beta - h_\epsilon{}^\alpha{}_d h_\beta{}^\epsilon{}_c + h_\epsilon{}^\alpha{}_c h_\beta{}^\epsilon{}_d, \end{aligned}$$

$$(3.3) \qquad \tilde{R}_{d\gamma b}{}^\alpha = -{}^*\nabla_d h_\gamma{}^\alpha{}_b + {}^{**}\nabla_\gamma L_{db}{}^\alpha + L_d{}^e{}_\gamma L_{eb}{}^\alpha + h_\gamma{}^\epsilon{}_d h_\epsilon{}^\alpha{}_b,$$

$$(3.4) \qquad \tilde{R}_{\delta\gamma\beta}{}^\alpha = \bar{R}_{\delta\gamma\beta}{}^\alpha + h_{\delta\beta}{}^e h_\gamma{}^\alpha{}_e - h_{\gamma\beta}{}^e h_\gamma{}^\alpha{}_e,$$

where we have put

$$(3.5) \qquad {}^*\nabla_d h_{\gamma\beta}{}^a = \partial_d h_{\gamma\beta}{}^a + \Gamma^a_{de} h_{\gamma\beta}{}^a - Q_{d\gamma}{}^\epsilon h_{\epsilon\beta}{}^a - Q_{d\beta}{}^\epsilon h_{\gamma\epsilon}{}^a,$$

$$(3.6) \qquad {}^{**}\nabla_\delta L_{cb}{}^\alpha = \partial_d L_{cb}{}^\alpha + \bar\Gamma^\alpha_{\delta\epsilon} L_{cb}{}^\epsilon - L_c{}^e{}_\delta L_{eb}{}^\alpha - L_b{}^e{}_\delta L_{ce}{}^\alpha,$$

$$(3.7) \qquad Q_{c\beta}{}^\alpha = P_{c\beta}{}^\alpha - h_\beta{}^\alpha{}_c,$$

and $P_{c\beta}{}^\alpha$ are local functions related to

$$\mathcal{L}_{C_\beta} C^\alpha = P_{d\beta}{}^\alpha E^d,$$

where $\{E^a, C^\alpha\}$ are dual to the local frame $\{E_b, C_\beta\}$ and $\mathcal{L}_{C_\beta}$ is a Lie derivation with respect to C_β.

The Ricci curvature and scalar curvature are given by

$$(3.8) \qquad \tilde S_{cb} = S_c b - 2 L_{ce}{}^\epsilon L_b{}^e{}_\epsilon - h_{\beta ac} h^{\beta\alpha}{}_b$$

$$+ \frac{1}{2}({}^*\nabla_c h_\epsilon{}^\epsilon{}_b + {}^*\nabla_b h_\epsilon{}^\epsilon{}_c),$$

$$(3.9) \qquad \tilde S_{\gamma\beta} = \bar S_{\gamma\beta} - h_{\gamma\beta}{}^e h_\epsilon{}^\epsilon{}_e + {}^*\nabla_e h_{\gamma\beta}{}^e + L_{ae\gamma} L^{ae}{}_\beta,$$

$$(3.10) \quad \tilde K = K + \bar K - ||L_{cb}{}^\alpha||^2 - ||h_{\gamma\beta}{}^a||^2 - h_\gamma{}^\gamma{}_e h_\beta{}^{\beta e} + 2{}^*\nabla_e h_\epsilon{}^{\epsilon e}.$$

The following lemma is well known [5].

LEMMA 3.1. *If the components* $L = (L_{cb}{}^\alpha)$ *and* $h = (h_{\gamma\beta}{}^a)$ *vanish identically in a fibred Riemannian space, then the fibred space is loally the Riemannian product of the base space and a fibre.*

4. Fibred Riemannian space with cosymplectic structure

In [6], we proved that if $\tilde{M}$ is a cosymplectic manifold with $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$, then B is a Kähler manifold with complex structure (J, g), F is a cosymplectic manifold with $(\bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$, $L = 0$ and each fibre is minimal in $\tilde{M}$. Furthermore, the identity

$$(4.1) \qquad h_\gamma{}^\beta{}_b \bar{\phi}_\beta{}^\alpha = h_\gamma{}^\alpha{}_a J_b{}^a$$

holds. Suppose that a fibred Riemannian space is conformally flat. Then we get

$$(4.2) \qquad \tilde{R}_{kji}{}^h = \frac{1}{(m-2)}(\delta_k^h \tilde{S}_{ji} - \delta_j^h \tilde{S}_{ki} + \tilde{S}_k{}^h \tilde{g}_{ji} - \tilde{S}_j{}^h \tilde{g}_{ki})$$

$$- \frac{\tilde{K}}{(m-1)(m-2)}(\delta_k^h \tilde{g}_{ji} - \delta_j^h \tilde{g}_{ki}).$$

By using (3.2) and (4.2), we obtain

$$(4.3) \qquad {}^*\nabla_c h_\beta{}^\alpha{}_d - {}^*\nabla_d h_\beta{}^\alpha{}_c - h_\epsilon{}^\alpha{}_d h_\beta{}^{\epsilon c} + h_\epsilon{}^\alpha{}_c h_\beta{}^{\epsilon d} = 0.$$

Transvecting $\bar{\Phi}^{\beta\alpha}$ and J^{cd} to (4.3) successively, we obtain $||h_{\beta\alpha}{}^b||^2 = 0$, that is $h = 0$. From this fact and Lemma 3.1, we have

THEOREM 4.1. *If the fibred Riemannian space is a cosymplectic manifold with flat conformal curvature tensor, then $\tilde{M}$ is the locally product manifold of a Kähler manifold and cosymplectic manifold.*

From the equations (3.1), (3.2), (3.3) and (4.2), we obtain

$$(4.4) \qquad R_{dcb}{}^a = \frac{1}{(m-2)}(\delta_d^a S_{cb} - \delta_c^a S_{db} + S_d{}^a g_{cb} - S_c{}^a g_{db})$$

$$- \frac{K + \bar{K}}{(m-1)(m-2)}(\delta_d^a g_{cb} - \delta_c^a g_{db}),$$

(4.5) $$\frac{1}{(m-2)}(\delta_\gamma^\alpha S_{db} - \bar{S}_\gamma{}^\alpha g_{db}) + \frac{K + \bar{K}}{(m-1)(m-2)}\delta_\gamma^\alpha g_{db} = 0,$$

(4.6)

$$\bar{R}_{\delta\gamma\beta}{}^\alpha = \frac{1}{(m-2)}(\delta_\delta^\alpha \bar{S}_{\gamma\beta} - \delta_\gamma^\alpha \bar{S}_{\delta\beta} + \bar{g}_{\gamma\beta}\bar{S}_\delta{}^\alpha - \bar{g}_{\delta\beta}\bar{S}_\gamma{}^\alpha)$$

$$+ \frac{K + \bar{K}}{(m-1)(m-2)}(\delta_\delta^\alpha \bar{g}_{\gamma\beta} - \delta_\gamma^\alpha \bar{g}_{\delta\beta}).$$

Contracting γ and α, and transvecting g^{db} successively in (4.5), we get

(4.7) $$p(m + n - 1)K = n(n-1)\bar{K}.$$

The equation (4.4) and (4.7) induce

(4.8) $$S_{cb} = -(m-1)g_{cb}K,$$

and that

(4.9) $$\{1 + n(m-1)\}K = 0.$$

Since $1 + n(m-1) \neq 0$, we see that $K = 0$. So, the equations (4.7) and (4.8) give $\bar{K} = 0$ and $S_{cb} = 0$. Henceforth, the curvature tensor $R_{dcb}{}^\alpha$ of the base space is identically zero, that is, locally Euclidean. On the other hand, if we contract (4.6) with respect to δ and α, then we have $\bar{S}_{\gamma\beta} = 0$, so that the curvature tensor $\bar{R}_{\delta\gamma\beta}{}^\alpha$ of the fibre is identically zero. Hence by the theorem 4.1 and above results, we have

THEOREM 4.2. *If the fibred Riemannian space $\tilde{M}$ is a cosymplectic manifold with flat conformal curvature tensor, then $\tilde{M}$ is the locally product manifold of locally Euclidean spaces, that is locally Euclidean.*

5. Critical Riemannian metric

In this section, we investigate the fibred Riemannian space with cosymplectic structure admits a critical Riemannian metric. Let G be the set of all Riemannian metrics g on M which satisfy $\int_M dV_g = 1$, where dV_g is the volume element measured by g. It is well known that [1,7] $g \in G$ is the critical point of the Riemannian functional

$$(5.1) \qquad A(g) = \int_M K \, dV_g$$

if and only if g is an Einstein metric. For this reason, if we assume that the Riemannian metric $\tilde{g}$ is an Einstein metric, then the second equation of (2.7) and the second Bianchi's identity give $\tilde{K} = 0$ and that $\tilde{S}_{ji} = 0$. Hence the equation (3.8)-(3.10) induce

$$(5.2) \qquad S_{cb} = h_{\beta\alpha c} h^{\beta\alpha b},$$

$$(5.3) \qquad \bar{S}_{\gamma\beta} = {}^* \nabla_e h_{\gamma\beta}{}^e.$$

From these equations, we can easily see that the scalar curvature K and $\bar{K}$ are respectively given by $K = ||h_{\beta\alpha c}||^2$ and $\bar{K} = 0$. Thus we have

PROPOSITION 5.1. *Let $\tilde{M}$ be a fibred Riemannian space with cosymplectic structure. If the Riemannian metric $\tilde{g}$ on $\tilde{M}$ is a critical Riemannian metric of the function A, then the Ricci tensor on $\tilde{M}$ vanish and the Ricci tensors of the base space and each fibre are given by the equations (5.2) and (5.3) respectively.*

COROLLARY 5.2. *Under the same assumptions of proposition 5.1, we have*

(1) g on B is a critical Riemannian metric of A if and only if

$$nh_{\beta\alpha c} h^{\beta\alpha}{}_b = ||h_{\beta\alpha e}||^2 g_{cb},$$

(2) $\bar{g}$ on F is a critical Riemannian metric of A if and only if $\bar{S} = 0$.

References

[1] M. Berger, *Quelques formules de variation pour une structure Riemanniane*, Ani. Sci. Ecole Norm. Sup. 4^e **serie 3** (1970), 285-294.

[2] A. Besse, *Einstein manifolds*, Springer-Verlag, Berlin, 1987.

[3] D. E. Blair, *The theory of quasi Sasakian structure*, J. Diff. Geom. **1** (1967), 331-345.

[4] D. Chinea, M. De Leon and J. C. Marrero, *Topology of cosymplectic manifolds*, J. Math. Pures. Appl. **72** (1993), 567-591.

[5] S. Ishihara and M. Konishi, *Differential geometry of fibred spaces*, Study Group of Differential Geometry vol. **7**, Tokyo, 1973.

[6] B. H. Kim, *Fibred Riemannian space with quasi Sasakian structure*, Hiroshima Math. J. **20** (1990), 477-513.

[7] Y. Muto, *Riemannian submersion and critical Riemannian metrics*, J. Math. Soc. Japan **29** (1977), 493-511.

[8] B. O'Neill, *The fundamental equations of submersion*, Michigan Math. J. **13** (1966), 459-469.

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