ON IRREDUCIBLE 3-MANIFOLDS

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ABSTRACT. This paper deals with certain conditions under which irreducibility of a 3-manifold is preserved under attaching a 2-handle along a simple closed curve (and then, if necessary, capping off a 2-sphere boundary component by a 3-ball).

1. Introduction

Let $\mathbb{M}$ be a 3-manifold and $\gamma$ a simple closed curve in $\partial \mathbb{M}$. Let $N(\gamma)$ be a regular neighborhood of $\gamma$ in $\partial \mathbb{M}$ and endow the 3-ball $\mathbb{B}^3$ with the product structure $D^2 \times I$. If $\phi : \partial D^2 \times I \to N(\gamma)$ is a homeomorphism, we define a new manifold $\mathbb{M}_\gamma$ to be $\mathbb{M}$ with a 2-handle attached along $\gamma$: that is,

$$\mathbb{M}_\gamma = \mathbb{M} \cup_\phi \mathbb{B}^3$$

If $\{\gamma_i\}_{i=1}^n$ is a finite collection of pairwise disjoint simple closed curves on $\partial \mathbb{M}$, then $\mathbb{M}_{\gamma_1, \ldots, \gamma_n}$ is defined to be $((\mathbb{M}_{\gamma_1})_{\gamma_2}) \cdots)_{\gamma_n}$. The homeomorphism type does not depend on the ordering of the $\gamma_i$. Hence, we often denote $\mathbb{M}_{\gamma_1, \ldots, \gamma_n}$ by $\mathbb{M}_{(\gamma_i)}_{i=1}^n$. $\mathbb{M}_\gamma^+$ denotes the manifold obtained from $\mathbb{M}_\gamma$ by capping off each 2-sphere boundary component of $\mathbb{M}_\gamma$ with a 3-ball. Kneser [5] proved that every closed orientable 3-manifold, different from $S^3$, can be built up as a finite connected sum of prime 3-manifolds, and according to Milnor [6] this decomposition is unique up to order and homeomorphism. With the exception of a 3-manifold homeomorphic to $S^3, S^1 \times S^2$, a 3-manifold is prime if and only if it is irreducible. Thus, the classification problem for compact 3-manifold is reduced to that for irreducible 3-manifolds. Assuming a 3-manifold to be irreducible avoids certain technical difficulties as well as the Poincaré Conjecture. In this
paper, it will be proved that under certain conditions, irreducibility is preserved under attaching a 2-handle along a simple closed curve (and then, if necessary, capping off a 2-sphere boundary component by a 3-ball).

2. Definitions and Remarks

This chapter deals with some definitions and remarks related to our main topic. Some geometric concepts are also introduced.

**Definition 2.1** [1]. Let \( M \) be a 3-manifold and \( F \) a connected surface which is either properly embedded in \( M \) or contained in \( \partial M \). We say that \( F \) is compressible if one of the followings is satisfied:

1. \( F \) is a 2-sphere which bounds a 3-ball in \( M \),
2. \( F \) is a 2-disk and either \( F \subset \partial M \) or there is a 3-ball \( B \subset M \) with \( \partial B \subset F \cup \partial M \),
3. there is a 2-disks \( D \subset M \) such that \( \partial F \cap F = \partial D \) and \( \partial D \) is not contractible in \( F \).

The 2-disk \( D \) in (3) is called a compressing disk for \( F \) in \( M \). We say that \( F \) is incompressible if it is not compressible. If \( F \) is not connected, we say that \( F \) is incompressible in the case that all components of \( F \) are incompressible.

**Definition 2.2** [7]. Let \( \gamma_1, \ldots, \gamma_n \) be pairwise disjoint 2-sided simple closed curves in the boundary of a 3-manifold \( M \). \( \gamma \) in \( \partial M \) is said to be coplanar with \( \{ \gamma_i \}_{i=1}^n \) if \( \gamma \) bounds a disk in \( \partial M_{\gamma_1,\ldots,\gamma_n} \).

**Remark** [7]. Let \( \gamma \) and \( \gamma' \) be disjoint simple closed curves in the boundary of a 3-manifold \( M \). If \( M \) is compact and \( \gamma \) is coplanar with \( \gamma' \), then precisely one of the following possibilities occurs:

1. \( \gamma \) bounds a disk in \( \partial M - \gamma' \),
2. \( \gamma \) is parallel in \( \partial M \) to \( \gamma' \),
3. \( \gamma \) separates \( \partial M \) into two components, one of which is a punctured torus \( T \) and \( \gamma' \subset T \) does not separate \( T \).
DEFINITION 2.3 [7]. Let $\mathcal{M}$ be a 3-manifold and $P$ a disk with holes properly embedded in $\mathcal{M}$, and $\gamma \subset \partial\mathcal{M}$ a 2-sided simple closed curve. $P$ is said to be a pre-sphere with respect to $\gamma$ in $\mathcal{M}$ if

1. $\partial P \subset \partial\mathcal{M} \setminus \gamma$,
2. each component of $\partial P$ is coplanar with $\gamma$ in $\partial\mathcal{M}$, and
3. $P^+$ does not bound a 3-ball in $M_{\gamma}^+$, where $P^+$ is a natural extension of $P$ to a 2-sphere in $M_{\gamma}^+$.

REMARK: In condition (3) in Definition 2.3, let $D_0, \ldots, D_n$ be the disks in $M_{\gamma}^+$ attached to $P$ along respective boundary components of $P$ to produce $P^+$. If it were the case that some of the $D_i$ are nested in a disk, say $D_0$, then we choose an innermost disk, $D_1$ say, among all of the disks $D_i$ which are nested in $D_0$ and introduce a collar on $D_0, c_1 : D_0 \times I \rightarrow M_{\gamma}^+$, with $c_1(x,0) = x$ such that $c_1(D_0 \times I) \subset \text{Int} M_{\gamma}$ and $c_1(D_0 \times 1)$ cuts $P^+$ into two disks, one of which, denoted by $D_1'$ contains $D_1$, and the other one contains all $\partial D_i$ except $\partial D_1$. Then, push $D_1'$ onto the disk $D_1^*$ bounded by $\partial D_1'$ in $c_1(D_0 \times 1)$ by an isotopy across the 3-ball $c_1(D_0 \times 1)$. And then choose the innermost disk, $D_2$ say, among all $D_i \subset D_0$ except $D_1$ and introduce a collar on $D_0$ in $M_{\gamma}^+, c_2 : D_0 \times I \rightarrow M_{\gamma}^+$, with $c_2(x,0) = x$ such that $c_2(D_0 \times I) \subset \text{Int} M_{\gamma}, c_2(D_0 \times I)$ does not contain $D_1^*$, and $c_2(D_0 \times 1)$ cuts $P^+$ into two disks, one of which, denoted by $D_2'$ contains $D_2$ and the other one contains all $\partial D_i$ except $\partial D_2$. Then push $D_2'$ onto the disk $D_2^*$ bounded by $\partial D_2'$ in $c_2(D_0 \times 1)$ by an isotopy across the 3-ball $c_2(D_0 \times 1)$. In the same way, we push the other disks $D_i$ into $\text{Int} M_{\gamma}$ sequentially so that the more inner in $D_0$ a disk, the deeper into $\text{Int} M_{\gamma}$ it is pushed. Keeping this process until all the $D_i$ are embedded in $\text{Int} M_{\gamma}$, we can assume that $P^+$ is embedded. (2) $P^+$ is unique up to isotopy in $M_{\gamma}^+$.

3. Irreducibility

Even though the following observation was made by Jaco [3], we prove it here with hopes of enhancing the understanding of the geometry appearing in our main topic.

Lemma 3.1. Let $M$ be a 3-manifold with $\partial M$ compressible and $\gamma$ a simple closed curve in $\partial M \setminus \gamma$ incompressible. Let $\delta$ be a simple closed
curve in $\partial M - \gamma$ which is not contractible in $M$. If $\delta$ is coplanar with $\gamma$, then $\partial M - \delta$ is incompressible in $M$.

**Proof.** If $\delta$ is parallel in $\partial M$ to $\gamma$, then the result is trivial. Hence, we assume $\delta$ is not parallel in $\partial M$ to $\gamma$. Since $\delta$ is not contractible in $\partial M - \gamma$, $\delta$ must separate $\partial M$ into two components, one of which is a once-punctured torus $T$ with $\gamma$ a non-separating curve in $T$. Let $\partial M = S \cup_\delta T$ for a surface $S$. Suppose $\partial M - \delta$ is compressible in $M$. If $S$ is compressible in $M$, there is a properly embedded disk $D_1$ in $M$ such that $\partial D_1 \subset S$ and $\partial D_1$ does not bound a disk in $S$. Since $\partial M - \gamma = S \cup_\delta (T - \gamma)$, $\partial D_1$ does not bound a disk in $\partial M - \gamma$. Hence, $D_1$ is a compressing disk for $\partial M - \gamma$ in $M$, which leads to a contradiction. Thus, $S$ is incompressible in $M$, and so $T$ must be compressible in $M$. Let $D$ be a compressing disk for $T$ in $M$. If $\partial D$ is parallel to $\delta$ in $T$, there is an annulus $A$ in $T$ with $\partial A = \partial D \cup \delta$, and so $D' = A \cup_{\partial D} D$ is a 2-disk with $\partial D' = \delta$. If $\partial D$ is not freely homotopic to $\delta$ in $T$, $\partial D$ is non-separating in $T$. Hence, $D$ compresses $T$ to a disk. Now, let $N(D)$ denote a regular neighborhood of $D$ with a product structure $D \times [-1, 1]$ in $M$. Then, $D' = (T - \partial D \times (-1, 1)) \cup (D \times \{-1, 1\})$ is a 2-disk and $\partial D' = \delta$. However, this is absurd because $\delta$ is not contractible in $M$. □

The following fact was asserted, without proof, by Przytycki [7]. Our proof is a modification of Jaco's argument used in proving his Handle Attaching Theorem [3]. However, it will turn out that the translation of Jaco's argument into this context requires considerable effort. This is, to the best of our knowledge, the first time the proof of this potentially valuable result has appeared in detail.

**Theorem 3.1.** Let $M$ be a 3-manifold with $\partial M$ compressible and $\gamma$ a 2-sided simple closed curve with $\partial M - \gamma$ incompressible in $M$. If $M$ is irreducible, then so is $M_\gamma^+$. 

**Proof.** Suppose that $M_\gamma^+$ is not irreducible. Then, there exists an embedded 2-sphere $S$ in $M_\gamma^+$ such that $S$ does not bound a 3-ball in $M_\gamma^+$. Since $S$ can be made disjoint from 3-handles by general position, we assume $S \subset M_\gamma$. Moreover, by general position, $S$ meets the 2-handle attached along $\gamma$ in parallel disks. Since $M$ is irreducible, $S$ cannot be embedded in $M$ and so $S \cap \partial M \neq \emptyset$. Since $S$ is compact and $S$ meets
the 2-handle in disks, $S \cap \partial M$ is a finite collection of simple closed curves parallel to $\gamma$ in $\partial M$. Let $P = S \cap M$. Then, $P$ is a properly embedded disk with holes in $M$ such that

1. $\partial P \subset \partial M - \gamma$,
2. each component of $\partial P$ is coplanar with $\gamma$ in $\partial M$, and
3. $P^+ = S$ in $M^+$. 

That is, there is a pre-sphere with respect to $\gamma$ in $M$. Hence, to prove the theorem, it suffices to show that there is no pre-sphere in $M$ with respect to $\gamma$. Let $\mathcal{P}$ be the collection of properly embedded disks with holes in $M$ so that $P \in \mathcal{P}$ if and only if there exists a simple closed curve $\gamma \subset \partial M$ such that $\partial M - \gamma$ is incompressible in $M$ and $P$ is a pre-sphere with respect to $\gamma$ in $M$. We will show that $\mathcal{P}$ is empty, thus establishing the theorem. Suppose that $\mathcal{P} \neq \emptyset$. Choose an element $P \in \mathcal{P}$ so that the Euler characteristic $\chi(P)$ is maximal. Then, there exists a simple closed curve $\gamma \subset \partial M$ such that $\partial M - \gamma$ is incompressible in $M$ and $P$ is a pre-sphere with respect to $\gamma$. Let $\mathcal{D}$ be the collection of compressing disks for $\partial M$ in $M$. Since $\partial M$ is compressible in $M$, the collection $\mathcal{D}$ is not empty. Each disk in $\mathcal{D}$ meets $\gamma$ because $\partial M - \gamma$ is incompressible in $M$. If a simple closed curve $\delta \subset \partial M$ is coplanar with $\gamma$ and is not contractible in $M$, then by lemma 3.1, $\partial M - \delta$ is incompressible in $M$. Since maximality of $\chi(P)$ implies that no component of $\partial P$ is contractible in $M$, for any component $\delta$ of $\partial P$, $\partial M - \delta$ is incompressible in $M$. Hence, it follows that any disk in $\mathcal{D}$ must meet all components of $\partial P$ and, moreover, $P$. Without loss of generality, we can assume that each disk is transverse to $P$. Choose a disk $D \in \mathcal{D}$ so that the number of components of intersection $D \cap P$ is minimal. We prove, by analyzing $D \cap P$, that this leads to the desired contradiction.

1. Suppose that $D \cap P$ contains a simple closed curve. Then, choose a simple closed curve $\alpha$ from the components of $D \cap P$ which is innermost in $D$: that is, $\alpha$ bounds a disk $\Delta$ in $D$ and $\text{Int}\Delta \cap P = \emptyset$. Furthermore, since $P$ is planar, $\alpha$ separates $P$ into two surfaces $F_1$ and $F_2$. If one, $F_1$ say, of the $F_i$ is a disk, then there is a simple closed curve $\beta$ which is a component of $D \cap P$ and innermost in $F_1$. That is, $\beta$ bounds a disk $\Delta'$ in $F_1$ and $\text{Int}\Delta' \cap D = \emptyset$. Moreover, the curve $\beta$ separates $D$ into an annulus $D_1$ and a disk $D_2$ with $\partial D \subset D_1$. The disk $D_1 \cup \Delta'$ is a compressing disk for $\partial M$ in $M$ and has the property that after a small
isotopy, the number of components of the intersection \((D_1 \cup \Delta') \cap P\) is less than that of \(D \cap P\). This contradicts the choice of \(D\). Hence, we can assume neither \(F_1\) nor \(F_2\) is a disk. Let \(P_i = F_i \cup \Delta\), and let \(F_i^+\) denote the 2-disk obtained from \(F_i\) by attaching 2-disks along boundary components except \(\alpha \quad i = 1, 2\) and desingularizing them as mentioned in the remark following definition 2.3. Then each \(P_i\) satisfies conditions (1) and (2) of definition 2.3. Observe that

1. \(F_i^+\) is a disk with \(\partial F_i^+ = \alpha\), \(i = 1, 2\),
2. \(P_i^+ = F_i^+ \cup_{\alpha} \Delta\), \(i = 1, 2\) and
3. \(P^+ = F_1^+ \cup_{\alpha} F_2^+\).

Since \(P^+\) does not bound a 3-ball in \(M^+_\gamma\), from 2) and 3) in the above observation it follows that one of the 2-spheres \(P_i^+\) does not bound a 3-ball in \(M^+_\gamma\), and so \(P_i\) is a pre-sphere with respect to \(\gamma\) in \(M\) and is of Euler characteristic greater than \(\chi(P)\). this contradicts the choice of \(P\). Hence, we can conclude that no component of \(D \cap P\) is a simple closed curve.

(2) Suppose that \(D \cap P\) contains an inessential spanning arc in \(P\). In other words, ther exists a component \(\alpha\) of \(D \cap P\) and an arc \(\beta\) in \(\partial P\) such that \(\partial \alpha = \partial \beta\) and the curve \(\alpha \cup \beta\) bounds disk \(\Delta\) in \(P\) with \(Int\Delta \cap D = \phi\). Now, we boundary-compress \(D\) along the disk \(\Delta\) to get two properly embedded disks \(D_1\) and \(D_2\) with the property that the number of components of \(D_i \cap P\) is less than that of \(D \cap P\). Let \(\delta_1\) and \(\delta_2\) be arcs in \(\partial D\) such that \(\partial D = \delta_1 \cup \delta_2, \partial \delta_1 = \partial \delta_2 = \partial \beta\), and \(\partial D_i = \delta_i \cup \beta, i = 1, 2\). With appropriate orientations of \(\delta_1, \delta_2\) and \(\beta\), we have \(\partial D_1 = \beta \delta_1\) and \(\partial D_2 = \delta_2 \beta^{-1}\) in \(\pi_1(\partial M)\). If each \(\partial D_i\) is contractible in \(\partial M\), then \(\partial D = \delta_1 \delta_2 = (\beta \delta_1)(\delta_2 \beta^{-1}) = 1\) in \(\pi_1(\partial M)\) and so \(\partial D\) is also contractible in \(\partial M\). This contradicts the fact that \(D\) is a compressing disk for \(\partial M\) in \(M\). Now, one of the \(D_i\) has non-contractible boundary of \(D\). Hence, we can conclude that no component of \(D \cap P\) is an inessential spanning arc in \(P\).

(3) suppose \(D \cap P\) contains an essential spanning arc in \(P\). Let \(\alpha\) be an essential arc which is a component of \(D \cap P\) and outmost on \(D\); that is, there is an arc \(\beta \subset \partial D\) such that \(\partial \alpha = \partial \beta\) and the imple closed curve \(\alpha \cup \beta\) bounds a disk \(\Delta\) in \(D\) with \(Int\Delta \cap P = \phi\). At this point of time, we require some preparations to analyze case (3): To perform a boundary compression of \(P\) along \(\Delta\), it is necessary to prove the following lemma.
LEMMA 3.2. $\beta$ may be assumed disjoint from $\gamma$ (by redefining $\gamma$ to be parallel in $\partial M$ to some component of $\partial P$, if necessary.)

PROOF. Recall That $M$ is irreducible, $\partial M$ is compressible and $\partial M - \gamma$ is incompressible in $M$. Let $S$ be the boundary component of $\partial M$ containing $\gamma$. If $\gamma$ is parallel in $\partial M$ to a component, $\mu$ say, of $\partial P$, then there is an annulus $A$ in $\partial M$ with $\gamma, \mu \subset A$, along which $\gamma$ can be isotoped to an appropriate side of $\mu$, off $\beta$. Suppose no component of $\partial P$ is parallel in $\partial M$ to $\gamma$. Since no component of $\partial P$ is contractible in $M$, every component of $\partial P$ separates $S$ into two components one of which is a once-punctured torus which contains $\gamma$, but is not separated by $\gamma$. Hence, there is a simple closed curve $\gamma'$ in $\partial M$ such that $\gamma'$ separates $S$ into two components $W$ and a once-punctured torus $T$ with $\gamma \subset T$ not separating, and all components of $\partial P$ are parallel to $\gamma'$ in $\partial M$. By lemma 3.1, $\partial M - \gamma'$ is incompressible in $M$ and so $W$ and $T$ are. Note that $W$ is not a disk. Hence, neither $M_\gamma$ nor $M_{\gamma'}$ has 2-sphere boundary components; that is, $M_\gamma = M_{\gamma}^{+}$ and $M_{\gamma'} = M_{\gamma'}^{+}$. Since $\gamma'$ bounds a disk in $\partial M_{\gamma}$, $(M_{\gamma})_{\gamma'}^{+}$ is homeomorphic to $M_{\gamma}$. Thus, we get

\[ M_{\gamma} = M_{\gamma'}^{+} \simeq M_{\gamma\gamma'}^{+} = M_{\gamma'}^{+}. \]

Now, let $P_{\gamma'}^{+}$ be a natural extension of $P$ in $M_{\gamma}^{+} = M_{\gamma'}$. Let $\xi$ be an arbitrary component of $\partial P$. Let $D_{\gamma}$ (resp. $D_{\gamma'}$) denote a 2-disk bounded by $\xi$ in

\[ P_{\gamma}^{+} \subset M_{\gamma} = M_{\gamma}^{+} (\text{resp. } P_{\gamma'}^{+} \subset M_{\gamma'} = M_{\gamma'}^{+}). \]

Then, there is an embedded 3-ball $B^{3} \subset \text{Int} M_{\gamma\gamma'}^{+}$ which meets $D_{\gamma}$ and $D_{\gamma'}$ in complementary faces, and $D_{\gamma}$ agrees with $D_{\gamma'}$ away from $B^{3}$. This implies that $P_{\gamma'}^{+}$ is (ambient) isotopic to $P^{+}$ in $M_{\gamma\gamma'}^{+}$ and so there is a homeomorphism of $M_{\gamma\gamma'}^{+}$ onto $M_{\gamma}$ which carries $P_{\gamma'}^{+}$ to $P^{+}$. Since $P^{+}$ does not bound a 3-ball in $M_{\gamma}^{+}$, $P_{\gamma'}^{+}$ does not bound a 3-ball in $M_{\gamma'}^{+}$. Therefore, $P$ is a pre-sphere in $M$ with respect to $\gamma'$. Now, we see that $\partial M - \gamma'$ is incompressible, $P$ is a pre-sphere in $M$ with respect to $\gamma'$, and every component of $\partial P$ is parallel to $\gamma'$ in $\partial M$. Therefore, by redefining $\gamma$ to be $\gamma'$, we can assume that some component $\mu$ of $\partial P$ is parallel in
\( \partial M \) to \( \gamma \). Now, we can assume that \( \gamma \) is disjoint from \( \beta \) because \( \gamma \) can be isotoped off \( \beta \) along the appropriate annulus in \( \partial M \) containing \( \gamma \) and \( \beta \). This establishes the lemma.

Now, we return to our proof.

Claim (1). It is not the case that both end points of \( \alpha \) are contained in one component of \( \partial P \).

Suppose that the component \( \alpha \) of \( D \cap P \) has both end points in a component \( \delta \) of \( \partial P \). Then a boundary compression of \( P \) at \( \alpha \) along \( \Delta \) results in two new disks with holes, \( P_1 \) and \( P_2 \). Since \( \alpha \) is essential, \( \chi(P) < \chi(P_i), i = 1, 2 \). Let the end points of \( \alpha \) separate \( \delta \) into two arcs \( \delta_1 \) and \( \delta_2 \). Each simple closed curve \( \delta_i \cup \beta \) \( (i = 1, 2) \) is a new boundary component of \( P_i \). Since \( \beta \) and \( \gamma \) are disjoint by lemma 3.2, we have \( \partial P_1 \subset \partial M - \gamma \). If one, \( \delta_1 \cup \beta \) say, of the \( \delta_i \cup \beta \) is not coplanar with \( \gamma \), then \( P_1^+ \) is compressing disk for \( \partial M^+_\gamma \) in \( M^+_\gamma \). This contradicts Jaco's Handle Attaching Theorem. Hence, each \( P_i \) has a boundary whose components are all coplanar with \( \gamma \). We want to show that at least one of the 2-sphere \( P_i^+ \) does not bound a 3-ball. Assume both \( P_i^+ \) bound 3-balls \( B^3_i \), respectively. Let \( D_i \) be a disk bounded by \( \delta_i \cup \beta \) in \( M^+_\gamma \) to produce \( P_i^+ \).

Let \( \alpha \) separate the planar surface \( P \) into two surfaces \( F_1 \) and \( F_2 \) and \( F_i^+ \) denote the 2-disk obtained from \( F_i \) by attaching 2-disks in \( M^+_\gamma \) along corresponding boundary components except \( \alpha \cup \delta_i \) and desingularizing them. Then, we have \( P_i = F_i \cup_{\alpha} \Delta \) and \( P_i^+ = (F_i^+ \cup_{\alpha} \Delta) \cup_{\delta_i \cup \beta} D_i \).

If one of the \( D_i \) is contained in the other one, \( D_1 \subset D_2 \) say, then let \( D = cl(D_2 - D_1) \). We can assume that \( \delta \) bounds the disk \( D \) in \( P^+ \).

Now, we observe that \( D \cup \Delta \) isotopes onto \( F_i^+ \) across the 3-ball \( B^3_i \) and so \( P_2^+ \) isotopes onto \( P^+ \). This is, however, impossible because \( P^+ \) does not bound a 3-ball in \( M^+_\gamma \). If it is not the case that one of the \( D_i \) is contained in the other one, then \( \delta \) bounds the disk \( D_1 \cup_{\beta} D_2 \) in \( M^+_\gamma \).

Since \( P_1 \cap P_2 = \Delta \) and \( F_i^+ \cup D_1 \) isotopes to \( \Delta \) across \( B^3_i \), \( P^+ \) isotopes to \( P_2^+ \). However, this is impossible because \( P^+ \) does not bound a 3-ball. Now, we see that one of the \( P_i \) is a pre-sphere in \( M \) with respect to \( \gamma \), of Euler characteristic greater than \( \chi(P) \). This contradicts the choice of \( P \). So, we conclude that no component is a spanning arc with end points in one component of \( \partial P \).
CLAIM (2). It is not the case that each end point of $\alpha$ is contained in a different component of $\partial P$.

Suppose that the component of $D \cap P$ is such an arc. This is, the end points of $\alpha$ lie in distinct components of $\partial P$, $\xi_1$ and $\xi_2$ say. Then a boundary compression of $P$ along $\Delta$ results in a disk with holes, $P'$, such that $\chi(P') > \chi(P)$ and $\partial P' = (\partial P - (\xi_1 \cup \xi_2)) \cup \xi$ where $\xi = \xi_1 \beta \xi_2 \beta^{-1}$, assuming $\xi_1$ and $\xi_2$ are appropriately oriented. We want to show that $P'$ is a pre-sphere in $M$ with respect to $\gamma$. Since $\beta$ does not meet $\gamma$, $\xi \subset \partial M - \gamma$. Since $\xi_1$ and $\xi_2$ are contractible in $\partial M_\gamma$, $\xi$ is contractible in $\partial M_\gamma$; that is, $\xi$ is coplanar with $\gamma$ in $M$. Let $D_i$ be disks bounded by $\xi_i$ in $M_\gamma^+$ with $D_i \subset P^+$. If each $D_i$ is disjoint from $Int \beta$ (after a small isotopy if necessary), then $\xi$ bounds the 2-disk $D_1 \cup_{l_1} (\beta \times [-1, 1] \cup_{l_2} D_2)$ (where $l_1 \cup l_2 = \partial \beta \times [-1, 1]$), and $P'^+$ isotopes onto $P^+$ across $\Delta \times [-1, 1]$. Now, we assume $\beta \subset D_2$ without loss of generality. Since $P$ is planar, $\xi_1 \subset D_2$. In this case, we push $D_1$ into $c(D_2 \times I) \subset Int M_\gamma^+$, where $c$ is collar on $D_0$ in $M_\gamma^+$ as described in the remark following definition 2.3. Performing bouddary-compressing along $\Delta$, we observe that $\xi$ bounds a disk, and $P^+$ isotopes onto $P'^+$ across the 3-ball, a boundary connected sum of the products $D_1 \times [0, 1]$ and $\Delta \times [-1, 1]$. Thus, $P'^+$ does not bound a 3-ball in $M_\gamma^+$, and so $P'$ is a pre-sphere in $M$ with respect to $\gamma$, with $\chi(P) < \chi(P')$. This contradicts the choice of $P$. Therefore, no component of $D \cap P$ can be a spanning arc which does not have both ends points in one component of $\partial P$.

By claim(1) and claim(2), no component of $D \cap P$ can be an essential spanning arc in $P$. Finally, it follows that $P$ is empty. 

\begin{corollary}
If $\gamma$ is a simple closed curve in $\partial H_k$ with $\partial H_k - \gamma$ incompressible, then $(H_k)_\gamma^+$ is irreducible. (Here, $H_k$ denotes a handlebody of genus $k$.)
\end{corollary}

References


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