ON THE FUNCTIONAL CENTRAL LIMIT THEOREM FOR A CLASS OF NONLINEAR AUTOREGRESSIVE PROCESSES

CHANHO LEE

ABSTRACT. A functional central limit theorem for a class of nonlinear stationary Markov processes which are geometrically Harris ergodic is derived.

1. Introduction

Consider a k-dimensional *Markov* process $\{Y_n : n \geq 0\}$ on $\mathbb{R}^k$ defined by

$$(1.1) \qquad Y_{n+1} := f(Y_n) + \varepsilon_{n+1} \ (n \geq 0),$$

where f is $\mathbb{R}^k$-valued Borel measurable function on $\mathbb{R}^k$, and Y_0 is an arbitrarily specified random vector with values in $\mathbb{R}^k$, independent of the random forcing terms, $\{\varepsilon_n : n \geq 1\}$.

We are mainly interested in the case $k > 1$. $\{\varepsilon_n : n \geq 1\}$ are assumed to be of the following form:

$$(1.2) \qquad \varepsilon_n := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \eta_n \end{pmatrix}$$

where $\{\eta_n : n \geq 1\}$ is a sequence of i.i.d. real-valued random variables.

In this article, we are interested in a nonlinear kth-order autoregressive process $\{X_n : n \geq 0\}$ defined by

$$(1.3) \qquad X_{n+1} = h(X_{n+1-k}, \cdots, X_n) + \eta_{n+1} (n \geq k-1),$$

where h is a real-valued Borel measurable function on $\mathbb{R}^k$, $\{\eta_n : n \geq 1\}$ is a sequence of i.i.d. real-valued random variables, and $\{X_0, X_1, \cdots, X_{k-1}\}$ are arbitrarily prescribable real-valued random variables independent of $\{\eta_n : n \geq 1\}$.

The equation (1.1) can be thought of as obtained by vectorization of the process (1.3) with $\{\varepsilon_n : n \geq 1\}$ of the form (1.2) and $f(y)$ defined by

$$(1.4) \qquad f(y) := \begin{pmatrix} y_2 \\ \vdots \\ y_k \\ h(y) \end{pmatrix}$$

Here $y = (y_1, y_2, \cdots, y_k)' \in \mathbb{R}^k$. So we limit ourselves to the k-dimensional Markov process (1.1) throughout this paper.

Let $p^{(n)}(x, dy)$ denote the n-step transition probability of $Y_n (n \geq 1)$ and $p(x, dy) \equiv p^{(1)}(x, dy)$.

A probability measure π on $(\mathbb{R}^k, \mathcal{B}^k)$ is said to be *invariant* for $\{Y_n : n \geq 0\}$, or for $p(x, dy)$, if

$$(1.5) \qquad \int_{\mathbb{R}^k} p(x, A)\pi(dx) = \pi(A), \qquad \forall A \in \mathcal{B}^k.$$

A ϕ-irreducible aperiodic Markov process with transition probability $p(x, dy)$ is said to be (*Harris*) *ergodic* if there exists a probability measure π such that

$$(1.6) \qquad ||p^{(n)}(x, dy) - \pi(dy)|| \longrightarrow 0 \quad \text{as} \quad n \to \infty, \forall x \in \mathbb{R}^k.$$

Here $|| \cdot ||$ denotes the variation norm on the Banach space of finite signed measure on $(\mathbb{R}^k, \mathcal{B}^k)$.

If the convergence in (1.6) is exponentially fast then the process is said to be *geometrically (Harris) ergodic.*

Recently there have been considerable works on kth-order nonlinear autoregressive models, most of which provide some verifiable criteria for geometric ergodicity (see, e.g., Chan and Tong (1985), $Tj\phi$stheim (1990), Bhattacharya and Lee (1995), (1995), Lee (1996)).

If (1.6) holds, then π is necessarily the unique invariant probability for $p(x, dy)$, and the process having π as the initial distribution is stationary.

We assume that the process $\{Y_n : n \geq 0\}$ is geometrically (Harris) ergodic and that Y_0 has the unique invariant π as its distribution. In this article, by identifying a subset of the range of the infinitesimal generator on $L^2(\mathbb{R}^k, \pi)$, we have derived a functional central limit theorem for the process $\{Y_n\}$.

3. Main result

Consider a real-valued function ψ on $\mathbb{R}^k$ such that $E\psi^2(Y_0) < \infty$. Write $\tilde{\psi} = \psi - \bar{\psi}$, where $\bar{\psi} = \int \psi d\pi$.

In $L^2(\mathbb{R}^k, \pi)$, consider the identity operator I and the transition operator T,

$$(Tg)(x) := \int g(y)p(x, dy).$$

Then $(T^m\tilde{\psi})(x) = (T^m\psi)(x) - \bar{\psi}$ for all $m \geq 0$. Also, $E\tilde{\psi}(X_0) = 0$.

The ergodicity of the process $\{Y_n : n \geq 0\}$ implies that the kernel of the operator $I - T$ is one dimensional.

We shall need the following results of Gordin and Lifšic (1978).

PROPOSITION 1. *Assume $p(x, dy)$ admits an invariant probability π and, under the initial distribution π, $\{Y_n\}$ is ergodic. Assume also that $\tilde{\psi} = \psi - \bar{\psi}$ is in the range of $I - T$. Then*

$$(2.1) \quad n^{-1/2} \left[\sum_{j=0}^{[nt]} (\psi(Y_j) - \bar{\psi}) + (nt - [nt])(\psi(Y_{[nt]+1}) - \bar{\psi}) \right] (t \geq 0)$$

converges weakly to a Brownian motion with mean zero and variance parameter $||h||_2^2 - ||Th||_2^2$, where $(I - T)h = \tilde{\psi}$ and $[nt]$ is the integer part of nt.

LEMMA 1. Let $\rho(n)$ denote the maximal correlation coefficient of the process $\{Y_n\}$. Then, if $\rho(n) \to 0$ as $n \to \infty$, $\tilde{\psi}$ is in the range of $I - T$.

For the proofs, see Gordin and Lifšic (1978).

Our main result is the following theorem.

THEOREM 1. Assume that the process $\{Y_n : n \geq 0\}$ in (1.1) is geometrically (Harris) ergodic and that Y_0 has the unique invariant π as its distribution. Then for every ψ in $L^2(\mathbb{R}^k, \pi)$ such that $\int \psi d\pi = 0$, ψ belongs to the range of $I - T$, i.e., (2.1) holds for every ψ in $L^2(\mathbb{R}^k, \pi)$ such that $\int \psi d\pi = 0$.

PROOF. Given the one-sided (strictly stationary) process $\{Y_n : n \geq 0\}$, we can always construct a two-sided $\{Y_n : n = 0, \pm 1, \pm 2, \cdots\}$ with the same finite-dimensional distribution. We may call $\{Y_n : n = 0, \pm 1, \pm 2, \cdots\}$ the doubly infinite extension of $\{Y_n : n \geq 0\}$ on a probability space $(\Omega, \mathcal{F}, P)$.

For $a \leq b$, define $\mathcal{F}_a^b$ as the σ-field generated by the random variables $Y_a, \cdots, Y_b$: define $\mathcal{F}_{-\infty}^a$ as the σ-field generated by $\cdots, Y_{a-1}, Y_a$: and define $\mathcal{F}_a^\infty$ as the σ-field generated by $Y_a, Y_{a+1}, \cdots$.

We claim that for each m $(-\infty < m < \infty)$ and for each n $(n \geq 1)$, $A \in \mathcal{F}_{-\infty}^m$, $B \in \mathcal{F}_{m+n}^\infty$ together imply

(2.2) $|P(A \cap B) - P(A)P(B)| \leq \varphi(n)P(A)$,

where $\varphi(n)$ is a nonnegative function of positive integers.

Let $g(Y_{m+n})$ denote $E(I_B|\mathcal{F}_{-\infty}^{m+n})$, where I_B is the indicator function of B. Then

$$E[I_A E(g(Y_{m+n})|\mathcal{F}_{-\infty}^m)]$$
$$= E[E(I_A g(Y_{m+n})|\mathcal{F}_{-\infty}^m)]$$
$$= E[I_A g(Y_{m+n})]$$
$$= E[I_A g(Y_m)]$$
$$= E[I_A I_B]$$
$$= P(A \cap B).$$

On the other hand,

$$P(A)P(B) = E[I_A \int g(y)\pi(dy)].$$

Therefore,

$$
\begin{aligned}
&|P(A \cap B) - P(A)P(B)| \\
(2.3) \quad &= \left| E\left[I_A E(g(Y_{m+n})|\mathcal{F}_{-\infty}^m) \right] - E\left[I_A \int g(y)\pi(dy) \right] \right| \\
&= \left| E\left[I_A \left(\int g(y)p^{(n)}(x,dy) - \int g(y)\pi(dy) \right) \right] \right| \\
&\leq E[I_A ||g||_\infty ||p^{(n)}(x,dy) - \pi(dy)||]
\end{aligned}
$$

Since the process $\{Y_n : n \geq 0\}$ is geometrically ergodic, there exist constants $\varepsilon > 0$, $0 < \rho < 1$, such that

$$||p^{(n)}(x,dy) - \pi(dy)|| < \varepsilon \rho^n$$

for all sufficiently large n, uniformly for $x \in \mathbb{R}^k$. Thus, taking $\varepsilon\rho^n$ as our $\varphi(n)$, we have shown that for every sufficiently large n, (2.2) holds, and from the inequality of (2.3), a constant function $\varphi(n)$ can be taken for the other finite many number of n's, justifying our claim. From our claim we see that the process $\{Y_n : n \geq 0\}$ is φ-mixing with coefficient $\varphi(n)$. Therefore, since the maximal correlation $\rho(n)$ of the process is less than or equal to $2\sqrt{\varphi(n)}$ and $\varphi(n) \to 0$ exponentially fast as $n \to \infty$, $\rho(n) \to 0$ as $n \to \infty$, and thus by Proposition 1 and Lemma 1, the conclusion of Theorem 1 holds.   □

REMARK 1. As references for earlier work on a class of nonlinear autoregressive processes, see Bhattacharya and Lee (1988), (1988), and Nummelin (1984), where they have provided some verifiable conditions under which the functional central limit theorem (FCLT) holds for a class of functions ψ in $L^2(\mathbb{R}^k, \pi)$.

References

[1] Bhattacharya, R. N., *On the funtional central limit theorem and the law of iterated logarithm for Markov processes*, Z. Wahr. Verw. Geb. **60** (1982), 185–201.

[2] Bhattacharya, R. N. and Lee, C. H., *Ergodicity of Nonlinear First Order Autoregressive Models*, J. Theoretical Probability **8** (1995), 207–219.

[3] ———, *On geometric ergodicity of nonlinear autoregressive models*, Statistics & Probability Letters **22** (1995), 311–315.

[4] Bhattacharya, R. N. and Lee, O., *Asymptotics of a class of Markov processes which are not in general irreducible*, Ann. Probab., **16** (1988), 1333–1347.

[5] ———, *Ergodicity and Central Limit Theorems for a class of Markov Processes*, J. Multivariate Analysis, **27** (1988), 80–90.

[6] Billingsley, P., *Convergence of Probability Measures, Wiley, New York*, 1968.

[7] Chan, K. S. and H. Tong paper On the use of the deterministic Lyapunov function for the ergodicity of stochastic difference equations, Adv. Appl. Probab. **17** (1985), 666–678.

[8] Gordin, M. I. and Lifšic, B. A. (1978), *The central limit theorem for stationary ergodic Markov process*, Dokl. Akad. Nauk. SSSR **19**, 392–393.

[9] Lee, C. H., *Asymptotics of a class pth-order Nonlinear Autoregressive processes*, preprints. (1997).

[10] Nummelin, E., *General Irreducible Markov chains and Nonnegative Operators*, (Cambridge Univ. Press. New York), 1984.

[11] Tjøstheim, D., *Nonlinear time series and Markov chains*, Adv. Appl. Probab. **22** (1990), 587–611.

Department of Mathematics
Hannam University
Taejon 300-791, Korea