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REMARKS ON VOLTERRA EQUATIONS IN BANACH SPACES

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ABSTRACT. Existence and Uniqueness for Volterra equations (VE) with a weak regularity assumption on A, the relative closedness of A are investigated by means of the Laplace transform theory. Also, (VE) are studied by means of the method of convoluted solution operator families.

0. Introduction

The objective of this paper is to study abstract Volterra integral equations

$$\text{(VE)} \qquad v(t) = A\int_0^t v(t-s)d\mu(s) + f(t), \qquad t \geq 0$$

by means of the Laplace transform theory. Here we assume that A is a linear operator on a Banach space X, μ is a numerical function of local bounded variation, and $f : [0, \infty) \to X$ is assumed to be locally Bochner integrable. In Section 2 we investigate the existence and uniqueness for (VE) under the assumption that A is a relatively closed linear operator. The properties of relatively closed operators are studied and applied to the integrated Cauchy problems in B. Bäumer and F. Neubrander ([4]). The following definition is taken from [4].

DEFINITION. A linear operator A with domain $D(A)$ and range in a Banach space X is called relatively closed if there exists an auxiliary Banach space $(X_A, \|\cdot\|_{X_A})$ which is continuously embedded in X (denoted as $X_A \hookrightarrow X$) such that

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(i) $D(A) \subset X_A$ and
(ii) the graph of A is closed in $X_A \times X$, i.e., if $x_n \in D(A)$ converges to x in X_A and Ax_n converges to y in X, then $x \in D(A)$ and $Ax = y$.

In this case we more specifically say that A is $(X_A \hookrightarrow X)$-closed.

Obviously, closed operators are relatively closed and relatively closed operators have continuity properties such as the commutativity of operators and the integral for sufficiently regular functions. Particular classes of operators which satisfy this rather weak regularity assumption are sums, compositions, and limits of closed operators (see [4]). In this view the results for (VE) in Section 2 extend those in [9].
As usual, we denote by $v * d\mu$ the Stieltjes convolution $t \mapsto \int_0^t v(t-s)\, d\mu(s)$ and by $v * \mu$ the convolution : $t \mapsto \int_0^t v(t-s)\mu(s)\, ds$ of sufficiently regular functions $v : [0,\infty) \to X$ and $\mu : [0,\infty) \to \mathbb{C}$.

DEFINITION. A function v is called a solution of (VE) if
(i) $v : [0,\infty) \to X$ is continuous,
(ii) $v * d\mu : [0,\infty) \to X_A$ is locally Bochner integrable, and
(iii) $\int_0^t v(t-s)d\mu(s) \in D(A)$ for all $t \geq 0$ and (VE) holds.

In order to be able to apply Laplace transform methods to (VE) we will restrict our discussion to Laplace transformable forcing terms f, exponentially bounded kernels μ, and exponentially bounded solutions v such that $v * d\mu$ is Lapalce transformable in X_A. In this case, the Laplace transform converts (VE) into the characteristic equation

$$\text{(CE)} \qquad (I - \widehat{d\mu}(\lambda)A)y(\lambda) = \hat{f}(\lambda) \qquad \text{for } \lambda > \omega$$

where $\widehat{d\mu}(\lambda) := \int_0^\infty e^{-\lambda t} d\mu(t)$, $\hat{f}(\lambda) := \int_0^\infty e^{-\lambda t} f(t)dt$, $y(\lambda) = \hat{v}(\lambda)$, and the number ω depends on the growth of the functions v, μ, and f. Thus, the Laplace transform method simplifies the problem (VE) by eliminating the time variable from the characteristic equation (CE).

Whereas Section 2 is centered around the solvability of (VE) for a given forcing term f, we investigate in Section 3 those pairs (A, μ) for which (VE) has unique solutions for *all* sufficiently regular f. From Section 3 on A is assumed to be a closed linear operator on X.
In recent years, the methods of integrated or convoluted semigroups

with generators A have been applied successfully to the abstract Cauchy problem $u'(t) = Au(t)$, $u(0) = x$. The main idea there is to regularize the equation by convoluting it with a sufficiently nice numerical function k and to study the special Volterra equation

$$v(t) = A \int_0^t v(s)\, ds + k(t)x$$

instead, where $v = (u * k)'$ (for references, see [3], [5], or [6]). Extending this method to equations of the form

$$v(t) = A \int_0^t v(t-s)\, d\mu(s) + k(t)x,$$

we will show that the notion of convoluted solution operator families with generators (A, μ) is suitable for studying generalized wellposedness for the Volterra equation (VE). A convoluted solution operator family coincides with an n-times integrated solution operator family for $k(t) = \frac{t^n}{n!}$ for $n \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}$.
Section 4 contains Trotter-Kato type approximations for Volterra equations and finally in Section 5 the method of analytic convoluted solution operator families which is a generalization of analytic integrated solution operator families is introduced with a characterization of them.

Whereas the method of convoluted solution operator families assumes the existence of $(I - \widehat{d\mu}(\lambda)A)^{-1}$ as bounded operators on X for $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega$ for some constant ω, there are various cases especially in applications to illposed problems where the assumption is not satisfied. Thus, Section 2 studies (VE) without assuming the existence of $(I - \widehat{d\mu}(\lambda)A)^{-1}$ for any $\lambda \in \mathbb{C}$. In particular, we characterize those forcing terms f for which there exist solutions of (VE) for a given pair (A, μ), where the graph of A is not necessarily closed in $X \times X$, but only in $X_A \times X$.

1. Preliminaries

In this section we introduce notation briefly and list some elementary facts concerning integration for vector-valued functions and results from the Laplace transform theory. Refer to [1], [3], [8], [9], or [10] for details.

Let X be a Banach space. $BV([a,b];X)$ denotes the space consisting of X-valued functions f of bounded variation on $[a,b]$. We define $BV_{loc}([0,\infty);X) := \bigcup_{b>0} BV([0,b];X)$ and $BV_\epsilon([0,\infty);X) :=$ the space of all $f \in BV_{loc}([0,\infty);X)$ such that $f(0)=0$ and for some constants $M, \epsilon \geq 0$, the variation $var_{[0,t]}(f)$ of f on the interval $[0,t]$ is less than or equal to $Me^{\epsilon t}$ for all $t \geq 0$. As usual, $L^1_{loc}([0,\infty);X)$ denotes the space of locally Bochner integrable functions from $[0,\infty)$ to X. $\mathbb{C}_\omega$ denotes the set of all complex numbers λ with $\operatorname{Re}\lambda > \omega$. We define the exponential growth bound of a function $f \in L^1_{loc}([0,\infty);X)$ as

$$\omega(f) := \inf\{\omega \in \mathbb{R} \mid \sup_{t \geq \tau} \|e^{-\omega t} f(t)\| < \infty \text{ for some } \tau \geq 0\}.$$

Let one of f and g be an X-valued and the other a $\mathbb{C}$-valued function. If one of f and g is continuous and the other is of bounded semivariation on $[a,b]$, then the Riemann-Stieltjes integral $\int_a^b f(s)dg(s)$ of f and g exists and $f * dg \in L^1([a,b];X)$. If $\int_a^b f(s)dg(s)$ exists, then so does $\int_a^b g(s)df(s)$ and the integration by parts formula

$$(1.1) \qquad \int_a^b f(s)dg(s) = f(b)g(b) - f(a)g(a) - \int_a^b g(s)df(s)$$

holds. It follows from (1.1) that if f and g are defined on $[0,\infty)$ and if $f * dg(t)$ exists for some $f : [0,\infty) \to X$ and $g : [0,\infty) \to \mathbb{C}$, then

$$(1.2) \qquad \int_0^t f(t-s)dg(s) = f(0)g(t) - f(t)g(0) + \int_0^t g(t-s)df(s)$$

for $t \geq 0$. The n-th normalized antiderivative of $f \in L^1_{loc}([0,\infty);X)$ is denoted by

$$t \mapsto f^{[n]}(t) := \underbrace{1 * 1 * \cdots * 1}_{n-\text{times}} * f(t) = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f(s)ds.$$

If $f \in L^1([0,b];\mathbb{C})$ and $g \in Lip([0,b];X)$, then $u := f * (dg^{[1]})$ is differentiable a.e. on $[0,b]$, and

$$u'(t) = \int_0^t f(t-s)dg(s) + f(t)g(0)$$

for almost all $t \in [0,b]$. Moreover, u is continuously differentiable if $g(0) = 0$.

For $f \in L^1_{loc}([0,\infty);X)$, we define

$$\text{abs}(f) := \inf\{\text{Re}\,\lambda \mid \widehat{f}(\lambda) = \lim_{T\to\infty} \int_0^T e^{-\lambda t} f(t)dt \text{ exists}\}$$

called the abscissa of f and f is said to be Laplace transformable if $\text{abs}(f) < \infty$.
If $\text{Re}\,\lambda > \max\{\text{abs}(f),\ 0\}$, then $\widehat{f}(\lambda)$ and $\widehat{f^{[1]}}(\lambda)$ exist, and by integration by parts,

$$\lambda\widehat{f^{[1]}}(\lambda) = \widehat{f}(\lambda). \tag{1.3}$$

For $f \in BV_{loc}([0,\infty);X) \bigcup C([0,\infty);X)$, we define

$$\text{abs}(df) := \inf\{\text{Re}\,\lambda \mid \widehat{df}(\lambda) = \lim_{T\to\infty} \int_0^T e^{-\lambda t} df(t) \text{ exists}\}$$

called the abscissa of df and f is said to be Laplace-Stieltjes transformable if $\text{abs}(df) < \infty$. If $f \in BV_{loc}([0,\infty);X) \bigcup C([0,\infty);X)$ is exponentially bounded and if $f(0) = 0$, then for a nonnegative number $\omega \geq \omega(f)$, it follows from (1.2) that

$$\widehat{df}(\lambda) = \lambda\widehat{f}(\lambda) \tag{1.4}$$

for all $\lambda \in \mathbb{C}_\omega$.
Let $(X, \|\cdot\|)$ be a Banach space and $(X_1, \|\cdot\|_1)$ a continuously embedded Banach space in X (i.e., $X_1 \hookrightarrow X$). Then $x_n \longrightarrow x$ in $(X_1, \|\cdot\|_1)$ implies that $x_n \longrightarrow x$ in $(X, \|\cdot\|)$. Let $f \in L^1_{loc}([0,\infty);X_1)$. We denote the

abscissa of f in X_1 by $\mathrm{abs}_{X_1}(f)$ to specify the space X_1 later in Section 2. The basic fact will be used in Theorem 2.1 that if $\mathrm{abs}_{X_1}(f) < \infty$, then $\mathrm{abs}(f) < \infty$ and $\int_0^\infty e^{-\lambda t} f(t)dt$ is a limit both in X and X_1 for $\lambda \in \mathbb{C}_\omega$ with $\omega \geq \mathrm{abs}_{X_1}(f)$.
The Laplace transform of a Stieltjes convolution has a multiplicative property, which is essential for transforming the Volterra equation (VE) to the characteristic equation (CE).

PROPOSITION 1.1. *Suppose that* $f \in C([0,\infty); X)$ *with* $\omega(f) < \infty$ *and that* $g \in BV_\epsilon([0,\infty); \mathbb{C})$ *for some* $\epsilon \geq 0$. *Let* ω *be a number such that* $\omega \geq \max\{\omega(f),\ \epsilon\}$. *Then* $\mathrm{abs}(f * dg) \leq \omega$ *and for* $\lambda \in \mathbb{C}_\omega$,

$$\widehat{f * dg}(\lambda) = \widehat{f}(\lambda)\widehat{dg}(\lambda).$$

The details of the Laplace transform results which follow can be found in [1], [3], or [10]. A fundamental fact from Functional Analysis follows at the end of this section.

THEOREM 1.2. (Uniqueness Theorem) *Let* $f \in L^1_{loc}([0,\infty); X)$ *with* $\mathrm{abs}(f) < \infty$. *If there exists an* $\omega \geq \mathrm{abs}(f)$ *such that* $\widehat{f} \equiv 0$ *on* (ω, ∞), *then* $f(t) = 0$ *for almost all* $t \geq 0$.

The space $Lip_\omega([0,\infty); X)$ for $\omega \in \mathbb{R}$ is defined as the space consisting of those functions $F : [0,\infty) \to X$ with $F(0) = 0$ and for which $\|F\|_{Lip_\omega}$ defined as

$$\inf\{M \mid \|F(t+h) - F(t)\| \leq M \int_t^{t+h} e^{\omega r} dr \ \text{ for } t,\ h \geq 0\}$$

is finite. It is clear that if $\omega \geq 0$ and $F \in Lip_\omega([0,\infty); X)$, then $\omega(F) \leq \omega$. If $f \in L^1_{loc}([0,\infty); X)$ with $\omega(f) < \infty$, then for any number $\omega > \omega(f)$, $f^{[1]} \in Lip_\omega([0,\infty); X)$.
The following inversion theorem of the Laplace-Stieltjes transform will be crucial in characterizing the solutions of Volterra equations.

THEOREM 1.3. (Phragmén-Doetsch Inversion Theorem). *Let $F \in Lip_\omega([0,\infty);X)$ and let $r := \widehat{dF}$ on (ω,∞). Then*

$$\|F(t) - \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j!} e^{tnj} r(nj)\| \leq \frac{2}{n} \|F\|_{Lip_\omega}$$

for all $n \in \mathbb{N}$ with $n > \omega$ and all $t \geq 0$.

The Widder space $C_W^\infty((\omega,\infty);X)$ is defined as the space consisting of all those functions $r \in C^\infty((\omega,\infty);X)$ for which

$$\|r\|_{W,\omega} := \sup_{k\in\mathbb{N}_0,\ \lambda>\omega} \|(\lambda-\omega)^{k+1} \frac{1}{k!} r^{(k)}(\lambda)\| < \infty.$$

THEOREM 1.4. (Widder's Theorem). *The Laplace-Stieltjes transform is an isometric isomorphism from $Lip_\omega([0,\infty);X)$ onto $C_W^\infty((\omega,\infty);X)$.*

The following theorem is due to B. Hennig and F. Neubrander [7].

THEOREM 1.5. *Let $F_n \in Lip_\omega([0,\infty);X)$ for every $n \in \mathbb{N}$ for which there exists a constant $M \geq 0$ such that $\|F_n\|_{Lip_\omega} \leq M$ for all $n \in \mathbb{N}$. Then the following are equivalent.*

(i) *There exist constants $a > \omega$ and $b > 0$ such that $\lim_{n\to\infty} \widehat{dF_n}(\lambda_k)$ exists for all $k \in \mathbb{N}_0$ where $\lambda_k := a + kb$.*
(ii) *There exists an $F \in Lip_\omega([0,\infty);X)$ such that $\|\widehat{dF}\|_{W,\omega} \leq M$ and $\{\widehat{dF_n}(\cdot)\}_n$ converges uniformly to $\widehat{dF}(\cdot)$ on compact subsets of (ω,∞).*
(iii) *$\lim_{n\to\infty} F_n(t)$ exists for every $t \geq 0$.*
(iv) *There exists an $F \in Lip_\omega([0,\infty);X)$ with $\|F\|_{Lip_\omega} \leq M$ such that $\{F_n(\cdot)\}_n$ converges uniformly to $F(\cdot)$ on compact subsets of $[0,\infty)$.*

Let $\omega \in \mathbb{R}$ and $0 < \theta \leq \pi$. By $\Sigma_{\omega,\theta}$ we denote the open sector $\{z \in \mathbb{C} \mid |\arg(z-\omega)| < \theta\}$.

THEOREM 1.6. *Let* $0 < \theta_0 \le \frac{\pi}{2}$, $\omega \in \mathbb{R}$, *and let* q *be a function from* $(\omega,\ \infty)$ *to* X. *Then the following are equivalent.*

(i) *There exists an analytic function* $f : \Sigma_{0,\theta_0} \to X$ *such that* $q = \widehat{f}$ *on* $(\omega,\ \infty)$, *and* $\sup_{z\in\Sigma_{0,\theta}} \|e^{-\omega z} f(z)\| < \infty$ *for every* $\theta \in (0,\ \theta_0)$.

(ii) *The function* q *admits an analytic extension* $\widetilde{q} : \Sigma_{\omega,\theta_0+\frac{\pi}{2}} \to X$ *for which* $\sup_{\lambda\in\Sigma_{\omega,\theta+\frac{\pi}{2}}} \|(\lambda-\omega)\widetilde{q}(\lambda)\| < \infty$ *for every* $\theta \in (0,\ \theta_0)$.

Moreover, if (i) *holds, then for every* $\theta \in (0,\ \theta_0)$, *there exists a constant* $C_\theta > 0$ *such that*

$$\|z^k f^{(k)}(z)\| \le C_\theta e^{\omega \mathrm{Re}\, z}(|\omega||z| + 1)^k$$

for all $z \in \Sigma_{0,\theta}$.

THEOREM 1.7. (Uniform Exponential Boundedness Theorem). *Let* $F : [0,\infty) \longrightarrow L(X;Y)$ *be a function such that* $F(\cdot)x$ *is exponentially bounded for each* $x \in X$. *Then there exist constants* $M > 0$ *and* $\omega \in \mathbb{R}$ *such that* $\|F(t)\| \le Me^{\omega t}$ *for all* $t \ge 0$.

2. Existence and Uniqueness of solutions of (VE)

Let X be a Banach space. This section contains a solution characterization and uniqueness for (VE).The methods used in their proofs are modified from [1], [3], [4], or [9].

THEOREM 2.1. *Let* A *be an* $(X_A \hookrightarrow X)$*-closed linear operator,* $\mu \in BV_\epsilon([0,\infty);\mathbb{C})$ *for some* $\epsilon \ge 0$, *and* f *a function in* $L^1_{loc}([0,\infty);X)$ *with* $\mathrm{abs}(f) < \infty$. *Let* $v \in C([0,\infty);X)$ *such that* $\omega(v) < \infty$, $v * d\mu \in L^1_{loc}([0,\infty);X_A)$ *with* $\mathrm{abs}_{X_A}(v * d\mu) < \infty$, *and* $v * d\mu(t) \in D(A)$ *for every* $t \ge 0$. *Let* $\omega \ge \max\{\epsilon,\ \mathrm{abs}(f),\ \omega(v),\ \mathrm{abs}_{X_A}(v * d\mu)\}$. *Then the following are equivalent.*

(i) v *is a solution of* (VE).

(ii) $\widehat{d\mu}(\lambda)\widehat{v}(\lambda) \in D(A)$ *and* $(I - \widehat{d\mu}(\lambda)A)\widehat{v}(\lambda) = \widehat{f}(\lambda)$ *for all* $\lambda \in \mathbb{C}_\omega$.

(iii) $\widehat{d\mu}(l)\widehat{v}(l) \in D(A)$ *and* $(I - \widehat{d\mu}(l)A)\widehat{v}(l) = \widehat{f}(l)$ *for all* $l \in \mathbb{N}$ *with* $l > \omega$.

PROOF. Suppose that (i) holds. Let $\lambda \in \mathbb{C}_\omega$. Since $\omega(v) \leq \omega$ and since $\text{abs}_{X_A}(v * d\mu) \leq \omega$, it follows from the $(X_A \hookrightarrow X)$-closedness of A and Proposition 1.1 that

$$\begin{aligned}
\widehat{v}(\lambda) &= \int_0^\infty e^{-\lambda t} v(t)dt \\
&= \int_0^\infty e^{-\lambda t}(A\int_0^t v(t-s)d\mu(s) + f(t))dt \\
&= \int_0^\infty e^{-\lambda t} A \int_0^t v(t-s)d\mu(s)dt + \int_0^\infty e^{-\lambda t} f(t)dt \\
&= A\int_0^\infty e^{-\lambda t} \int_0^t v(t-s)d\mu(s)dt + \int_0^\infty e^{-\lambda t} f(t)dt \\
&= A\widehat{v * d\mu}(\lambda) + \widehat{f}(\lambda) \\
&= A\widehat{d\mu}(\lambda)\widehat{v}(\lambda) + \widehat{f}(\lambda).
\end{aligned}$$

Thus, (i) implies (ii). Clearly, (ii) implies (iii). We show the implication (iii) $\Longrightarrow$ (i) by using the Phragmén inversion formula(Theorem 1.3) $F(t) = \lim_{n\to\infty} \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j!} e^{tnj} r(nj)$ for $t \geq 0$ and $n > \omega$ where $F \in Lip_\omega([0,\infty); X)$ and $r(\cdot) = \widehat{dF}(\cdot)$ on (ω, ∞). Suppose that (iii) holds. Let $\omega' > \omega$. It follows from $\omega(v^{[1]})$, $\omega(f^{[1]}) \leq \omega'$ that $v^{[2]}$, $f^{[2]} \in Lip_{\omega'}([0,\infty); X)$ and from $\omega_{X_A}((v * d\mu)^{[1]}) \leq \omega'$ that $(v * d\mu)^{[2]} \in Lip_{\omega'}([0,\infty); X_A)$. Hence it follows from (iii) and the relations (1.3) and (1.4) that

$$\widehat{dv^{[2]}}(l) - \widehat{df^{[2]}}(l) = A(d(v * d\mu)^{[2]})^\wedge(l)$$

for every $l \in \mathbb{N}$ with $l > \omega'$ (we denote the Laplace-Stieltjes tansform $\int_0^\infty e^{-lt} d(v * d\mu)^{[2]}(t)$ of the more or less long expression $(v * d\mu)^{[2]}$ at l by $(d(v * d\mu)^{[2]})^\wedge(l)$ for accuracy because $\widehat{}$ does not spread enough in tex). Then it follows from the $(X_A \hookrightarrow X)$-closedness of A and the Phragmén inversion formula that

$$v^{[2]}(t) - f^{[2]}(t) = A(v * d\mu)^{[2]}(t), \qquad t \geq 0.$$

It follows from the twice differentiability of $(v * d\mu)^{[2]}$ in X_A and the $(X_A \hookrightarrow X)$-closedness of A that

$$v(t) - f(t) = A(v * d\mu)(t), \qquad t \geq 0.$$

Thus, (iii) implies (i). □

The linearity of A and Theorem 2.1 imply that the exponentially bounded solutions of the Volterra equation (VE) are unique if and only if for any $\omega \geq \epsilon$, the equation

$$(I - \widehat{d\mu}(\lambda)A)y(\lambda) = 0 \qquad (\lambda > \omega)$$

has no nonzero solution y which has a Laplace transform representation $y(\lambda) = \widehat{v}(\lambda)$ for some $v \in C([0,\infty); X)$ such that $v * d\mu(t) \in D(A)$ for every $t \geq 0$ and $v * d\mu : [0,\infty) \to X_A$ is Laplace transformable. Another uniqueness theorem is given in terms of spectral properties of A and μ. $\sigma_p(A)$ denotes the point spectrum of an operator A.

THEOREM 2.2. *Let A be an $(X_A \hookrightarrow X)$-closed linear operator and $\mu \in BV_\epsilon([0,\infty); \mathbb{C})$ for some $\epsilon \geq 0$. Suppose that there exists a sequence $\{\lambda_k\}_k$ in $\mathbb{C}_\epsilon$ such that* Re $\lambda_k \longrightarrow \infty$ *as $k \longrightarrow \infty$, and for which $\widehat{d\mu}(\lambda_k)^{-1} \notin \sigma_p(A)$ for all $k \in \mathbb{N}$. Then the equation* (VE) *has at most one exponentially bounded solution v for which* $\text{abs}_{X_A}(v * d\mu) < \infty$.

PROOF. Since A is linear, it suffices to show that $v \equiv 0$ is the only exponentially bounded solution with $\text{abs}_{X_A}(v * d\mu) < \infty$ to the equation

$$v(t) = A \int_0^t v(t-s)d\mu(s), \qquad t \geq 0. \tag{2.1}$$

Let $v \in C([0,\infty); X)$ be a solution of (2.1) with $\omega(v) < \infty$. Let $\omega \geq \max\{\epsilon,\ \omega(v),\ \text{abs}_{X_A}(v * d\mu)\}$. Since Re $\lambda_k \longrightarrow \infty$, it follows from Theorem 2.1 that there exists a $K \in \mathbb{N}$ such that

$$\widehat{v}(\lambda_k) - A\widehat{d\mu}(\lambda_k)\widehat{v}(\lambda_k) = 0$$

for all $k \geq K$. Since $\widehat{d\mu}(\lambda_k)^{-1} \notin \sigma_p(A)$, $\widehat{v}(\lambda_k) = 0$ for all $k \geq K$. Fix a $k \geq K$. We claim that $\widehat{v} \equiv 0$ on $\mathbb{C}_\omega$. Assume not. Let $m \in \mathbb{N}$

be the order of the zero λ_k of the analytic function $\widehat{v}$ on $\mathbb{C}_\omega$. Since $\widehat{d\mu}\widehat{v}(\lambda_k) = \widehat{v * d\mu}(\lambda_k) \in X_A$, it follows from the $(X_A \hookrightarrow X)$-closedness and linearity of A that $(\widehat{d\mu}\widehat{v})^{(m)}(\lambda_k) \in D(A)$ as a limit in X_A and

$$\begin{aligned}\widehat{v}^{(m)}(\lambda_k) &= A(\widehat{d\mu}\,\widehat{v})^{(m)}(\lambda_k)\\ &= A\sum_{j=0}^{m}\binom{m}{j}\widehat{d\mu}^{(j)}(\lambda_k)\,\widehat{v}^{(m-j)}(\lambda_k)\\ &= \widehat{d\mu}(\lambda_k)A\widehat{v}^{(m)}(\lambda_k).\end{aligned}$$

Since $\widehat{v}^{(m)}(\lambda_k) \neq 0$, it follows that $\widehat{d\mu}(\lambda_k)^{-1} \in \sigma_p(A)$, which is a contradiction. Thus, $\widehat{v} \equiv 0$ on $\mathbb{C}_\omega$. It follows from Theorem 1.2 and the continuity of v that $v \equiv 0$ on $[0,\infty)$. □

3. Convoluted solution operator families and the wellposedness of (VE)

This section studies convoluted solution operator families for (VE) where A is a closed linear operator on a Banach space X. The method of convoluted solution operator families is a generalization of integrated solution operator families for (VE) and integrated or convoluted semigroups for abstract Cauchy problems. It is suitable for studying the wellposedness of (VE). Most properties of integrated solution operator families in [9] are hereditary. The proofs of some properties of convoluted solution operator families are omitted which can be obtained almost directly from the corresponding results of integrated solution operator families in [9].

DEFINITION 3.1. Let A be a closed linear operator with domain $D(A)$ and range in a Banach space X and $\mu \in BV_\epsilon([0,\infty);\mathbb{C})$ for some $\epsilon \geq 0$. Let $k \in L^1_{loc}([0,\infty);\mathbb{C})$ be Laplace transformable. Let $M > 0$ and $\omega \geq \max\{\epsilon,\ \mathrm{abs}(k)\}$ be some constants. Suppose that $(I-\widehat{d\mu}(\lambda)A)^{-1} \in L(X)$ for all $\lambda > \omega$. A strongly continuous mapping $S : [0,\infty) \to L(X)$ is said to be a k-convoluted solution operator family (k-c.s.o.f. for short) of exponential type $(M;\omega)$ with generator (A,μ) if the following hold.

(i) $\|S(t)\| \le Me^{\omega t}$ for all $t \ge 0$.
(ii) $\widehat{k}(\lambda)(I - \widehat{d\mu}(\lambda)A)^{-1}x = \widehat{S}(\lambda)x = \int_0^\infty e^{-\lambda t}S(t)x\,dt$ for every $\lambda > \omega$ and every $x \in X$.

REMARK. Analogously to integrated solution operator families (see [9]), if S is a k-c.s.o.f. of exponential type $(M;\omega)$ with generator (A,μ), then $(I - \widehat{d\mu}(\lambda)A)^{-1}$ exist in $L(X)$ and $\widehat{k}(\lambda)(I - \widehat{d\mu}(\lambda)A)^{-1}x = \widehat{S}(\lambda)x$ holds for all $x \in X$ and $\lambda \in \mathbb{C}_\omega$. It follows from the uniqueness of Laplace transform (Theorem 1.2) that for each Laplace transformable k in $L^1_{loc}([0,\infty);\mathbb{C})$, the pair (A,μ) generates at most one k-c.s.o.f..

LEMMA 3.2. *If S is a k-c.s.o.f. with generator (A,μ), then the following hold.*
(i) *$S(t)x \in D(A)$ and $AS(t)x = S(t)Ax$ for every $t \ge 0$ and every $x \in D(A)$.*
(ii) *S satisfies the equation*

$$S(t)x = \int_0^t S(t-s)Axd\mu(s) + k(t)x \quad (t \ge 0,\ x \in D(A)). \tag{3.1}$$

(iii) *$\int_0^t S(t-s)xd\mu(s) \in D(A)$ and*

$$S(t)x = A\int_0^t S(t-s)xd\mu(s) + k(t)x \quad (t \ge 0,\ x \in X). \tag{3.2}$$

Equation (3.2) says that if S is a k-c.s.o.f. with generator (A,μ), then $S(\cdot)x$ is an exponentially bounded solution with values in X of the Volterra equation

$$\text{(VE}; k, x) \qquad v(t) = A\int_0^t v(t-s)d\mu(s) + k(t)x, \qquad t \ge 0$$

for all $x \in X$. The necessary conditions (i) and (ii) in Lemma 3.2 are sufficient for an exponentially bounded, strongly continuous operator family $S = \{S(t)\}_{t\ge 0}$ in $L(X)$ to be a k-c.s.o.f. with generator (A,μ).

PROPOSITION 3.3. *Let A be a closed linear operator on X and let $\mu \in BV_\epsilon([0,\infty);\mathbb{C})$ for some $\epsilon \geq 0$. Let $k \in L^1_{loc}([0,\infty);\mathbb{C})$ be Laplace transformable. Let $S = \{S(t)\}_{t\geq 0}$ be a strongly continuous operator family in $L(X)$ for which there exist constants $M > 0$ and $\omega \geq \max\{\epsilon, \text{ abs}(k)\}$ such that $\|S(t)\| \leq Me^{\omega t}$ for all $t \geq 0$. Then S is a k-c.s.o.f. with generator (A,μ) if and only if S satisfies the following statements.*

(i) *$S(t)x \in D(A)$ and $AS(t)x = S(t)Ax$ for every $t \geq 0$ and $x \in D(A)$.*
(ii) *$S(t)x = A\int_0^t S(t-s)x\, d\mu(s) + k(t)x$ for every $t \geq 0$ and $x \in X$.*

Analogously to integrated solution operator families, a Hille-Yosida type characterization of convoluted solution operator families with generator (A,μ) is possible if additionally, A is densely defined and μ is absolutely continuous. The following is modified from and improves Theorem 3.1.10 in [9] and Theorem 2.2 in [2].

THEOREM 3.4. *Let $k \in L^1_{loc}([0,\infty);\mathbb{C})$ be Laplace transformable. Suppose that A is a densely defined closed linear operator on X and that μ is an absolutely continuous function in $BV_\epsilon([0,\infty);\mathbb{C})$ for some $\epsilon \geq 0$. Let $M > 0$ and $\omega \geq \max\{\epsilon, \text{ abs}(k)\}$ be some constants. Then the following are equivalent.*

(i) *The pair (A,μ) generates a k-c.s.o.f. of exponential type $(M;\omega)$.*
(ii) *For every $\lambda > \omega$, $(I - \widehat{d\mu}(\lambda)A)^{-1}$ exists in $L(X)$ and the function $H : (\omega,\infty) \to L(X)$ defined by $H(\lambda) = \widehat{k}(\lambda)(I - \widehat{d\mu}(\lambda)A)^{-1}$ belongs to $C^\infty((\omega,\infty);L(X))$ and satisfies the estimates*

$$(3.3)\qquad \left\|\frac{H^{(j)}(\lambda)}{j!}\right\| \leq \frac{M}{(\lambda-\omega)^{j+1}} \qquad \textit{for all } j \in \mathbb{N}_0 \textit{ and } \lambda > \omega.$$

PROOF. Suppose that S is the k-c.s.o.f. of exponential type $(M;\omega)$ with generator (A,μ). Let $\lambda > \omega$ and $x \in X$. From Definition 3.1,

$$H(\lambda)x = \widehat{k}(\lambda)(I - \widehat{d\mu}(\lambda)A)^{-1}x = \widehat{S}(\lambda)x = \left[\int_0^\infty e^{-\lambda t}dS^{[1]}(t)\right]x$$

for all $x \in X$. Since $S^{[1]} \in Lip_\omega([0,\infty);L(X))$, it follows from Widder's Theorem (Theorem 1.4) that $H \in C^\infty_W((\omega,\infty);L(X))$. Thus, (i) implies (ii). Suppose that (ii) holds. Then $H \in C^\infty_W((\omega,\infty);L(X))$. Hence, by Widder's Theorem, there exists $T \in Lip_\omega([0,\infty); L(X))$ such that $\widehat{dT} = H$ on (ω, ∞) and $\|T\|_{Lip_\omega} = \|H\|_{W,\omega} \leq M$. It follows from the relations (1.3) and (1.4) that

$$\widehat{T}(\lambda) = \widehat{k^{[1]}}(\lambda)(I - \widehat{d\mu}(\lambda)A)^{-1} \qquad \text{for } \lambda > \omega.$$

Hence, T is the $k^{[1]}$-c.s.o.f. with generator (A, μ). Thus, by Lemma 3.2,

$$T(t)x = \int_0^t T(t-s)Axd\mu(s) + k^{[1]}(t)x$$

for all $t \geq 0$ and $x \in D(A)$. Let $g(t) := T(t)Ax$ for $t \geq 0$. Then

$$T(t)x = \int_0^t g(t-s)\mu'(s)ds + k^{[1]}(t)x = \int_0^t \mu'(t-s)dg^{[1]}(s) + k^{[1]}(t)x.$$

Hence, $t \mapsto T(t)x$ is continuously differentiable (see Section 1) and

$$\frac{dT(t)x}{dt} = \int_0^t \mu'(t-s)dg(s) + k(t)x = \int_0^t \mu'(t-s)dT(s)Ax + k(t)x$$

for all $x \in D(A)$. Next, we show that $T(\cdot)x$ is differentiable for all $x \in X$. Let $t \geq 0$. Since $T \in Lip_\omega([0,\infty); L(X))$, the difference quotients $D_h := \frac{T(t+h)-T(t)}{h}$ are uniformly bounded for h such that $0 < |h| \leq 1$ and $t + h \geq 0$. Since $\lim_{h\to 0} D_h x$ exists for $x \in D(A)$, we obtain from Banach-Steinhaus Theorem that there exist operators $S(t) \in L(X)$ such that $S(t)x = \lim_{h\to 0} D_h x = \lim_{h\to 0} \frac{T(t+h)x-T(t)x}{h}$ for all $x \in X$. Notice that S is of exponential type $(M;\omega)$. To show that S is strongly continuous on $[0,\infty)$, let $x \in X$, $\epsilon > 0$, and $z \in D(A)$ with $\|x - z\| < \epsilon$. It follows from $\|S(t)x - S(t_0)x\| \leq \|S(t)\|\,\|x-z\| + \|S(t)z - S(t_0)z\| + \|S(t_0)\|\,\|x-z\|$, the continuity of $t \mapsto S(t)z$ for $z \in D(A)$, and the exponential boundedness of S that $t \mapsto S(t)x$ is continuous on $[0,\infty)$. Thus, the operator family T is strongly differentiable on $[0,\infty)$ and $\frac{dT(t)x}{dt} = S(t)x$ for all $t \geq 0$ and $x \in X$. Therefore,

$$\widehat{k}(\lambda)(I - \widehat{d\mu}(\lambda)A)^{-1}x = \lambda \int_0^\infty e^{-\lambda t}T(t)xdt = \int_0^\infty e^{-\lambda t}S(t)xdt$$

for all $x \in X$ and $\lambda > \omega$. This shows that S is a k-c.s.o.f. with generator (A, μ). □

If (A, μ) generates a k-c.s.o.f. with $k(t) = \frac{t^n}{n!}$, i.e., if (A, μ) is an n-times integrated solution operator family, then unique solutions of (VE) are found for all sufficiently regular f. See Theorem 3.1.11, [9] for the following.

THEOREM 3.5. *Suppose that (A, μ) generates an n-times integrated solution operator family S for some $n \in \mathbb{N}_0$. Let f be a Laplace transformable function in $C([0, \infty); X)$. Define $w(t) := \int_0^t S(t-s)f(s)ds$ for every $t \geq 0$. Then the following hold.*

(i) *If $w \in C^{n+1}([0, \infty); X)$, then $w^{(n+1)}$ is a solution of* (VE).
(ii) *If $v \in C([0, \infty); X)$ is a solution of* (VE), *then $w \in C^{n+1}([0, \infty); X)$ and $v = w^{(n+1)}$.*

Now, we discuss the wellposedness of (VE) with respect to convoluted solution operator families. Let $k \in C([0, \infty); \mathbb{C})$ such that $k \not\equiv 0$ and k is Laplace transformable. As mentioned after Lemma 3.2, if S is a k-c.s.o.f. with generator (A, μ), then $S(\cdot)x$ is an exponentially bounded solution of the Volterra equation (VE $; k, x$) for all $x \in X$. In fact, the following Proposition 3.6 implies that $S(\cdot)x$ is the unique expontially bounded solution to the equation (VE $; k, x$) for every $x \in X$. By modifying Proposition 2.1 in H. Oka [11], we obtain the following.

PROPOSITION 3.6. *Let A be an $(X_A \hookrightarrow X)$-closed linear operator and $\mu \in BV_\epsilon([0, \infty); \mathbb{C})$ for some $\epsilon \geq 0$. Suppose that there exists a Laplace transformable $k \in C([0, \infty); \mathbb{C})$ such that $k \not\equiv 0$ and for which there exists a k-c.s.o.f. with generator (A, μ). Then the equation* (VE) *has at most one solution.*

PROOF. Let k be a nontrivial Laplace transformable function in $L^1_{loc}([0, \infty); \mathbb{C})$ for which there exists a k-c.s.o.f. S with generator (A, μ). Let $v \in C([0, \infty); X)$ be a solution to the equation $v(t) = A \int_0^t v(t -$

$s)d\mu(s)$. Then

$$\begin{aligned}
S * v(t) &= S * A(v * d\mu)(t) \\
&= \int_0^t S(r)A\int_0^{t-r} v(t-r-s)d\mu(s)dr \\
&= \int_0^t A\int_0^{t-r} S(r)v(t-r-s)d\mu(s)dr \\
&= \int_0^t A\int_0^{t-r} S(t-r-s)v(r)d\mu(s)dr \\
&= \int_0^t \big[S(t-r)-k(t-r)\big]v(r)dr \\
&= S * v(t) - k * v(t)
\end{aligned}$$

for all $t \geq 0$. Thus, $k * v \equiv 0$. Therefore, $v \equiv 0$. □

We will show that if $k \in C([0,\infty);\mathbb{C})$ with $k \not\equiv 0$, then (VE ; k, x) has a unique, exponentially bounded solution for every $x \in X$ if and only if (A, μ) is a generator of a k-c.s.o.f.. For this the following lemma is crucial.

LEMMA 3.7. *The following statements are equivalent.*

(i) *The equation* (VE ; k, x) *has a unique, exponentially bounded solution for all* $x \in X$.

(ii) *The equation* (VE ; k, x) *has a unique, exponentially bounded solution* $v(\cdot) = v(\cdot, x)$ *for every* $x \in X$ *and there exist constants* $M > 0$ *and* $\omega \geq \max\{\epsilon,\ \mathrm{abs}(k)\}$ *such that* $\|v(t)\| \leq Me^{\omega t}\|x\|$ *for all* $x \in X$ *and* $t \geq 0$.

PROOF. Clearly, (ii) implies (i). Suppose that (i) holds. For every $x \in X$, let $v(\cdot, x)$ be the unique exponentially bounded solution of the equation (VE ; k, x). Considering $C([0,\infty); X)$ as the Fréchet space with the seminorms $p_T(f) = \sup_{0\leq t\leq T} \|f(t)\|$, $T \geq 0$, define a map $\phi : X \to C([0,\infty); X)$ by $x \mapsto v(\cdot, x)$. Then ϕ is linear since $v(\cdot, x)$ is a unique solution of (VE ; k, x) for each $x \in X$. We show that ϕ is continuous. Since ϕ is defined on X, it suffices to show that ϕ is closed. Suppose that a sequence $\{x_m\}_m$ converges to x in X and $\{v(\cdot, x_m)\}_m$ converges to u in $C([0,\infty);X)$. Since $v(\cdot, x_m) * d\mu(t) \in D(A)$ and the sequence $\{\int_0^t v(t-s, x_m)d\mu(s)\}_m$ converges to $\int_0^t u(t-s)d\mu(s)$ in X for every $t \geq 0$, we obtain from the closedness of A that $(u * d\mu)(t) \in D(A)$ and

$$u(t) = A\int_0^t u(t-s)d\mu(s) + k(t)x$$

for every $t \geq 0$ and $x \in X$. Hence ϕ is closed. For every $t \geq 0$, define $S(t) : x \mapsto v(t,x)$ on X. Clearly, the operators $S(t)$ are linear. Since $S(t)x = v(t,x) = \phi(x)(t)$, we obtain that $S(t) \in L(X)$. Since $t \mapsto S(t)x = v(t,x)$ is exponentially bounded for each $x \in X$, it follows from the Uniform Exponential Boundedness Theorem (Theorem 1.7) that there exist constants $M > 0$ and $\omega \geq \max\{\epsilon, \text{ abs}(k)\}$ such that $\|S(t)x\| = \|v(t,x)\| \leq Me^{\omega t}\|x\|$ for all $x \in X$ and $t \geq 0$. □

THEOREM 3.8. *Let A be a closed linear operator on X and let $\mu \in BV_\epsilon([0,\infty);\mathbb{C})$ for some $\epsilon \geq 0$. Let $k \in C([0,\infty);\mathbb{C})$ such that $k \not\equiv 0$ and k is Laplace transformable. Then the following are equivalent.*

(i) *The equation* (VE ; k,x) *has unique, exponentially bounded solutions for all $x \in X$.*

(ii) *(A,μ) generates a k-c.s.o.f..*

PROOF. The implication (ii) $\Longrightarrow$ (i) holds by Proposition 3.6. Suppose that (i) holds. It follows from the proof of Lemma 3.7 that there exists a strongly continuous operator family $\{S(t)\}_{t\geq 0} \subset L(X)$ for which there exist constants $M > 0$, $\omega \geq 0$ such that $\|S(t)\| \leq Me^{\omega t}$ for all $t \geq 0$ and that

$$S(t)x = A\int_0^t S(t-s)xd\mu(s) + k(t)x, \qquad t \geq 0$$

and for every $x \in X$. Hence $S(\cdot)x$ is an exponentially bounded solution of (VE ; k,x) for every $x \in X$. Let $x \in D(A)$ and $t \geq 0$. Define $v(t) := \int_0^t S(t-s)Axd\mu(s) + k(t)x$. Then $v(t) \in D(A)$ and

$$\begin{aligned} Av(t) &= A\int_0^t S(t-s)Axd\mu(s) + k(t)Ax \\ &= S(t)Ax - k(t)Ax + k(t)Ax \\ &= S(t)Ax. \end{aligned}$$

Thus,

$$v(t) = \int_0^t Av(t-s)d\mu(s) + k(t)x = A\int_0^t v(t-s)d\mu(s) + k(t)x.$$

Then, by the uniqueness of the solutions of (VE ; k, x), $v(t) = S(t)x$. Thus, $AS(t)x = S(t)Ax$ for all $t \geq 0$ and $x \in D(A)$. Therefore, by Proposition 3.3, S is a k-c.s.o.f. with generator (A, μ). □

4. Trotter-Kato type approximations of convoluted solution operator families

Let X be a Banach space and let $M > 0$ and $\omega \geq 0$ be some constants. A sequence $\{S_n\}_n$ of functions $S_n : [0, \infty) \to L(X)$ is said to be $(M; \omega)$-stable (or simply stable) if $\|S_n(t)\| \leq Me^{\omega t}$ for all $n \in \mathbb{N}$ and $t \geq 0$. Trotter-Kato type approximations of integrated solution operator families in [9] extend to convoluted solution operator families.

THEOREM 4.1. *Let A, A_n be closed linear operators on X and let μ, $\mu_n \in BV_\epsilon([0,\infty); \mathbb{C})$ for some $\epsilon \geq 0$ for all $n \in \mathbb{N}$. Let $k \in L^1_{loc}([0,\infty); \mathbb{C})$ be Laplace transformable. Let $M > 0$ and $\omega \geq \max\{\epsilon, \operatorname{abs}(k)\}$ be some constants. Let $\{S_n\}_{n \in \mathbb{N}}$ be an $(M;\omega)$-stable sequence of k-convoluted solution operator families S_n with generators (A_n, μ_n). Suppose that $(I - \widehat{d\mu}(\lambda)A)^{-1}$ exists in $L(X)$ and $\lim_{n\to\infty}(I - \widehat{d\mu_n}(\lambda)A_n)^{-1}x = (I - \widehat{d\mu}(\lambda)A)^{-1}x$ for every $\lambda > \omega$ and $x \in X$. Then there exists a $k^{[1]}$-c.s.o.f. $T \in Lip_\omega([0,\infty); L(X))$ with generator (A, μ) such that $\|T\|_{Lip_\omega} \leq M$. Moreover, for every $x \in X$, $\{S_n^{[1]}(t)x\}_n$ converges uniformly to $T(t)x$ on compact subsets of $[0,\infty)$. If in addition, A is densely defined and μ is absolutely continuous, then there exists a k-c.s.o.f. S of exponential type $(M;\omega)$ with generator (A,μ). In fact, $S(t)x := \frac{dT(t)x}{dt}$ for all $t \geq 0$ and $x \in X$.*

PROOF. Define $T_n(t)x := \int_0^t S_n(s)xds$ for every $n \in \mathbb{N}$, $t \geq 0$, and $x \in X$. Then the $(M;\omega)$-stability of $\{S_n\}_n$ implies that $T_n \in Lip_\omega([0,\infty); L(X))$ with $\|T_n\|_{Lip_\omega} \leq M$ for all $n \in \mathbb{N}$. It follows from the hypothesis $\lim_{n\to\infty}(I - \widehat{d\mu_n}(\lambda)A_n)^{-1}x = (I - \widehat{d\mu}(\lambda)A)^{-1}x$ that $\widehat{dT_n}(\lambda)x =$

$\widehat{S_n}(\lambda)x = \widehat{k}(\lambda)(I - \widehat{d\mu_n}(\lambda)A_n)^{-1}x$ converges to $\widehat{k}(\lambda)(I - \widehat{d\mu}(\lambda)A)^{-1}x$ for every $\lambda > \omega$ and $x \in X$. Since $\|T_n(\cdot)x\|_{Lip_\omega} \leq M\|x\|$ for all $n \in \mathbb{N}$ and $x \in X$, it follows from Theorem 1.5 that for each $x \in X$, there exists $T_x \in Lip_\omega([0,\infty);X)$ with $\|T_x\|_{Lip_\omega} \leq M\|x\|$ such that the sequence $\{T_n(\cdot)x\}_n$ converges uniformly to $T_x(\cdot)$ on compact subsets of $[0,\infty)$. Define $T(t)x := T_x(t)$ for every $t \geq 0$ and $x \in X$. Then, by the uniqueness of a limit, $T(t) : X \to X$ is linear for every $t \geq 0$. Moreover, $T \in Lip_\omega([0,\infty);L(X))$ with $\|T\|_{Lip_\omega} \leq M$. It follows from Theorem 1.5 that $\{\widehat{dT_n}(\lambda)x\}_n$ converges uniformly to $\widehat{dT}(\lambda)x$ on compact subsets of (ω,∞). By the uniqueness of limits,

$$\widehat{k^{[1]}}(\lambda)(I - \widehat{d\mu}(\lambda)A)^{-1}x = \frac{\widehat{k}(\lambda)}{\lambda}(I - \widehat{d\mu}(\lambda)A)^{-1}x = \frac{1}{\lambda}\widehat{dT}(\lambda)x = \widehat{T}(\lambda)x$$

for every $\lambda > \omega$ and $x \in X$. Thus, T is a $k^{[1]}$-c.s.o.f. with generator (A,μ). Assuming that A is densely defined and μ is absolutely continuous, it follows from the second half of the proof of Theorem 3.4 that $\frac{dT(t)x}{dt}$ exists for all $t \geq 0$ and $x \in X$ and there exists an operator family $S = \{S(t)\}_{t\geq 0}$ in $L(X)$ such that $S(t)x = \frac{dT(t)x}{dt}x$ and which is a k-c.s.o.f. with generator (A,μ). □

The previous theorem says that if $\{S_n\}_n$ is a stable sequence of k-convoluted solution operator families S_n with generators (A_n,μ_n) where A_n are closed linear operators on X and $\mu_n \in BV_\epsilon([0,\infty);\mathbb{C})$ for all $n \in \mathbb{N}$, and if A is a densely defined closed linear operator on X and $\mu \in BV_\epsilon([0,\infty);\mathbb{C})$ is absolutely continuous, then the strong convergence of the sequence $\{(I - \widehat{d\mu_n}(\lambda)A_n)^{-1}\}_n$ to $(I - \widehat{d\mu}(\lambda)A)^{-1}$ on a right half line of $\mathbb{R}$ implies the existence of a k-c.s.o.f. S with generator (A,μ) and the uniform convergence of the sequence $\{S_n^{[1]}(\cdot)x\}_n$ to $S^{[1]}(\cdot)x$ on compact subsets of $[0,\infty)$. For every $x \in X$, the sequence $\{S_n(\cdot)x\}_n$ converges uniformly to $S(\cdot)x$ on compact subsets of $[0,\infty)$ under additional assumptions on k, and (A,μ) and (A_n,μ_n) for $n \in \mathbb{N}$.

THEOREM 4.2. *Let A, A_n be densely defined closed linear operators on X and let μ, μ_n be both absolutely continuous and in $BV_\epsilon([0,\infty);\mathbb{C})$ for some $\epsilon \geq 0$ for all $n \in \mathbb{N}$. Let $k \in L^1_{loc}([0,\infty);\mathbb{C})$ be a Laplace transformable function which is bounded on compact subsets of $[0,\infty)$.*

Let $M > 0$ and $\omega \geq \max\{\epsilon, \text{ abs}(k)\}$ be some constants. Let $\{S_n\}_n$ be an $(M;\omega)$-stable sequence of k-convoluted solution operator families S_n with generators (A_n, μ_n) for $n \in \mathbb{N}$. Suppose that $\widehat{d\mu} \not\equiv 0$ on (ω, ∞) and that $(I - \widehat{d\mu}(\lambda)A)^{-1}$ exists in $L(X)$ for every $\lambda > \omega$ and $\lim_{n\to\infty}(I - \widehat{d\mu_n}(\lambda)A_n)^{-1}x = (I - \widehat{d\mu}(\lambda)A)^{-1}x$ for every $\lambda > \omega$ and $x \in X$. Additionally, suppose firstly that $D(A) \cap \bigcap_{n\in\mathbb{N}} D(A_n)$ contains a dense subset D of X, secondly that $\mu_n' \in Lip_\omega([0,\infty);\mathbb{C})$ with $\|\mu_n'\|_{Lip_\omega} \leq L$ for all $n \in \mathbb{N}$, and finally that $\{\mu_n'(t)\}_n$ converges uniformly to $\mu'(t)$ on compact subsets of $[0,\infty)$. Then, there exists a k-c.s.o.f. S of exponential type $(M;\omega)$ with generator (A, μ) for which for every $x \in X$, the sequence $\{S_n(\cdot)x\}_n$ converges uniformly to $S(\cdot)x$ on compact subsets of $[0,\infty)$.

PROOF. By Theorem 4.1, there exists a k-c.s.o.f. S of exponential type $(M;\omega)$ with generator (A,μ). For the uniform convergence of $\{S_n(t)x\}_n$ to $S(t)x$, we first show that for every $y \in D$, the sequence $\{S_n(t)y\}_n$ converges uniformly to $S(t)y$ on compact subsets of $[0,\infty)$. Let $y \in D$. Since S and S_n are k-convoluted solution operator families with generators (A,μ) and (A_n,μ_n), respectively, by Lemma 3.2,

$$S_n(t)y = \int_0^t S_n(t-s)A_n y d\mu_n(s) + k(t)y \tag{4.1}$$

and

$$S(t)y = \int_0^t S(t-s)Ayd\mu(s) + k(t)y \tag{4.2}$$

for every $t \geq 0$. Define $h(\lambda) := (I - \widehat{d\mu}(\lambda)A)^{-1}$ and $h_n(\lambda) := (I - \widehat{d\mu_n}(\lambda)A_n)^{-1}$ for every $\lambda > \omega$ and $n \in \mathbb{N}$. Then from the hypothesis, $\lim_{n\to\infty} h_n(\lambda)x = h(\lambda)x$ for every $\lambda > \omega$ and $x \in X$. Let $\lambda_0 > \omega$ such that $\widehat{d\mu}(\lambda_0) \neq 0$ and let $z := (I - \widehat{d\mu}(\lambda_0)A)y$. Then $y = h(\lambda_0)z$ and

$$\begin{aligned} &\| S_n(t)y - S(t)y\| \\ &\leq \|S_n(t)(h(\lambda_0)z - h_n(\lambda_0)z)\| + \|S_n(t)h_n(\lambda_0)z - S(t)h(\lambda_0)z\|. \end{aligned}$$

Since $S_n(t)$ are uniformly bounded on compact subsets of $[0,\infty)$, it suffices to estimate the convergence of the second term in this expression. It follows easily from the conditions $\mu_n' \in Lip_\omega([0,\infty);\mathbb{C})$ and $\|\mu_n'\|_{Lip_\omega} \le L$ for all $n \in \mathbb{N}$ that $\mu_n \in Lip_\omega([0,\infty);\mathbb{C})$ and $\|\mu_n\|_{Lip_\omega} \le \frac{L}{\omega}$ for all $n \in \mathbb{N}$. The condition that $\mu_n'(t)$ converges uniformly to $\mu'(t)$ on compact subsets of $[0,\infty)$ implies that $\lim_{n\to\infty} \mu_n(t) = \mu(t)$ for all $t \ge 0$. Thus, by Theorem 1.5, $\lim_{n\to\infty} \widehat{d\mu_n}(\lambda_0) = \widehat{d\mu}(\lambda_0)$. Since $\lim_{n\to\infty} \widehat{d\mu_n}(\lambda_0) = \widehat{d\mu}(\lambda_0)$ and $\widehat{d\mu}(\lambda_0) \ne 0$, to estimate the convergence of $\{\|S_n(t)h_n(\lambda_0)z - S(t)h(\lambda_0)z\|\}_n$ is equivalent to estimate the convergence of $\{\|\widehat{d\mu}(\lambda_0)\widehat{d\mu_n}(\lambda_0)\Big(S_n(t)h_n(\lambda_0)z - S(t)h(\lambda_0)z\Big)\|\}_n$. By (4.1) and (4.2),

$$
\begin{aligned}
&\|\widehat{d\mu}(\lambda_0)\widehat{d\mu_n}(\lambda_0)\Big(S_n(t)h_n(\lambda_0)z - S(t)h(\lambda_0)z\Big)\| \\
&= \|\widehat{d\mu}(\lambda_0)\widehat{d\mu_n}(\lambda_0)\Big(\int_0^t S_n(t-s)A_nh_n(\lambda_0)z\mu_n'(s)ds \\
&\qquad - \int_0^t S(t-s)Ah(\lambda_0)z\mu'(s)ds\Big)\| \\
&\quad + \|\widehat{d\mu}(\lambda_0)\widehat{d\mu_n}(\lambda_0)k(t)(h_n(\lambda_0)z - h(\lambda_0)z)\|.
\end{aligned}
\tag{4.3}
$$

Since the second term in (4.3) converges uniformly to 0 on compact subsets of $[0,\infty)$, it suffices to estimate the convergence of the first term in (4.3).

$$
\begin{aligned}
&\|\widehat{d\mu}(\lambda_0)\widehat{d\mu_n}(\lambda_0)\Big(\int_0^t S_n(t-s)A_nh_n(\lambda_0)z\mu_n'(s)ds \\
&\qquad - \int_0^t S(t-s)Ah(\lambda_0)z\mu'(s)ds\Big)\| \\
&= \|\widehat{d\mu}(\lambda_0)\int_0^t S_n(t-s)(h_n(\lambda_0)-I)z\mu_n'(s)ds \\
&\qquad - \widehat{d\mu_n}(\lambda_0)\int_0^t S(t-s)(h(\lambda_0)-I)z\mu'(s)ds\|
\end{aligned}
$$

$$(4.4)\quad \begin{aligned} &\le \|(\widehat{d\mu}(\lambda_0) - \widehat{d\mu_n}(\lambda_0)) \int_0^t S_(t-s)(h_n(\lambda_0) - I)z\mu_n'(s)ds\| \\ &\quad + |\widehat{d\mu_n}(\lambda_0)| \, \| \int_0^t S_n(t-s)(h_n(\lambda_0) - I)z\mu_n'(s)ds \\ &\qquad - \int_0^t S(t-s)(h(\lambda_0) - I)z\mu'(s)ds\|. \end{aligned}$$

Since $\int_0^t S_n(t-s)(h_n(\lambda_0)-I)z\mu_n'(s)ds$ are uniformly bounded on compact subsets of $[0,\infty)$ and since $\lim_{n\to\infty} \widehat{d\mu_n}(\lambda_0) = \widehat{d\mu}(\lambda_0)$, it suffices to estimate the convergence of the term $\| \int_0^t S_n(t-s)(h_n(\lambda_0)-I)z\mu_n'(s)ds - \int_0^t S(t-s)(h(\lambda_0) - I)z\mu'(s)ds\|$ in (4.4).

$$\begin{aligned} &\| \int_0^t S_n(t-s)(h_n(\lambda_0) - I)z\mu_n'(s)ds \\ &\qquad - \int_0^t S(t-s)(h(\lambda_0) - I)z\mu'(s)ds\| \\ &= \| \int_0^t S_n(s)(h_n(\lambda_0) - I)z\mu_n'(t-s)ds \\ &\qquad - \int_0^t S(s)(h(\lambda_0) - I)z\mu'(t-s)ds\| \end{aligned}$$

$$(4.5)\quad \begin{aligned} &\le \| \int_0^t S_n(s)(h_n(\lambda_0)z - h(\lambda_0)z)\mu_n'(t-s)ds\| \\ &\quad + \| \int_0^t \Big(S_n(s) - S(s)\Big)(h(\lambda_0) - I)z\mu_n'(t-s)ds\| \\ &\qquad + \| \int_0^t S(s)(h(\lambda_0) - I)z(\mu_n'(t-s) - \mu'(t-s))ds\|. \end{aligned}$$

Since $S_n(t)\mu_n'(t)$ are uniformly bounded on compact subsets of $[0,\infty)$ and since $\lim_{n\to\infty} h_n(\lambda_0) = h(\lambda_0)$, the first term in (4.5) converges uniformly to 0 on compact subsets of $[0,\infty)$. Since $S(s)(h(\lambda_0) - I)z$ is bounded on compact subsets of $[0,\infty)$ and since $\mu_n'(s)$ converges uniformly to $\mu'(s)$ on compact subsets of $[0,\infty)$, the third term converges uniformly to 0 on compact subsets of $[0,\infty)$. Thus, it suffices to estimate the convergence of the second term in (4.5). By integration by parts,

$$\begin{aligned}
&\|\int_0^t \Big(S_n(s)-S(s)\Big)(h(\lambda_0)-I)z\mu'(t-s)ds\| \\
\le&|\mu'(t)|\ \|\Big(S_n^{[1]}(t)-S^{[1]}(t)\Big)(h(\lambda_0)-I)z\| \\
&+\operatorname*{ess\,sup}_{s\in[0,t]}|\mu''(s)|\int_0^t \|\Big(S_n^{[1]}(s)-S^{[1]}(s)\Big)(h(\lambda_0)-I)z\|ds.
\end{aligned} \tag{4.6}$$

Since $\sup_{n\in\mathbb{N}}\|\mu_n'\|_{Lip_\omega}\le L$ implies that $\sup_{n\in\mathbb{N}}\operatorname*{ess\,sup}_{0\le s\le t}|\mu_n''(s)|<\infty$ and since for every $x\in X$, $S_n^{[1]}(s)x$ converges uniformly to $S^{[1]}(s)x$ on compact subsets of $[0,\infty)$, expression (4.6) converges uniformly to 0 on compact subsets of $[0,\infty)$. Thus, $\{S_n(t)y\}_n$ converges uniformly to $S(t)y$ on compact subsets of $[0,\infty)$ for every $y\in D$. Since $\overline{D}=X$ and $S_n(t)$ are uniformly bounded on compact subsets of $[0,\infty)$, we conclude that $S_n(\cdot)x$ converges uniformly to $S(\cdot)x$ on compact subsets of $[0,\infty)$ for every $x\in X$. □

5. Analytic convoluted solution operator families

In this section we extend the method of analytic integrated solution operator families for (VE) in [9] to analytic convoluted solution operator families and characterize them. They coincide with analytic resolvents in [12] when $k(t)=1$ and and $\mu=a^{[1]}$ where $a\in L^1_{loc}([0,\infty);\mathbb{C})$. Compared to the characterization of convoluted solution operator families (Theorem 3.4) the conditions for $H(\lambda):=\widehat{k}(\lambda)(I-\widehat{d\mu}(\lambda)A)^{-1}$ in the characterization of analytic convoluted solution operator families are relatively much easier to be checked.

DEFINITION 5.1. Let A be a closed linear operator on X and let $\mu\in BV_\epsilon([0,\infty);\mathbb{C})$ for some $\epsilon\ge 0$. Let $k\in L^1_{loc}([0,\infty);\mathbb{C})$ with $\text{abs}(k)<\infty$. Let $0<\theta_0\le\frac{\pi}{2}$ and let $M>0$ and $\omega\ge\max\{\epsilon,\ \text{abs}(k)\}$ be some constants. Let S be a function from $\{0\}\cup\Sigma_{0,\theta_0}$ to $L(X)$ satisfying the following conditions.

(i) Its restriction $S|_{[0,\infty)}$ is a k-c.s.o.f. of exponential type $(M;\omega)$ with generator (A,μ).
(ii) S is analytic on the sector Σ_{0,θ_0}.
(iii) For every $\theta \in (0,\ \theta_0)$, there exists a constant $M_\theta \geq 0$ such that $\sup\limits_{z\in\Sigma_{0,\theta}} \|e^{-wz}S(z)\| \leq M_\theta$.

Then S is said to be an analytic k-convoluted solution operator family (analytic k-c.s.o.f. for short) of analyticity type $(\omega;\theta_0)$ with generator (A,μ).

THEOREM 5.2. *Let A be a densely defined closed linear operator on X and let $\mu \in BV_\epsilon([0,\infty);\mathbb{C})$ for some $\epsilon \geq 0$. Let $k \in L^1_{loc}([0,\infty);\mathbb{C})$ with* abs$(k) < \infty$. *Let $0 < \theta_0 \leq \frac{\pi}{2}$ and let $\omega \geq \max\{\epsilon,\ \text{abs}(k)\}$ be a constant. Suppose that $(I - \widehat{d\mu}(\lambda)A)^{-1}$ exist in $L(X)$ for all $\lambda > \omega$ and that $\widehat{d\mu} \not\equiv 0$ on and $\widehat{k}$ has no zero in $\mathbb{C}_\omega$. Define $H : (\omega,\infty) \to L(X)$ by $H(\lambda) = \widehat{k}(\lambda)(I - \widehat{d\mu}(\lambda)A)^{-1}$. Then the following are equivalent.*

(i) The pair (A,μ) generates an analytic k-c.s.o.f. S of analyticity type $(\omega;\theta_0)$.
(ii) The function H has an analytic continuation to the sector $\Sigma_{\omega,\theta_0+\frac{\pi}{2}}$ such that $\sup\limits_{\lambda\in\Sigma_{\omega,\theta+\frac{\pi}{2}}} \|(\lambda-\omega)H(\lambda)\| < \infty$ for every $\theta \in (0,\ \theta_0)$.

PROOF. It follows from the remark following Definition 3.1 that the function H extends to the half plane $\mathbb{C}_\omega$ (we denote the extension of H by the same letter H unambiguously) and $H(\lambda)x = \widehat{S}(\lambda)x$ for every $x \in X$ and $\lambda \in \mathbb{C}_\omega$. Suppose that (i) holds. Then by Theorem 1.6, the function H also admits an analytic continuation to the sector $\Sigma_{\omega,\theta_0+\frac{\pi}{2}}$ such that $\sup\limits_{\lambda\in\Sigma_{\omega,\theta+\frac{\pi}{2}}} \|(\lambda-\omega)H(\lambda)\| < \infty$ for every $\theta \in (0,\ \theta_0)$. Thus, (i) implies (ii). Suppose that (ii) holds. It follows from Theorem 1.6 that there exists an analytic function $S : \Sigma_{0,\theta_0} \to L(X)$ such that $\sup\limits_{z\in\Sigma_{0,\theta}} \|e^{-\omega z}S(z)\| < \infty$ for every $\theta \in (0,\ \theta_0)$ and

$$H(\lambda)x = \widehat{k}(\lambda)(I - \widehat{d\mu}(\lambda)A)^{-1}x = \int_0^\infty e^{-\lambda t}S(t)x\,dt$$

for all $\lambda > \omega$ and $x \in X$. It follows from the exponential boundedness of S on subsectors $\Sigma_{0,\theta}$ that there exists a constant $M > 0$ such that

$\|S(t)\| \leq Me^{\omega t}$ for all $t > 0$. Thus,

$$\|\frac{1}{j!}H^{(j)}(\lambda)x\| = \|\frac{1}{j!}\int_0^\infty e^{-\lambda t}t^j S(t)xdt\| \leq M\int_0^\infty \frac{t^j}{j!}e^{-(\lambda-\omega)t}dt\|x\|$$
$$= M\frac{1}{(\lambda-\omega)^{j+1}}\|x\|$$

for all $x \in X$, $j \in \mathbb{N}_0$, and $\lambda > \omega$. It follows from Theorem 3.4 that there exists a k-c.s.o.f. S_1 with generator (A, μ). By the Uniqueness Theorem (Theorem 1.2), $S = S_1$ on $(0, \infty)$. Thus, $S|_{(0,\infty)}$ can be continuously extended to $[0, \infty)$ as $S(0) = S_1(0)$. Thus, S is an analytic k-c.s.o.f. with generator (A, μ). □

The following is an immediate consequence of Theorem 1.6 and the estimate (5.1) improves Corollary 2.1 in [12].

REMARK. Let S be an analytic c.s.o.f. of analyticity type $(\omega; \theta_0)$ with generator (A, μ). Then, for each $\theta \in (0, \theta_0)$, there exists a constant $C_\theta > 0$ such that

$$\|z^k S^{(k)}(z)\| \leq C_\theta e^{\omega \mathrm{Re}\, z}(|\omega||z| + 1)^k \tag{5.1}$$

for all $z \in \Sigma_{0,\theta}$ (See Theorem 1.6).

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