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A CHARACTERIZATION OF SUPERSOLVABLE SIGNED GRAPHS

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ABSTRACT. We characterize supersolvable signed graphs by using simplicial nodes and some signed graphs Δ_2, L_k.

1. Introduction

Signed graphs were studied by Zaslavsky [5]. Also a characterization of supersolvable graphs was proved by Stanley [3]. This paper will give a characterization of supersolvable signed graphs.

A *graph* (or *unsigned graph*) Γ consists of a set of nodes, $N(\Gamma)$, and a set of arcs, $E(\Gamma)$. There are two kinds of arcs. A *link*, written $e : vw$, has two distinct endpoints, v and w. A *half arc*, written $e : v$, has one endpoint, v, the other end tailing off into space. We use the term *ordinary graph* to mean there are no half arcs or parallel links. A *path* in a graph is a finite sequence of distinct links of the form $e_1 e_2 \ldots e_k$ where $e_i : v_{i-1} v_i$ for each $i = 1, 2, \ldots, k$. It is *closed* if $v_0 = v_k$. And a path is said to be *simple* if it has no parallel links.

A *signed graph* Σ consists of an unsigned graph, denoted $|\Sigma|$, and a partial mapping $\sigma : E(|\Sigma|) \to \{+, -\}$, the arc labeling, which is defined on all links. A *circle* is a simple closed path. Let $C = e_1 e_2 \ldots e_k$ be a circle in Σ. Then C is said to be *balanced* if $\sigma(e_1)\sigma(e_2) \ldots \sigma(e_k) = +$. A signed graph Σ is said to be *connected* if every two nodes are connected by a path in Σ.

We assume familiarity with the basic concepts of the theory of (combinatorial) geometries [4]. To fix our terminology, let G be a finite geometric lattice. Let S be the set of points (or atoms) in G. The lattice

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structure of G induces the structure of a (combinatorial) *geometry*, also denoted by G, on S. The cardinality $|G|$ of the geometry G is the cardinality of the set S of points.

In a geometry G with a rank function r, a flat x is *modular* if $r(x) + r(y) = r(x \wedge y) + r(x \vee y)$ for all flats y of G. And a flat x is a *copoint* if $r(x) = r(G) - 1$. A geometry G is *supersolvable* if its lattice G contains a maximal chain of modular flats.

Given a signed graph Σ, we can define a corresponding *signed-graphic geometry* $M(\Sigma)$ whose points are the arcs of Σ in the following way.

If Σ has r nodes, then label $N(\Sigma)$ by $\{1, 2, \dots, r\}$. Let $\{b_1, b_2, \dots, b_r\}$ be a basis for a projective geometry $PG(r-1, 3)$ of dimension $r-1$, representable over $GF(3)$. Consider a representation $f : M(\Sigma) \to PG(r-1, 3)$ such that $f(M(\Sigma))$ is a subset of $\{b_i, b_j \pm b_k;\ 1 \leq i \leq r, 1 \leq j < k \leq r\}$. Let $\varphi : E(|\Sigma|) \to M(\Sigma)$ be the bijection such that

(1) e is a half arc incident on a node $i \iff f(\varphi(e)) = b_i$;

(2) e is a link signed $\pm$ incident on two nodes $j, k \iff f(\varphi(e)) = b_j \mp b_k$.

A node v is called *simplicial* if the following conditions are satisfied ;

(1) If two non-parallel links e, f are adjacent on v, then there exists a link g such that $\{e, f, g\}$ is a balanced circle.

(2) If a half arc e and a link f are adjacent on v, then there exists a half arc g not equal to e such that f and g are adjacent.

(3) If two parallel links e, f are incident on v and w, then there exists a half arc g incident on w.

Suppose that Σ is a connected signed graph. Then the *rank* of Σ, denoted by $r(\Sigma)$, is $|N(\Sigma)| - 1$ if Σ has no half arcs or parallel links. Otherwise $r(\Sigma)$ is $|N(\Sigma)|$.

An ordinary graph is called *chordal* if every circle of length at least four contains a chord, that is, a link joining two nonconsecutive nodes of the circle. The following characterization of supersolvable graphic geometries was proved by Stanley [3].

THEOREM 1.1 [3]. *A graphic geometry $M(\Gamma)$ is supersolvable if and only if the ordinary graph Γ is chordal.*

PROOF. ($\Longrightarrow$) Use induction on the rank of $M(\Gamma)$.
($\Longleftarrow$) Use the following classical result. □

THEOREM 1.2 [1]. *Every chordal ordinary graph* Γ *has a simplicial node. Moreover, if* Γ *is not a clique, then it has two nonadjacent simplicial nodes.*

2. Main Results

When we say H is a modular copoint of Σ, it means that the set of arcs in H is a modular copoint of $M(\Sigma)$.

LEMMA 2.1. *Let* Σ *be a connected signed graph.*
(1) *If* Σ *has a simplicial node* v, *then* $\Sigma - v$ *is a modular copoint of* Σ.
(2) *Let* H *be a modular copoint of* Σ. *Then*

(i) *If* Σ *contains a node* v *not in* H, *then* v *is simplicial and* $H = \Sigma - v$.

(ii) *If* H *contains all nodes of* Σ, *then* H *is an ordinary graph.*

PROOF. (1) By the definition of a simplicial node v, the set of arcs in $\Sigma - v$ is a modular copoint of $M(\Sigma)$.

(2)(i) Since Σ is connected, there is a link e between v and a node w in H. If there is another link f between v and a node w' in H, then $\{e, f, g\}$ is a balanced circle for some link $g : ww'$ in H. If there is a parallel link f to e, then there exists a half arc g on w. If there is a half arc f on v, then there exists a half arc g on w. Thus v is a simplicial node and $H = \Sigma - v$.

(2)(ii) Note that H contains all nodes of Σ. If H contains a half arc or parallel links, then the rank of H is the rank of Σ, a contradiction. Thus H contains no half arcs or parallel links. □

Let P_n be the cycle matroid of K_n where K_n is the complete graph on n vertices. By Δ_2, we denote the signed graph of P_4 without half arcs. By L_k, we denote the union of two signed graphs of P_{k+1} without half arcs such that their intersection is the signed graph of P_k without half arcs. A signed graph is *supersolvable* if its geometry is supersolvable.

LEMMA 2.2. *Let* Σ *be a connected supersolvable signed graph. If* Σ *contains no simplicial nodes, then* Σ *is either* Δ_2 *or* L_k *for* $k = 2, 3, \ldots, r(\Sigma) - 1$.

PROOF. If Σ contains no half arcs, no parallel links, then Σ is an ordinary graph. Thus Σ contains a simplicial node by Theorem 1.1 and Theorem 1.2. Therefore Σ contains a half arc or parallel links.

Assume that Σ contains a half arc. Let H be a modular copoint of Σ. Then Lemma 2.1 implies that H is an ordinary graph. By Theorem 1.2, either H is complete or H has two nonadjacent simplicial nodes. If H is complete, then each node is simplicial in H. Since no node is simplicial in Σ, every half arc is incident on all nodes of Σ, a contradiction. If H has two nonadjacent simplicial nodes, then these two nodes have half arcs. But H has no link between those half arcs. Thus Σ does not contain half arcs.

If Σ contains only one parallel-link set, then Σ is not connected. Thus Σ contains at least two parallel-link sets. Let H be a modular copoint of Σ. If two parallel-link sets are disjoint, then H is not modular. Thus every two parallel-link sets are adjacent. If parallel-link sets are not incident on some node, then that node would be simplicial. Thus the set of parallel-link sets span Σ. It follows that H is complete and Σ is either Δ_2 or L_k for $k = 2, 3, \ldots, r(\Sigma) - 1$. □

By Lemma 2.1 and Lemma 2.2, we have the following characterization for supersolvable signed graphs.

THEOREM 2.3. *Σ is a supersolvable signed graph if and only if one of the following conditions holds :*
(a) The signed graph Σ has a sequence of simplicial nodes $v_2, v_3, \ldots, v_{r(\Sigma)}$ such that $H_i = H_{i+1} - v_{i+1}$ is a rank-i subgraph for $i = 1, 2, \ldots, r(\Sigma) - 1$, or
(b) There exists a sequence of subgraphs $H_1, H_2, \ldots, H_{r(\Sigma)}$ and a unique integer j, $2 \leq j \leq r(\Sigma) - 1$ such that for all $i \neq j$, there exists a simplicial node v_{i+1} with $H_i = H_{i+1} - v_{i+1}$, and $H_j = \Delta_2$ or L_{j-1}.

Kahn and Kung [2] defined splitting in geometries. Let G be a geometry. Then G *splits* if G is the union of two of its proper flats. G is said to be *non-splitting* otherwise. To describe non-splitting supersolvable signed graphs, we define the *adjacency set* of a node v, denoted by $Adj(v)$, by the following conditions.

A half arc e on a node w is in $Adj(v) \iff$ either v, w are joined by a pair of parallel links or a half arc on v is joined by a link to e.

A link e is in $Adj(v) \iff$ there are two non-parallel links f, g adjacent on v such that $\{e, f, g\}$ is a balanced circle.

COROLLARY 2.4. *Σ is a non-splitting supersolvable signed graph if and only if a signed graph Σ has a sequence of simplicial nodes v_{i+1} in $H_{i+1} - H_i$ for $i = 1, 2, \ldots, r(\Sigma) - 1$ such that each H_i has rank i with $H_i = H_{i+1} - v_{i+1}$ and $r[Adj(v_{i+1})] = i$ except having at most one H_3 by Δ_2.*

COROLLARY 2.5. *A non-splitting supersolvable graphic geometry is a cycle matroid of a complete graph.*

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