구형 공동을 가진 횡 방향 등방성매체의 응력 분포에 관한 연구

Investigation of the Stress Distributions in a Transversely Isotropic Medium Containing a Spheroidal Cavity

이 윤 복* Rhie, Yoon-Bock　　전 종 균** Chun, Chong-Keun

요 약

본 연구에서는 구형 공동을 가진 횡 방향 등방성 매체(transversely isotropic medium)에 한 방향의 인장과 평면 전단력을 각각 가한 후의 응력 분포를 고찰하였다. 이 연구에서 사용된 접근 방법은 이론적 해석과 수치해석적 고찰을 병행하였다. 이론적 해석 방법은 1종, 2종 Legendre 함수를 이용한 potential function에 기초하였고, 수치해석적 방법은 유한차분법을 이용하였다. 다섯 가지 이방성 재료에 대하여 두 가지 하중 조건에 의한 수치해석 결과가 제시되었다.

Abstract

This study investigates the stress distribution in a transversely isotropic medium containing a spheroidal cavity where the medium is under uniaxial tension in z-direction in one case and pure shear in the plane of isotropy in another case. The technical approach used in this study combines exact analytical and numerical methods. The exact analytical method is based upon three potential functions taken in terms of the Legendre associated functions of the first and second kind. The numerical method is based upon the finite difference approach. Numerical results concerning the two loading conditions with five anisotropic materials are presented.

Keywords : stress distribution, transversely isotropic medium, spheroidal cavity, stress concentration factor(SCF), legendre potential function

1. INTRODUCTION

Three-dimensional stress concentration problems have been the subject of many investigation during the past hundred years or so. Investigators have successfully obtained exact solutions of many boundary value problems associated with a variety of anisotropic materials. Among these materials are those which are classified to be transversely isotropic. Exam-

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ples of transversely isotropic materials are crystalline materials, i.e., graphite, zinc, magnesium, beryllium, cadmium, titanium, cobalt, ice, honeycomb-structured materials, and laminate composites. Stratified rock and soils can also be modified as transversely isotropic(Fig. 1.1).

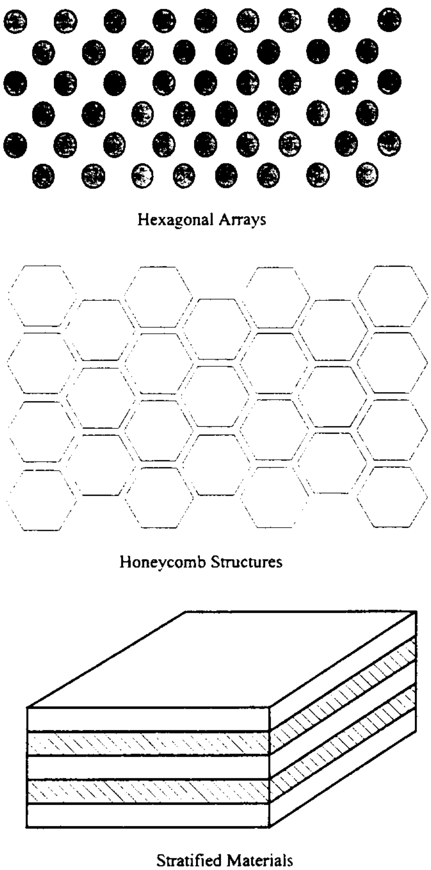

Fig. 1.1 Transversely Isotropic Materials

Stress concentration problems play an important role in studying fracture of materials. Thus, a considerable number of articles have been published on the phenomenon of stress concentration in elastic media during the past 15 decades[1-10, 13, 15]. The survey paper by Sternberg[17] summarizes most of these investigations prior to 1947, while the report of Zureik and Eubanks[19] reviews recent work related to the three dimensional stress concentration problems in infinite media containing spheroidal cavity.

The finite difference method is used to find the maximum stress and to investigate the stress distributions. This method is widely used for the numerical solution of a differential equation. The objective of this study is to investigate the stress distribution in a transversely isotropic medium containing a spheroidal cavity under various loading conditions for several transversely isotropic materials.

2. TRANSVERSELY ISOTROPIC MATERIALS

2.1. Stress-Strain Relationship

The stress-strain relationship for a transversely isotropic medium may be written as follows;

$$\{\sigma\} = [C]\{\varepsilon\} \tag{2.1}$$

where,

$\{\sigma\}$; A column matrix representing the stress components

$[C]$; A 6X6 matrix representing the elastic constants

$\{\varepsilon\}$; A column matrix representing the strain components

Equation(2.1) can be expressed explicitly in the following form

$$\begin{bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{zz} \\ \sigma_{\theta z} \\ \sigma_{rz} \\ \sigma_{r\theta} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{rr} \\ \varepsilon_{\theta\theta} \\ \varepsilon_{zz} \\ \gamma_{\theta z} \\ \gamma_{rz} \\ \gamma_{r\theta} \end{bmatrix} \tag{2.2}$$

where, c_{11}, c_{12}, c_{13}, c_{33} and are c_{44} five independent elastic constants characterizing the medium and(σ_{rr}, $\sigma_{\theta\theta}$, σ_{zz}, σ_{rz}, $\sigma_{r\theta}$) are the stress components. We may write for convenience, $c_{66}=\frac{1}{2}(c_{11}-c_{12})$.

In certain applications of the theory of elasticity of transversely isotropic materials, Eqs. (2.1) and (2.2) may be written in the form

$$\begin{bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{zz} \\ \sigma_{\theta z} \\ \sigma_{rz} \\ \sigma_{r\theta} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\ a_{12} & a_{11} & a_{13} & 0 & 0 & 0 \\ a_{13} & a_{13} & a_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{rr} \\ \varepsilon_{\theta\theta} \\ \varepsilon_{zz} \\ \gamma_{\theta z} \\ \gamma_{rz} \\ \gamma_{r\theta} \end{bmatrix} \quad (2.3)$$

where the coefficients a_{11}, a_{12}, a_{13}, a_{33} and a_{44} are called "compliances". We may also write, for convenience, $a_{66}=2(a_{11}-a_{12})$. It is evident from Eqs.(2.2) and (2.3) that the elastic constants (stiffness) can be converted to the compliances and vice versa[14]. By doing so, we obtain

$$\begin{aligned} a_{11} &= \frac{(c_{11}c_{33}-c_{13}^2)}{(c_{11}-c_{12})(c_{11}c_{33}+c_{12}c_{33}-2c_{13}^2)} \\ a_{12} &= \frac{(c_{11}c_{33}-c_{13}^2)}{(c_{11}-c_{12})(c_{11}c_{33}+c_{12}c_{33}-2c_{13}^2)} \\ a_{33} &= \frac{c_{13}}{(c_{11}-c_{12})(c_{11}c_{33}+c_{12}c_{33}-2c_{13}^2)} \\ a_{44} &= \frac{1}{c_{44}} \\ a_{66} &= 2(a_{11}-a_{12}) = \frac{1}{c_{66}} = \frac{2}{c_{11}-c_{12}} \end{aligned} \quad (2.4)$$

and

$$\begin{aligned} c_{11} &= \frac{(a_{11}a_{33}-a_{13}^2)}{(a_{11}-a_{12})(a_{11}a_{33}+a_{12}a_{33}-2a_{13}^2)} \\ c_{12} &= \frac{(a_{11}a_{33}-a_{13}^2)}{(a_{11}-a_{12})(a_{11}a_{33}+a_{12}a_{33}-2a_{13}^2)} \\ c_{33} &= \frac{a_{13}}{(a_{11}-a_{12})(a_{11}a_{33}+a_{12}a_{33}-2a_{13}^2)} \\ c_{44} &= \frac{1}{a_{44}} \\ c_{66} &= \frac{1}{2}(a_{11}-a_{12}) = \frac{1}{a_{66}} = \frac{1}{2(a_{11}-a_{12})} \end{aligned} \quad (2.5)$$

The elastic constants and the compliances can be written in terms of the engineering constants which can be interpreted to be Young's moduli, the shear moduli, and Poisson's ratios associated with various directions. If we introduce these constants which are related to the compliances by:

$$\begin{aligned} & a_{11} = \frac{1}{E}, \quad a_{12} = -\frac{\nu}{E}, \quad a_{13} = \frac{\nu}{E} \\ & a_{33} = \frac{1}{E}, \quad a_{44} = \frac{1}{G}, \quad \left(a_{66} = \frac{1}{G} = \frac{2(1+\nu)}{E}\right) \end{aligned} \quad (2.6)$$

Eq.(2.3) takes the form

$$\begin{bmatrix} \varepsilon_{rr} \\ \varepsilon_{\theta\theta} \\ \varepsilon_{zz} \\ \gamma_{\theta z} \\ \gamma_{rz} \\ \gamma_{r\theta} \end{bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & -\frac{\nu}{E} & -\frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{2(1+\nu)}{E} \end{bmatrix} \begin{bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{zz} \\ \sigma_{\theta z} \\ \sigma_{rz} \\ \sigma_{r\theta} \end{bmatrix} \quad (2.7)$$

where

E and E are Young's moduli in the plane of isotropy and perpendicular to it respectively.

ν is Poisson's ratio characterizing transverse contraction in the plane of isotropy when tension is applied in this plane.

ν is Poisson's ratio characterizing transverse contraction in the plane of isotropy when tension is applied in a direction normal to the plane of isotropy.

G and $\bar{G}$ are shear moduli for the plane of isotropy and any plane perpendicular to it respectively.

The engineering constants and the elastic constants can be related to each other as follows:

$$\begin{aligned} E &= 4c_{66}\left(1-\frac{c_{33}c_{66}}{c_{11}c_{33}-c_{13}^2}\right) \\ \bar{E} &= c_{33}-\frac{c_{13}^2}{c_{11}c_{33}-c_{13}^2} \\ \nu &= 1-\frac{2c_{33}c_{66}}{c_{11}c_{33}-c_{13}^2} \\ \bar{\nu} &= \frac{c_{13}}{c_{11}+c_{12}} \\ G &= c_{44} \\ \bar{G} &= \frac{E}{2(1+\upsilon)} = c_{66} \end{aligned} \tag{2.8}$$

For isotropic materials, the engineering constants and the elastic constants are reduced to

$$\bar{E} = E,\ \bar{\nu} = \nu,\ \bar{G} = G = \frac{E}{(1+\nu)} \tag{2.9a}$$

and

$$\begin{aligned} c_{13} &= c_{33} = \lambda+2\mu = \frac{2(1-\nu)\mu}{1-2\nu} \\ c_{12} &= c_{13} = \lambda = \frac{2\nu\mu}{1-2\nu} \\ c_{44} &= c_{66} = \mu = \frac{E}{2(1+\nu)} \end{aligned} \tag{2.9b}$$

where λ and μ are the well known Lame's constants, is Poisson's ratio, and E is Young's moduli. For positive definiteness of the strain energy density, the following conditions must be satisfied

$$\begin{aligned} &c_{11}>|c_{12}|>0,\quad c_{33}>0,\quad c_{44}>0,\quad c_{66}>0 \\ &c_{33}(c_{11}+c_{12})-2c_{13}^2 = c_{11}c_{33}-c_{13}^2-c_{13}c_{66}>0 \end{aligned} \tag{2.10a}$$

or

$$\begin{aligned} &a_{11}>|a_{12}|>0,\quad a_{33}>0,\quad a_{44}>0,\quad a_{66}>0 \\ &a_{33}(a_{11}+a_{12})-2a_{13}^2 = a_{11}a_{33}-a_{13}^2-a_{13}a_{66}>0 \end{aligned} \tag{2.10b}$$

or, equivalently,

$$\begin{aligned} &E>0,\quad \bar{E}>0,\quad \bar{G}>0,\quad \bar{G}=\frac{E}{2(1+\nu)}>0, \\ &(1-\nu)-2\bar{\nu}^2\frac{E}{\bar{E}}>0,\quad -1<\nu<1 \end{aligned} \tag{2.10c}$$

Negative values of Poissons ratio do theoretically exist but they may not be physically attainable. They have been reported indirectly by Hearmon[12] for zinc and cadmium sulfide. This data is questionable, since virtually all real materials have positive values of Poissons ratio. By equating the last equality of Eq.(2.10c), we can obtain the bounding surface between $\nu=1$ and $\nu=-1$ in $E/\bar{E}$, ν, $\bar{\nu}$ space. If we ignore the negative values of Poissons ratio, Eq.(2.10c) become:

$$\begin{aligned} &E>0,\quad \bar{E}>0,\quad \bar{G}>0, \\ &0<\nu<1-2\bar{\nu}^2\frac{E}{\bar{E}} \end{aligned} \tag{2.11}$$

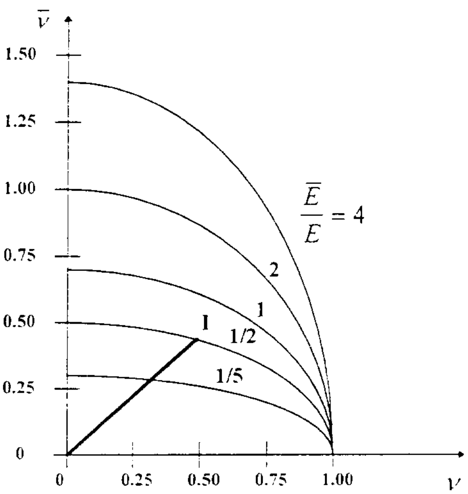

Fig. 2.1 Variation of the Engineering Constants

and the last inequality of Eq.(2.11) is plotted in Fig. 2.1. The interior area bounded by a parabola corresponding to a specific value of $E/\bar{E}$ represents the valid domain of variation of Poisson's ratio. Line OI represents the isotropic materials. It is noteworthy that for values of $E/\bar{E}>2$, Poisson's ratio $\bar{\nu}$ might exceed unity. For isotropic material, these ranges reduce to:

$$E>0,\quad 0\leq\nu\leq\frac{1}{2} \tag{2.12}$$

2.2. Value of the Elastic Constants

The numerical values of the elastic constants for a variety of different hexagonal crystalline and non-crystalline materials are reported by Hearmon(1961)[12]. Some of these materials are listed in Table 2.1 and 2.2. In both tables, the unit used for the elastic constants is gigapascal(GPa=10 dyne /cm^2) and the unit used for the compliances is the reciprocal of terapascal(TPa=10^3GPa).

3. BASIC FORMULAE FOR STRESSES AND FINITE DIFFERENCE FORMULATION

Consider a homogeneous transversely isotropic elastic medium occupying a region of the three-dimensional space referred to a fixed Cartesian coordinate system(x, y, z) in which the z-axis coincides with the axis of elastic sym-

Table 2.1 Elastic Constants for Hexagonal systems

Materials		Values of the elastic constants(Tpa)				
		c_{11}	c_{12}	c_{13}	c_{33}	c_{44}
(a) Crystals						
Beryllium, Be		292.0	24.0	6.0	349.0	163.0
Cadmium, Cd		116.0	42.0	41.0	50.9	19.6
Cobalt, Co		295.0	159.0	111.0	335.0	71.0
Graphite, C		1060.0	180.0	15.0	36.5	4.0
Hafnium, Hf		181.0	77.0	66.0	197.0	55.7
Ice, H_2O at(-16℃)		13.5	6.5	5.9	14.9	3.09
Magnesium, Mg		59.3	25.7	21.4	61.5	16.4
Quarts, β-SiO		117.0	16.0	33.0	110.0	36.0
Rhenium, Re		616.0	273.0	206.0	683.0	161.0
Silver Aluminum, Ag Al		142.0	85.0	75.0	168.0	34.1
Titanium, Ti		160.0	90.0	66.0	181.0	46.5
Zinc, Zn		165.0	31.1	50.0	61.8	39.6
Zinc Oxide(Zincite), ZnO		209.0	120.0	104.0	218.0	44.1
Zirconium, α-Zr		144.0	74.0	67.0	166.0	33.4
(b) Non-Crystalline						
Al-CuAl$_2$ composite		123.0	60.0	60.0	123.0	29.7
Bone(dried phalanx)		21.2	9.5	10.2	37.4	7.5
Bone(fresh phalanx)		19.7	12.1	12.6	32.0	5.4
Bone(dried femur)		23.8	10.2	12.2	33.4	8.2
Ceramics BatiO$_3$ (Barium-Titanate)		166.0	77.0	78.0	162.0	43.0
(c) Fiber Reinforced Resins						
Carbon fiber/epoxy resin						
Type of Fiber	Type of Fraction					
Graphite Thornel 50	0.55	10.0	4.8	5.6	186.0	5.9
Graphite Thornel 75	0.56	9.2	5.0	5.2	309.0	6.3

Table 2.2 Compliances for Hexagonal Systems

Materials		Values of the elastic constants(Tpa)				
		a_{11}	a_{12}	a_{13}	a_{33}	a_{44}
(a) Crystals						
Beryllium, Be		3.45	-0.28	-0.05	2.87	6.13
Cadmium, Cd		12.19	-1.32	-8.76	33.76	51.02
Cobalt, Co		4.99	-2.36	-0.87	3.56	14.08
Graphite, C		0.98	-0.16	-0.33	27.67	250.0
Hafnium, Hf		7.15	-2.47	-1.57	6.13	17.95
Ice, H_2O at(-16℃)		104.05	-38.81	-25.83	87.57	323.62
Magnesium, Mg		22.01	-7.75	-4.96	19.71	60.98
Quarts, β-SiO		9.37	-0.53	-2.65	10.68	27.78
Rhenium, Re		2.11	-0.80	-0.39	1.70	6.21
Silver Aluminum, Ag Al		11.90	-5.65	-2.79	8.44	29.33
Titanium, Ti		9.62	-4.67	-1.81	6.84	21.51
Zinc, Zn		8.07	0.61	-7.02	27.55	25.25
Zinc Oxide(Zincite), ZnO		7.79	-3.44	-2.08	6.57	22.68
Zirconium, α-Zr		10.19	-4.09	-2.46	8.01	29.94
(b) Non-Crystalline						
Al-CuAl$_2$ composite		11.95	-3.92	-3.92	11.95	33.67
Bone(dried phalanx)		62.63	-22.84	-10.85	32.66	133.33
Bone(fresh phalanx)		88.64	-42.93	-18.0	45.42	185.19
Bone(dried femur)		55.64	-17.89	-12.66	38.43	121.95
Ceramics BatiO$_3$ (Barium-Titanate)		8.60	-2.64	-2.87	8.93	23.26
(c) Fiber Reinforced Resins						
Carbon fiber/epoxy resin						
Type of Fiber	Type of Fraction					
Graphite Thornel 50	0.55	130.73	-61.58	-2.08	5.50	169.49
Graphite Thornel 75	0.56	154.70	-83.40	-1.20	3.28	158.73

metry of the material. In the absence of body forces, solutions of three-dimensional asymmetric problems can be expresses in terms of three potential functions ϕ_1(x, y, z), ϕ_2(x, y, z), and Ψ (x, y, z), where the displacement components (u_x, u_y, u_z) can be expressed as follows:

$$u_x = \frac{\partial\phi_1}{\partial x} + \frac{\partial\phi_2}{\partial x} + \frac{\partial\Psi}{\partial y}$$
$$u_y = \frac{\partial\phi_1}{\partial y} + \frac{\partial\phi_2}{\partial y} + \frac{\partial\Psi}{\partial x} \qquad (3.1)$$
$$u_z = k_1\frac{\partial\phi_1}{\partial z} + k_2\frac{\partial\phi_2}{\partial z}$$

These potential functions satisfy the equations

$$\nabla_1^2\phi_1 = \nabla_2^2\phi_2 = \nabla_3^2\Psi = 0 \qquad (3.2)$$

in which

$$\nabla_j^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \nu_j^2\frac{\partial^2}{\partial z^2} \qquad (3.3)$$

ν_1^2 and ν_2^2 and are the roots of the equation

$$c_{11}c_{44}\nu^4+[c_{13}(2c_{44}+c_{13})-c_{11}c_{33}]\nu^2+c_{33}c_{44}=0 \qquad (3.4)$$

while ν_3^2 is defined by

$$\nu_3^2 = \frac{2c_{44}}{c_{11}-c_{12}} = \frac{c_{44}}{c_{66}} \qquad (3.5)$$

Here, c_{11}, c_{12}, c_{13}, c_{33}, *and* c_{44} are five independent constants characterizing the transversely isotropic medium. The constants ν_1^2 and ν_2^2 and are either real or complex conjugate(with a real part different from zero) depending upon the values of the elastic constants, but the constant ν_3^2 is always real and positive. We also specify that ν_1, ν_2, and ν_3 always have positive real parts.

In these equations, k_1, k_2 are the roots of the equation[11]

$$c_{44}(c_{13}+c_{44})k_2+[(c_{13}+c_{44})^2+c_{44}^2-c_{11}c_{33}]$$
$$k+c_{44}(c_{44}+c_{13})=0 \qquad (3.6)$$

The constants k_1 and k_2 are related to ν_1^2 and ν_2^2 and respectively by

$$k_j = \frac{c_{11}\nu_j^2-c_{44}}{c_{13}+c_{44}} = \frac{\nu_j^2(c_{13}+c_{44})}{c_{33}-c_{44}\nu_j^2} \qquad (3.7a)$$

or equivalently

$$\nu_j^2 = \frac{k_j(c_{13}+c_{44})+c_{44}}{c_{11}} = \frac{k_jc_{33}}{c_{44}(1+k_j)+c_{13}} \qquad (3.7b)$$

Sometimes, it is convenient to employ three new variable defined by

$$z_j = \frac{z}{\nu_j} \quad (j=1,2,3) \qquad (3.8)$$

The stress components(σ_{xx}, σ_{yy}, σ_{zz}, σ_{yz}, σ_{xz}, σ_{xy}) can be expressed in terms of ϕ_1, ϕ_2, and Ψ as follows.

$$\sigma_{xx} = -c_{44}(1+k_1)\frac{\partial^2\phi_1}{\partial z^2}-c_{44}(1+k_2)\frac{\partial^2\phi_2}{\partial z^2}$$
$$-(c_{11}-c_{12})\frac{\partial^2}{\partial y^2}(\phi_1+\phi_2)+(c_{11}-c_{12})\frac{\partial^2\Psi}{\partial x\partial y}$$
$$\sigma_{yy} = -c_{44}(1+k_1)\frac{\partial^2\phi_1}{\partial z^2}-c_{44}(1+k_2)\frac{\partial^2\phi_2}{\partial z^2}$$
$$-(c_{11}-c_{12})\frac{\partial^2}{\partial y^2}(\phi_1+\phi_2)+(c_{11}-c_{12})\frac{\partial^2\Psi}{\partial x\partial y}$$
$$\sigma_{zz} = c_{44}\left[\nu_1^2(1+k_1)\frac{\partial^2\phi_1}{\partial z^2}+\nu_2^2(1+k_2)\frac{\partial^2\Psi}{\partial z^2}\right]$$

$$\sigma_{yz} = c_{44}\,[(1+k_1)\frac{\partial^2\phi_1}{\partial y\partial z}+(1+k_2)\frac{\partial^2\phi_2}{\partial y\partial z}-\frac{\partial^2\Psi}{\partial x\partial z}]$$

$$\sigma_{xz} = c_{44}\,[(1+k_1)\frac{\partial^2\phi_1}{\partial x\partial z}+(1+k_2)\frac{\partial^2\phi_2}{\partial x\partial z}+\frac{\partial^2\Psi}{\partial y\partial z}]$$

$$\sigma_{xy} = (c_{11}-c_{12})[\frac{\partial^2}{\partial x\partial y}(\phi_1+\phi_2)+\frac{1}{2}(\frac{\partial^2\Psi}{\partial y^2}-\frac{\partial^2\Psi}{\partial x^2})] \qquad (3.9)$$

By using finite difference method, the second partial derivatives in the above equations can be explained by the following procedures[18]. For the purpose of illustration, consider a cubic net with spheroidal cavity by taking $\Delta x=\Delta y=\Delta z=H$ (Fig. 3.1). The equations are

$$\frac{\partial^2\phi_n}{\partial x^2} = \frac{(\phi_n)_{i+1,j,k}-2(\phi)_{i,j,k}+(\phi_n)_{i-1,j,k}}{H^2}$$

$$\frac{\partial^2\phi_n}{\partial y^2} = \frac{(\phi_n)_{i,j+1,k}-2(\phi)_{i,j,k}+(\phi_n)_{i,j-1,k}}{H^2}$$

$$\frac{\partial^2\phi_n}{\partial x^2} = \frac{(\phi_n)_{i,j,k+1}-2(\phi)_{i,j,k}+(\phi_n)_{i,j,k-1}}{H^2}$$

$$(n=1,2) \qquad (3.10)$$

$$\frac{\partial^2\phi_n}{\partial x\partial z} = \frac{(\phi_n)_{i+1,j,k+1}-(\phi)_{i-1,j,k+1}-(\phi)_{i+1,j,k-1}+(\phi)_{i-1,j,k-1}}{4H^2}$$

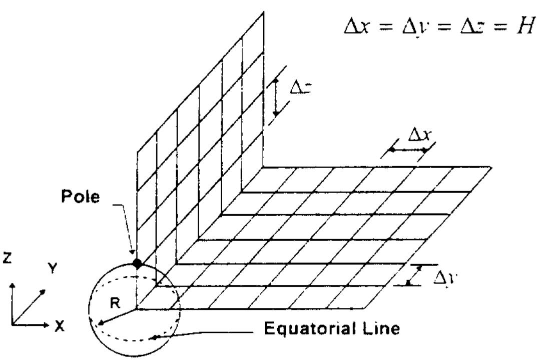

Fig. 3.1 The cubic net with a spheroidal cavity for finite difference method

$$\frac{\partial^2\phi_n}{\partial y\partial z} = \frac{(\phi_n)_{i,j+1,k+1}-(\phi)_{i,j-1,k+1}-(\phi)_{i,j+1,k-1}+(\phi)_{i,j-1,k-1}}{4H^2}$$

$$\frac{\partial^2\phi_n}{\partial x\partial y} = \frac{(\phi_n)_{i+1,j+1,k}-(\phi)_{i-1,j+1,k}-(\phi)_{i+1,j-1,k}+(\phi)_{i-1,j-1,k}}{4H^2}$$

Similar expressions can be derived from other derivations.

4. PROBLEM DESCRIPTION FOR INFINITE MEDIUM CONTAINING A SPHEROIDAL CAVITY

4.1. Statement of Problem

Consider an infinite elastic transversely isotropic medium, as shown in Fig. 4.1, containing a spheroidal inclusion whose surface S is defined by

$$\frac{z^2}{a^2} + \frac{r^2}{b^2} = 1 \qquad (4.1)$$

where a and b are two semi-axes of the spheroid.

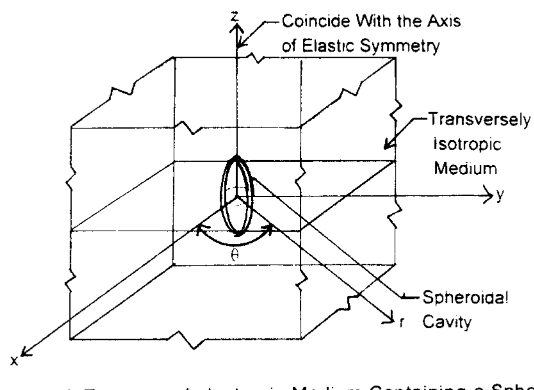

Fig. 4.1 Transversely Isotropic Medium Containing a Spheroidal Cavity

In this problem, the medium is subjected to uniaxial tension which act in z-direction and pure shear stress in plane of isotropy. It is required to determine the stress concentration factor(SCF) throughout the transversely isotropic medium.

4.2. The Coordinate System

Since we have a problem in which the natural boundaries are spheroids with center at the origin of the coordinates, it is suitable to employ the spheroidal coordinate systems[16] defined by

$$x = a_j\left(q_j^2-1\right)^{\frac{1}{2}}\left(1-p_j^2\right)^{\frac{1}{2}}\cos\theta$$
$$y = a_j\left(q_j^2-1\right)^{\frac{1}{2}}\left(1-p_j^2\right)^{\frac{1}{2}}\sin\theta \qquad (4.2a)$$
$$z_j = a_j q_j p_j$$

or equivalently,

$$r = a_j\left(q_1^2-1\right)^{\frac{1}{2}}\left(1-p_j^2\right)^{\frac{1}{2}}$$
$$z_j = a_j q_j p_j \qquad (4.2b)$$

where q_j and p_j are parameters which can be determined for any point whose coordinates(r, z) are known. The parameters q_j and p_j can be expressed in terms of r and z as follows;

$$q_j = \sqrt{Q_j}$$
$$p_j = \frac{z}{a_j\nu_j q_j} \qquad (4.3a)$$

where

$$Q_j = \frac{(a_j^2+r^2\nu_j^2z^2)+\sqrt{(a_j^2\nu_j^2+r^2\nu_j^2z^2)-4z^2(a_j^2\nu_j^2)}}{2a_j^2\nu_j^2}$$
$$(j=1, 2, 3) \qquad (4.3b)$$

Let ρ_j denote the value q_j on the spheroidal surface, then the three coordinate systems coincide on the surface of spheroid if the following equalities are satisfied

$$a_1^2\nu_1^2\rho_1^2 = a_2^2\nu_2^2\rho_2^2 = a_3^2\nu_3^2\rho_3^2 = a^2$$
$$a_1^2(\rho_1^2-1) = a_2^2(\rho_2^2-1) = a_3^2(\rho_3^2-1) = b^2 \qquad (4.4)$$

form which we obtain

$$a_j^2 = \frac{a^2}{\nu_j^2}-b^2$$
$$\rho_j^2 = \frac{a^2}{a^2-b^2\nu_j^2} \qquad (j=1, 2, 3) \qquad (4.5)$$

4.3. Loading Conditions

The first boundary value problem consists of finding stresses and displacements of an elastic body in equilibrium when the body forces are known and the displacements of the body are prescribed, and the second boundary value problem does when the body forces are known and the surface forces are prescribed.

In this chapter, we will consider the second boundary problem of a transversely isotropic medium containing a spheroidal cavity when the medium is subjected, at large distances from the cavity, to the following loading conditions. For the simplicity, the spheroidal cavity is changed to a sphere by $a=b$ from Eq.(4.1).

a) Uniaxial Tension in the z-direction

b) Pure Shear Stress in the Plane of Isotropy

4.3.1. Uniaxial Tension in the z-direction

This problem is an axisymmetric one in which $\sigma_{r\theta}=\sigma_{\theta z}=0$. Therefore, the boundary conditions in this directions in this case are

$$\sigma_{zz} = \tau_0,\ \sigma_{rr} = \sigma_{\theta\theta} = \sigma_{rz} = 0 \quad \text{at infinity}$$
$$t_r = t_z = 0 \quad \text{on the spheroidal surface} \qquad (4.6)$$

The potential functions in this case are

$$\phi_j = \frac{2a_j}{3} A_{j10}[P_2(p_j)Q_2(q_j) - Q_2(q_j)] \quad (n=1, 2) \tag{4.7}$$

where

$$A_{110} = \frac{\tau_0 b}{2(k_2-k_1)\Delta}\left[-\nu_3^2\frac{(1+k_2)}{\nu_2}Q_1(\rho_2)+\frac{a}{b}Q_1^1(\rho_2)\right]$$

$$A_{210} = \frac{-\tau_0 b}{2(k_2-k_1)\Delta}\left[-\nu_3^2\frac{(1+k_2)}{\nu_2}Q_1(\rho_1)+\frac{a}{b}Q_1^1(\rho_1)\right]$$

$$\Delta = c_{44}\frac{a}{b}Q_1^1(\rho_1)\,Q_1^1(\rho_2)\,F_1$$

$$F_1 = 1+\frac{bv_3^2(1+k_1)\,(1+k_2)}{a(k_2-k_1)}\left[\frac{Q_1(\rho_1)}{\nu_2\,Q_1^1(\rho_2)} - \frac{Q_1^1(\rho_1)}{\nu_1\,Q_1^1(\rho_1)}\right] \tag{4.8}$$

4.3.2. Pure Shear Stress in the Plane of Isotropy

Boundary conditions in this case are

$$\sigma_{xy} = \tau_0,\ \sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0 \quad \text{at infinity} \tag{4.9}$$

$t_x = t_y = t_z = 0$ on the spheroidal surface

The potential functions in this case are

$$\phi_j = \frac{2a_j}{3} B_{j12}[P_2^{-2}(p_j)Q_2^2)(q_j) - P_0^{-2}(p_j)Q_0^2(q_j)]\sin 2\theta$$

$$\psi_j = \frac{2a_j}{3} B_{312}[P_2^{-2}(p_3)Q_2^2)(q_3) - P_0^{-2}(p_3)Q_0^2(q_3)]\cos 2\theta \quad (j=1, 2) \tag{4.10}$$

where

$$B_{112} = -\frac{k_1 v_1 ab^2}{a_2^2(k_1-k_2)F_3}\frac{\tau_0}{c_{44}}$$

$$B_{212} = -\frac{k_1 v_1 ab^2}{a_2^2(k_1-k_2)F_3}\frac{\tau_0}{c_{44}}$$

$$B_{312} = -\frac{\nu_3 ab^2}{a_3^2 F_3}\frac{\tau_0}{c_{44}}$$

$$F_3 = -2 - \frac{ab}{\nu_3}\frac{Q_1^1(\rho_3)}{a_3^2} - \frac{ab}{\nu_3^2}\frac{k_1 v_2 Q_1^1(\rho_2)}{a_3^2(k_1-k_2)} + \frac{ab}{\nu_3^2}\frac{k_2 \nu_1 Q_1^1(\rho_1)}{a_3^2(k_1-k_2)} \tag{4.11}$$

5. RESULTS

5.1. Uniaxial Tension in z-direction

This problem was first investigated by Chen[3] who evaluated the stress concentration factor(SCF) for a few transversely isotropic materials. We have calculated the tensile and compressive SCF for a wide range of transversely isotropic materials. SCF is defined by the ratio of the resultant versus the applied stress. Results are presented for some particular materials. Some observations are summarized as follows;

1) As we expected, for almost all materials studied, the maximum tensile principal stress occurs at the equatorial line while the maximum compressive principal stress occurs at the pole.

2) For Graphite Thornel 50 and 75, which are highly anisotropic materials, much higher tensile stresses than those of other materials considered were observed.

Variations of principal stresses for five transversely isotropic materials are shown in Fig. 5.1

5.2. Pure Shear Stress in the Plane of Isotropy

In as much as previous case, observations for results are summarized as follows,

1) The maximum principal stresses occur at $\theta=90°$ for all considered materials except Grap-

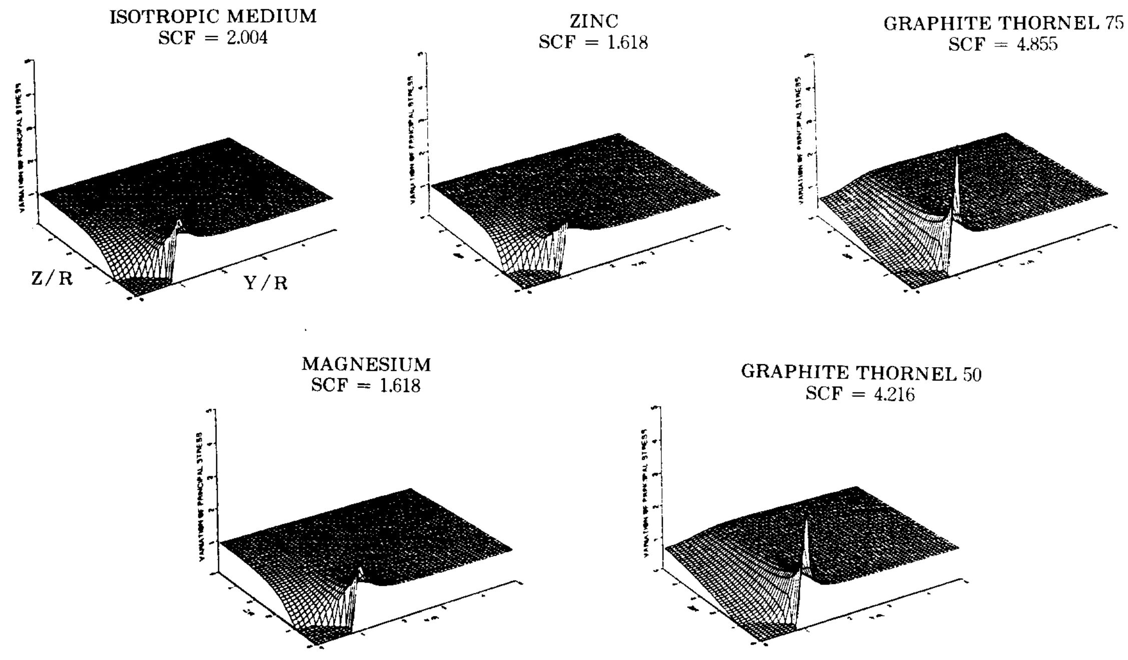

Fig. 5.1 Principal Stress under Uniaxial Tension in z-direction

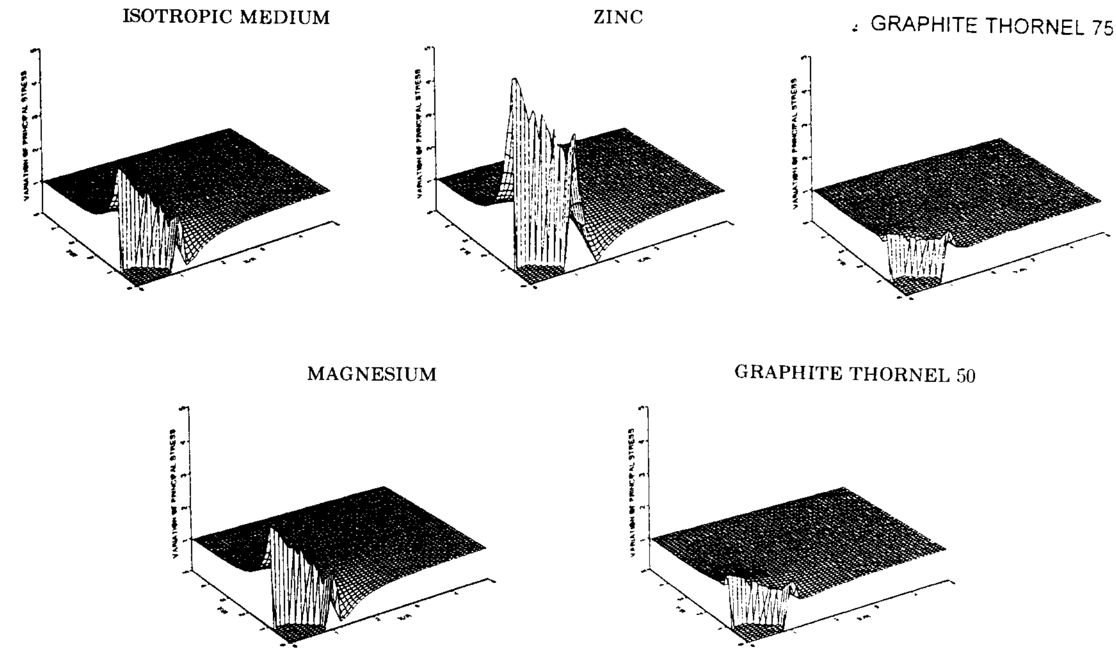

Fig.5.2.1 Principal Stress under Pure Shear Stress in the Plane of Isotropy

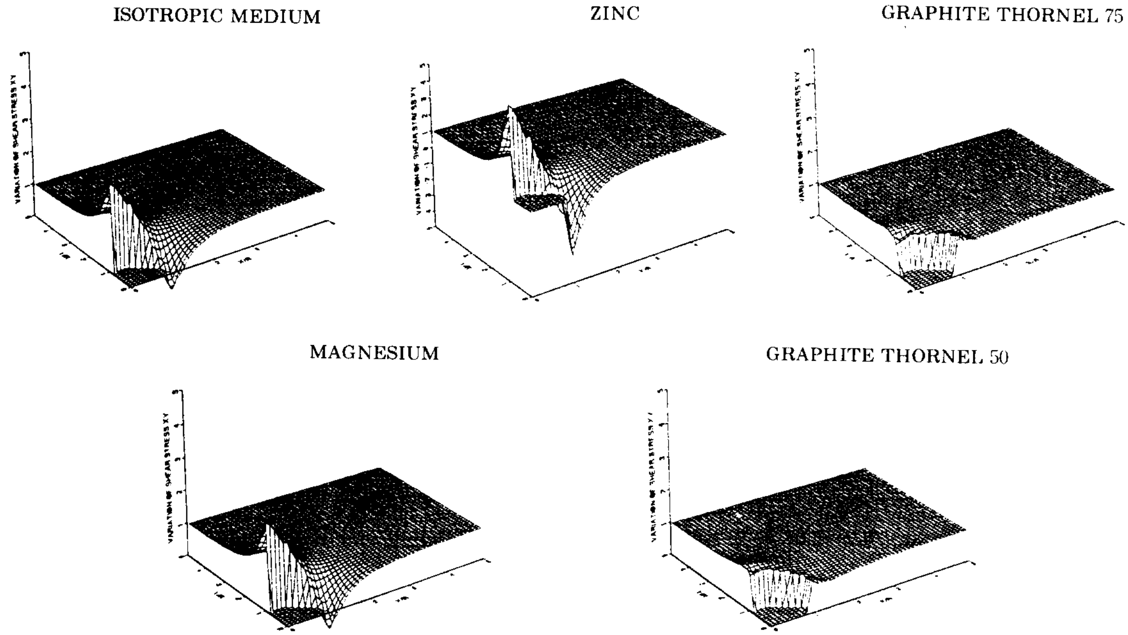

Fig.5.2.2 under Pure Shear Stress in the Plane of Isotropy

hite Thornel 75. The stress distributions on the plane of z=0 are shown in Fig. 5.2.1.

2) In this case of shear stress, σ_{xy}, the Stress Concentration Factors(SCF) according to the polar angle, , at the equator are shown. They have the maximum value at $\theta=90^\circ$ and the minimum value at $\theta=0^\circ$, but maximum and minimum values are reversed for Graphite Thornel 75. Also, it is noted that the value at $\theta=45^\circ$ is the same as the results as the applied stress. Variation of shear stresses, σ_{xy}, for five transversely isotropic materials on the plane of z=0 are shown in Fig. 5.2.2.

6. CONCLUSION

The technical approach used in this study combines exact analytical and numerical methods. The finite difference method has been used to obtain results for the problems of a transversely isotropic medium containing a spheroidal cavity under uniaxial tension in the direction of the axis of symmetry of the material in one case and pure shear in the plane of isotropy in another case. The results will converge to the exact value obtained from the closed-form solution as a finer mesh is selected. Thus, if one likes to obtain numerical results pertaining to different loading, the same procedure can be used.

As was shown in this study, the use of finite difference approach once the potential function are found reduces the algebraic manipulations which sometimes appear to be, in transversely isotropy, of formidable complexity.

New numerical results related to the case of a transversely isotropic medium subject to pure shear in the plane of isotropy were obtained.

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APPENDIX A.

Legendre Associated Functions of First Kind

$$P_0(p) = 1$$

$$P_1(p) = p$$

$$P_2(p) = \frac{1}{2}(3p^2 - 1)$$

$$P_3(p) = \frac{p}{2}(5p^2 - 3)$$

$$P_4(p) = \frac{1}{8}(35p^4 - 30p^2 + 3)$$

$$P_0^{\ 1}(p) = \frac{(1-p)}{(1-p^2)^{\frac{1}{2}}}$$

$$P_1^{\ 1}(p) = \frac{1}{2}(1-p^2)^{\frac{1}{2}}$$

$$P_2^{\ 1}(p) = \frac{1}{2}(1-p^2)^{\frac{1}{2}}$$

$$P_3^{\ 1}(p) = \frac{1}{8}(5p^2 - 1)(1 - p^2)^{\frac{1}{2}}$$

$$P_4^{\ 1}(p) = \frac{1}{8}(7p^2 - 1)(1 - p^2)^{\frac{1}{2}}$$

$$P_0^{\,2}(p) = \frac{(1-p)^2}{2(1-p^2)}$$

$$P_1^{\,2}(p) = \frac{(p^3-3p+2)^2}{6(1-p^2)}$$

$$P_2^{\,2}(p) = \frac{1}{8}(1-p^2)$$

$$P_3^{\,2}(p) = \frac{1}{8}p(1-p^2)$$

$$P_4^{\,2}(p) = \frac{1}{48}p(1-p^2(7-p^2-1)$$

$$P_1^{\,3}(p) = -\frac{(p^4-6p^2+8p-3)}{24(1-p^2)^{\frac{3}{2}}}$$

$$P_2^{\,3}(p) = -\frac{(p^8-15p+10p^3-3p^2)}{120(1-p^2)^{\frac{3}{2}}}$$

$$P_3^{\,3}(p) = \frac{1}{48}(1-p^2)^{\frac{3}{2}}$$

$$P_4^{\,3}(p) = \frac{1}{48}p(1-p^2)^{\frac{3}{2}}$$

APPENDIX B.

Legendre Associated Functions of Second Kind

$$Q_0(q) = \frac{1}{2}\,Ln\frac{q+1}{q-1}$$

$$Q_1(q) = \frac{q}{2}\,Ln\frac{q+1}{q-1}-1$$

$$Q_2(q) = \frac{1}{4}\,(3q^2-1)Ln\frac{q+1}{q-1}-\frac{3}{2}q$$

$$Q_3(q) = \frac{1}{4}\,(5q^2-3)Ln\frac{q+1}{q-1}-\frac{1}{6}(15q^2-4)$$

$$Q_4(q) = \frac{1}{16}\,(35q^4-30q^2+3)$$

$$Ln\frac{q+1}{q-1}-\frac{5}{24}q(21q^2-11)$$

$$Q_0^1(q) = -\frac{1}{(q^2-1)^{\frac{1}{2}}}$$

$$Q_1^1(q) = \frac{1}{2}(q^2-1)^{\frac{1}{2}}\,Ln\frac{q+1}{q-1}-\frac{q}{(q^2-1)^{\frac{1}{2}}}$$

$$Q_2^1(q) = \frac{3}{2}q(q^2-1)^{\frac{1}{2}}\,Ln\frac{q+1}{q-1}-\frac{3q^2-2}{(q^2-1)^{\frac{1}{2}}}$$

$$Q_3^1(q) = \frac{3}{4}(5q^2-1)(q^2-1)^{\frac{1}{2}}$$

$$Ln\frac{q+1}{q-1}-\frac{q(15q^2-13)}{(2(q^2-1)^{\frac{1}{2}}}$$

$$Q_0^2(q) = \frac{2q}{q^2-1}$$

$$Q_1^2(q) = \frac{2}{q^2-1}$$

$$Q_2^2(q) = \frac{3}{2}(q^2-1)\,Ln\frac{q+1}{q-1}+\frac{q(15q^2-13)}{q^2-1}$$

$$Q_3^2(q) = \frac{15}{2}q(q^2-1)Ln\frac{q+1}{q-1}-15q^2+10+\frac{2}{q^2-1}$$

$$Q_0^3(q) = \frac{2(3q^2+1)}{(q^2-1)^{\frac{3}{2}}}$$

$$Q_1^3(q) = \frac{8q}{(q^2-1)^{\frac{3}{2}}}$$

$$Q_2^3(q) = \frac{8}{(q^2-1)^{\frac{3}{2}}}$$

$$Q_3^3(q) = \frac{15}{2}(q^2-1)^{\frac{3}{2}}\,Ln\frac{q+1}{q-1}-15(q^2-1)^{\frac{1}{2}}+$$

$$\frac{10q}{(q^2-1)^{\frac{1}{2}}}-\frac{10p}{(q^2-1)^{\frac{1}{2}}}$$

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