단순지지 변화곡선 길이 보의 정확탄성곡선

Elastica of Simple Variable-Arc-Length Beams

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요 지

이 논문은 한개의 집중하중을 받는 단순지지 변화곡선길이 보에 관한 연구이다. Bernoulli-Euler 보 이론에 의하여 정확탄성곡선을 지배하는 미분방정식을 유도하고 이를 수치해석하여 정확탄성곡선의 거동값들을 예측하였다. 미분방정식을 적분하기 위하여 Runge-Kutta method를 이용하고, 단부의 회전각을 산출하기 위하여 Regula-Falsi method를 이용하였다. 본 연구에서의 수치해석 결과들은 문헌값들과 매우 잘 일치하여 본 연구방법의 타당성을 입증하였다. 수치해석의 결과로 정확탄성곡선의 거동값과 하중사이의 관계 및 한계거동값과 하중위치변수 사이의 관계를 각각 그림에 나타내었다. 수치해석의 결과를 분석하여 변화곡선길이 보에서 발생가능한 최대 단부회전각, 최대 처짐 및 최대 휨모멘트를 산정하였다.

Abstract

In this paper, numerical methods are developed for solving the elastica of simple beams with variable-arc-length subjected to a point loading. The beam model is based on Bernoulli-Euler beam theory. The Runge-Kutta and Regula-Falsi methods, respectively, are used to solve the governing differential equations and to compute the beam's rotation at the left end of the beams. Extensive numerical results of the elastica responses, including deflected shapes, rotations of cross-section and bending moments, are presented in non-dimensional forms. The possible maximum values of the end rotation, deflection and bending moment are determined by analyzing the numerical data obtained in this study.

Keywords : *Bernoulli-Euler beam theory, elastica, Runge-Kutta method, Regula-Falsi method, variable-arc-length beam.*

1. INTRODUCTION

The first studies of the elastica were published in 1774 by Euler[1]. A survey of the classical literature on this subject was published in 1971 by Schmidt and Da Deppo[2]. Present-day applications of the elastica involving variable-arc-length beam were discussed by Conway[3], Gospodnetic[4], Schile and Sierakowski[5], Wang and Kitipornchai[6] and

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Chucheepsakul, et al[7]. On the other hand, the works related to the constant deformed arc-length beam are in references[8~12]. Nowhere in the open literature were found solutions for the class of elastica of variable-arc-length beams considered herein : numerical method solving the elastica of such beam using the numerical integration technique combined with the Regula-Falsi method.

In the analysis of the elastica, one usually begins with classical Bernoulli-Euler beam theory, the non-linear differential equation that relates deflection to load. This beam theory is also used in the present analytical studies. The following assumptions are inherent in this theory : the beam is linearly elastic, the neutral axis for bending of the beam is inextensible, and the Poisson's ratio and transverse shear deformation are negligible. In addition, the point load is assumed to sustain its loading position and vertical direction, respectively.

Historically, solutions of the elastica have four forms : (1) closed-form solutions in terms of elliptic integrals ; (2) power series solutions ; (3) numerical solutions ; and (4) experimental solutions. The present study begins with an analysis involving numerical solutions using the Runge-Kutta and Regula-Falsi methods, and ends with numerical results and discussion including the comparison of the results between this study and references.

2. MATHEMATICAL MODEL

The undeflected simple variable-arc-length beam with a vertical point load P is shown in Fig. 1(a). The beam's span length is l, and the load position is αl away from the end A, which does not change after deforming. Here α is defined as the position parameter. This beam is supported on a frictionless support at the end A and is hinged at the end B.

The symbols for the deflected beam are depicted in Fig. 1(b). The shape of the elastica is defined by the (x, y) coordinate system whose origin is at A. At the material point (x, y), the beam's arc length measuring from the end A is s, the rotation of cross-section is θ, and the bending moment is M. At the ends A and B, the rotations of cross-section are θ_A and θ_B, respectively. Also both the horizontal and vertical components of reaction at the end A are presented as H and V, respectively.

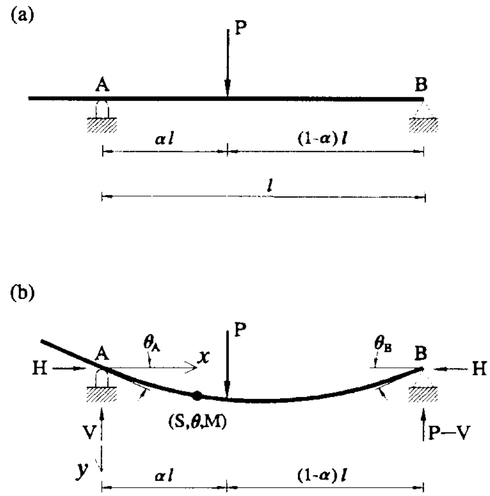

Fig. 1 (a) undeflected beam and
(b) deflected beam with its symbols

When the equilibrium equation is used, the vertical component V of reaction at the end A for the deflected beam becomes

$$V = P(1-\alpha) \tag{1}$$

Considering the rotation of cross-section at the end A, θ_A, the horizontal component H of reaction becomes

$$H = V\tan\theta_A$$
$$= P(1-\alpha)\tan\theta_A \quad (2)$$

Using equations (1) and (2), the bending moment M at (x, y) is obtained :

$$M = P(1-\alpha)(x + \tan\theta_A y), \quad 0 \le x \le \alpha l \quad (3.1)$$
$$M = P[\alpha(l-x) + (1-\alpha)\tan\theta_A y], \quad \alpha l \le x \le l \quad (3.2)$$

It is assumed that Bernoulli-Euler beam theory governs the beam behavior under load, for which the differential equation for the elastica[13] is

$$y^{ii} = -(M/EI)[1 + (y^i)^2]^{3/2} \quad (4)$$

where superscript i is the operator d/dx and EI is the flexural rigidity. The appropriate boundary conditions for the ends A (x=0) and B (x=l) are

$$y = 0 \quad \text{at} \quad x = 0 \quad (5)$$

$$y = 0 \quad \text{at} \quad x = l \quad (6)$$

which impose zero deflections at those ends.

Using the differential relationship for the arc length, or $(ds)^2 = (dx)^2 + (dy)^2$, the following equation is obtained :

$$s^i = [1 + (y^i)^2]^{1/2} \quad (7)$$

In term of the arc length variable s, the boundary condition of the end A is

$$s = 0 \quad \text{at} \quad x = 0 \quad (8)$$

To facilitate the numerical studies and to obtain the most general results for this class of problem, the geometric parameters, the loadings, and the governing differential equations with their boundary conditions above are cast in the following non-dimensional forms. First, the coordinates (x, y) and the arc length s are normalized by the span length l, or

$$\xi = x/l \quad (9)$$
$$\eta = y/l \quad (10)$$
$$\lambda = s/l \quad (11)$$

The three loading parameters are

$$p = Pl^2/EI \quad (12)$$

$$v = Vl^2/EI \quad (13)$$
$$= p(1-\alpha)$$

$$h = Hl^2/EI \quad (14)$$
$$= p(1-\alpha)\tan\theta_A$$

When equations (3. 1) and (3. 2) are substituted into equation (4), and equations (9), (10) and (12) are used, the non-dimensional form of equation (4) becomes

$$\eta'' = -p(1-\alpha)(\xi + \tan\theta_A\eta)[1 + (\eta')^2]^{3/2}, \quad 0 \le \xi \le \alpha \quad (15.1)$$

$$\eta'' = -p[\alpha(1-\xi) + (1-\alpha)\tan\theta_A\eta][1 + (\eta')^2]^{3/2}, \quad \alpha < \xi \le 1 \quad (15.2)$$

Also, with equations (10) and (11), the non-dimensional form for the arc length, equation (7) becomes

$$\lambda' = [1 + (\eta')^2]^{1/2} \quad (16)$$

In the above equations, prime denotes the operator $d/d\xi$.

Consider boundary conditions. Based on equations (5) and (8), with equations (9)-(11), the two boundary conditions for the end A become

$$\eta = 0 \quad \text{at} \quad \xi = 0 \tag{17}$$

$$\lambda = 0 \quad \text{at} \quad \xi = 0 \tag{18}$$

Based on equation (6), with equations (9) and (10), the boundary condition for the end B becomes

$$\eta = 0 \quad \text{at} \quad \xi = 1 \tag{19}$$

3. NUMERICAL METHODS

Numerical values were assigned for the load parameter p and position parameter α. Then the differential equations (15.1), (15.2) and (16), subject to equations (17) and (18), were solved in a straightforward way using the Runge-Kutta method[14]. These calculations yielded the shape of elastica $\eta = \eta(\xi)$, the slope $\eta' = \eta'(\xi)$ and the arc length $\lambda = \lambda(\xi)$. To compute the value of θ_A, the rotation of cross-section at the end A, the Regula-Falsi method[14] combined with equation (19) was used, which is one of the solution methods of non-linear equation.

The algorithm developed to solve these equations had two convergence criteria. This algorithm is summarized as follows.

(1) Assume a trial value θ_{At} and compute $\eta' = \tan\theta_{At}$ at $\xi = 0$. Choose the initial value of θ_{At} as zero.

(2) Integrate equations (15.1), (15.2) and (16) with equations (17) and (18), and the calculated value of η' for ξ in the range 0~1 using the Runge-Kutta method. The results give trial solutions for η, η' and λ.

(3) If the assumed value of θ_{At} is the characteristic value of the elastica, then $D = \eta(1)$ must be zero due to equation (19). The first criterion for convergence of the solution is

$$|D| < \text{tol}_1 \tag{20}$$

(4) If the value of D does not satisfy the first convergence criterion of equation (20), then increment the previous value of θ_{At} and compute $\eta' = \tan\theta_{At}$.

(5) Repeat steps 2~4 and note the sign of D in each iteration. If D changes sign between two consecutive values θ_{A1} and θ_{A2}, then the characteristic value θ_A lies between θ_{A1} and θ_{A2}.

(6) Compute the advanced value of θ_{At}, expressed in equation (21), based on trial values of θ_{A1} and θ_{A2} using the Regula-Falsi method.

$$\theta_{At} = \frac{\theta_{A2}|D_1| + \theta_{A1}|D_2|}{|D_1| + |D_2|} \tag{21}$$

where D_1 and D_2 are the values $D = \eta(1)$ for θ_{A1} and θ_{A2}, respectively. The second criterion for convergence of the solution is

$$|(\theta_{A1} - \theta_{A2})/\theta_{At}| < \text{tol}_2 \tag{22}$$

(7) Choose $\theta_A = \theta_{At}$ as the value that satisfies the convergence criteria, equations (20) and (21). Compute the corresponding final solutions to the elastica, the shape η, slope η' and arc length λ, using the Runge-Kutta method.

(8) Compute the v and h from the equations (13) and (14), the θ, θ_B and the non-di-

mensional bending moment m from the following equations, respectively :

$$\theta = \tan^{-1}(\eta') \quad (23)$$

$$\theta_B = \tan^{-1}[\eta'(1)] \quad (24)$$

$$m = Ml/EI = p(1-\alpha)(\xi + \tan\theta_A \eta), \quad 0 \le \xi \le \alpha \quad (25.1)$$

$$m = p[\alpha(1-\xi) + (1-\alpha)\tan\theta_A \eta], \quad \alpha < \xi \le 1 \quad (25.2)$$

Based on this algorithm, a general FORTRAN 77 computer program was written. All computations were carried out on a Personal Computer with graphics support. For nearly all of the numerical results presented herein, a step size of $\Delta\xi$=0.01 in the Runge-Kutta method was found to give convergence for θ_A to within three significant figures, when the tol_1 and tol_2 in equations (20) and (22) were fixed as 1×10^{-7} and 1×10^{-5}, respectively.

4. NUMERICAL RESULTS AND DISCUSSION

In the first series of studies, the numerical methods discussed above were used to calculate the elastica and the results were compared to those available in references[6,7]. The results of reference [6] and [7] are based on the Elliptical integration and Shooting optimization methods, respectively. It is noted that the solutions based on the Elliptic integration method were the closed-form solutions and those based on the Shooting optimization method were the numerical solutions. In the Shooting optimization method in reference [7], the three characteristic values, θ_A, θ_B and $\lambda_{\xi=1}$, of the elastica were calculated by the optimization algorithm in which the sum of error norms given by the differences in values of θ, ξ and η between the prescribed and computed terminal boundary conditions was minimized. In the numerical method presented herein which is a contrast with the Shooting optimization method, only one characteristic value θ_A of the elastica is calculated by simply using the Regula-Falsi method and the remaing characteristic values θ_B and $\lambda_{\xi=1}$ are not necessary to obtain the elastica.

These comparisons are summarized in Table 1 which shows the reference values are nearly identical to the present results. Such comparisons serve to validate the numerical results developed herein. Especially, it is found that both the present and Shooting optimization methods are very accurate since the Elliptic integration's solutions are exact ones.

In the second series of studies, the numerical method developed herein were used to compute the elastica for the varying load parameters p with the load position parameter α =0.3. Here the end rotations θ_A and θ_B, horizontal reaction component h, maximum bending moment m_{max}, maximum deflection η_{max} and beam's arc length λ_{max} were evaluated.

Table 1 Comparisons of responses reported in references to the present results(p=6.)

position parameter	data source	θ_A		θ_B		λ_{max}*	
		stable	unstable	stable	unstable	stable	unstable
α=0.25	present	0.4134	0.8013	0.3126	0.6804	1.0333	1.1534
	ref[6]	0.4134	0.8012	0.3126	0.6804	1.0333	1.1534
	ref[7]	0.4134	0.8011	0.3126	0.6804	1.0333	1.1534
α=0.5	present	0.4708	0.8760	0.4708	0.8760	1.0617	1.2391
	ref[6]	0.4708	0.8760	0.4708	0.8760	1.0617	1.2391
	ref[7]	0.4708	0.8760	0.4708	0.8760	1.0617	1.2391
α=0.75	present	0.2496	1.1634	0.3453	1.2991	1.0221	1.5593
	ref[6]	0.2496	1.1634	0.3452	1.2991	1.0221	1.5593
	ref[7]	0.2496	1.1634	0.3453	1.2991	1.0221	1.5593

* λ_{max}=non-dimensional arc length

These results are shown in Figs. 2-4.

From all these figures, it is seen that each load parameter p has two responses whose one is presented as solid line and another dashed one. This means that two shapes of elastica are possible for a given load parameter p except the critical load parameter p_{cr} =6.44 marked by in Figs. 2-4. Since a physical phenomenon will follow the easiest path whenever there is a choice between different paths, the shapes of elastica with the lower responses(solid line) are stable and the others(dashed line) unstable in this elastica problems. At the critical load parameter p_{cr}= 6.44, all responses are maximum so that the elastica subjected to the p exceeding 6.44 is impossible at the position parameter α=0.3.

In the third series of studies, the critical responses p_{cr}, θ_A, η_{max} and m_{max} against each α level were obtained and presented in Fig. 5 and 6. As shown in Fig. 5, it is noted that the minimum p_{cr}=6.31 exists at α=0.37 marked by □ in this figure. On the other hand, it is noted that maximum critical responses of θ_A, m_{max} and η_{max} exist as shown in Fig. 6. And it is concluded that the higest

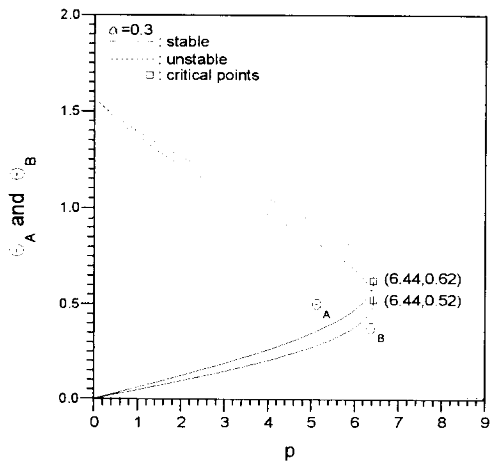

Fig. 2 θ_A and θ_B versus p curves

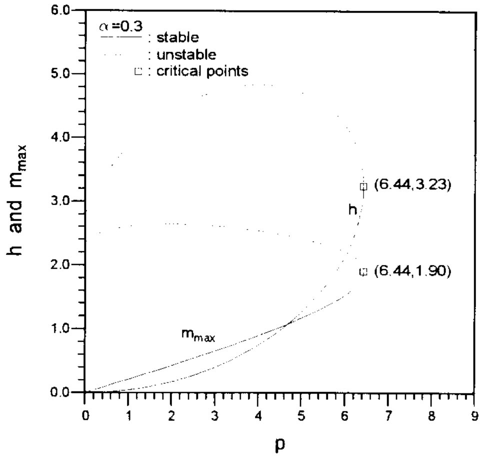

Fig. 3 h and m_{max} versus p curves

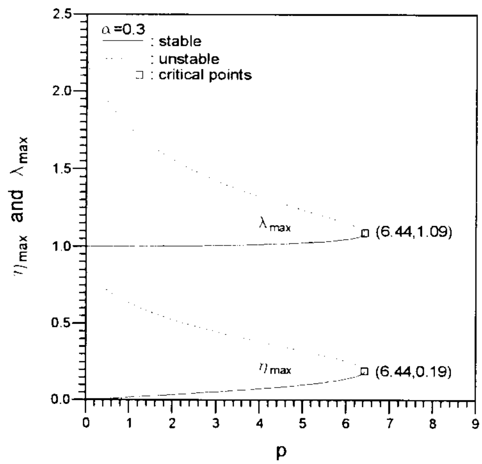

Fig. 4 η_{max} and λ_{max} versus p curves

values of θ_A, m_{max} and η_{max} in the simple variable-arc-length beam are 0.68, 0.27 and 2.51, respectively.

Finally, the elastica with p=5. and α=0.3 are shown in Fig. 7 in which the solid and dashed curves are the stable and unstable configurations, respectively.

5. CONCLUSIONS

The numerical methods developed herein for computing the elastica of simple variable

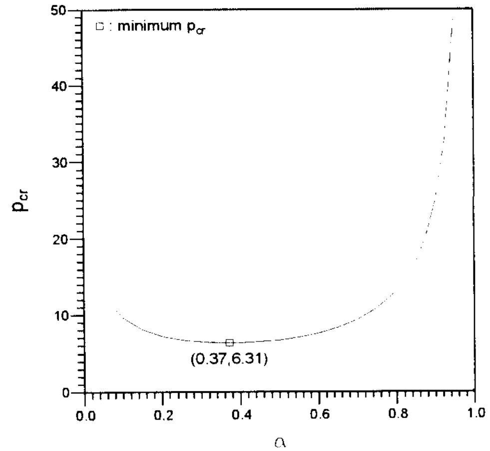

Fig. 5 p_{cr} versus α curve

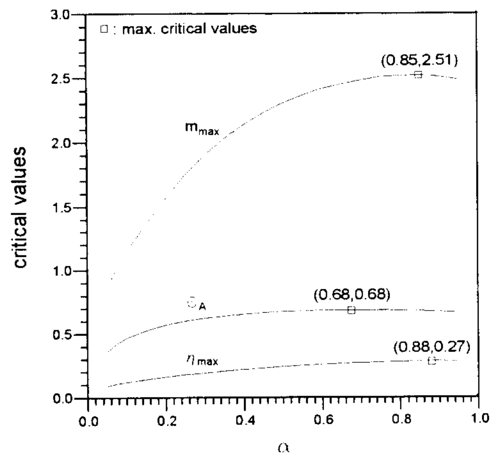

Fig. 6 θ_A, η_{max} and m_{max} versus α curves

-arc-length beams subjected to a point load were found to be efficient, accurate, and highly versatile. The responses computed by the present numerical methods compared favorably with previously published results based on more classical methods-Elliptic integral method and Shooting optimization method.

The responses versus load parameter curves for $\alpha=0.3$, and critical responses versus position parameter curves are shown in figures. It is found that the highest values of

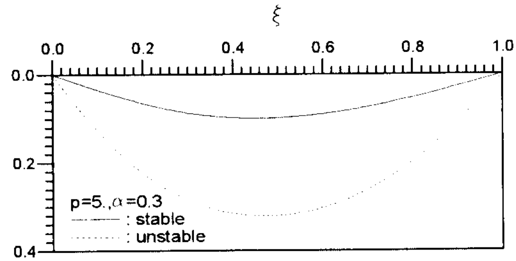

Fig. 7 Example of the elastica

θ_A, η_{max} and m_{max} in the simple variable-arc-length beam are 0.68, 0.27 and 2.51, respectively.

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