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An Approximation Theorem for Two-Parameter Wiener Process†

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Abstract

In this paper, a two-parameter version of Ikeda-Watanabe's mollifiers approximation of the Brownian motion is considered, and an approximation theorem corresponding to the one parameter case is proved. Using this approximation, we formulate Wong-Zakai type theorem in a Stochastic Differential Equation (SDE) driven by a two-parameter Wiener process.

Key Words : Wong-Zakai type theorem; Stochastic differential equation; Two-parameter Wiener process; Stratonovich integral.

1. INTRODUCTION

The solution $(X(t)), 0 \leq t \leq T$, of SDE's

$$dX(t) = b(t, X(t))dt + \sigma(t, X(t))dW(t), \quad X(0) = X_0$$

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is the solution of the following integral equation

$$X(t) = X_0 + \int_0^t b(s, X(s))\, ds + \int_0^t \sigma(s, X(s))\, dW(s) \tag{1.1}$$

on a probability space $(\Omega, \mathcal{F}, P)$ where $(W(t)), 0 \le t \le T$, is a Wiener process. The last integral in (1.1) is the Itô integral.

Suppose that the Wiener process in Equation (1.1) is replaced by smooth processes $\{W_\delta(t)\}_{\delta>0}$ such that for a.a. ω,

$$W_\delta(t,\omega) \longrightarrow W(t,\omega) \text{ as } \delta \to 0$$

in some sense in [0,T]. For each ω, let $X_\delta(t)$ be the solution of the corresponding ordinary differential equation

$$\frac{dX_\delta(t)}{dt} = b(t, X_\delta(t)) + \sigma(t, X_\delta(t))\frac{dW_\delta(t)}{dt}. \tag{1.2}$$

Then $X_\delta(t)$ converges to some process $X(t)$ in the same sense: For a.a. ω, we have that

$$X_\delta(t,\omega) \longrightarrow X(t,\omega) \text{ as } \delta \to 0.$$

It turns out that under suitable assumptions, this limit $X(t)$ coincides with the solution of Stratonovich formulation of (1.1) (Wong-Zakai Theorem, see e.g. Wong and Zakai (1965)), i.e., $X(t)$ is the solution of the following equation:

$$X(t) = X_0 + \int_0^t b(s, X(s))\, ds + \int_0^t \sigma(s, X(s)) \circ dW(s) \tag{1.3}$$

where $\circ$ denotes the Stratonovich integral. Indeed we have

$$\begin{aligned}\int_0^t \sigma(s, X(s)) \circ dW(s) &= \int_0^t \sigma(s, X(s))\, dW(s) \\ &+ \frac{1}{2}\int_0^t \sigma(s, X(s))\sigma'(s, X(s))\, ds\end{aligned}$$

where σ' denotes the derivative of $\sigma(t,x)$ with respect to x (see Stratonovich (1966)).

The new work is concerned with extensions of the basic result of Wong and Zakai (1965) to infinite dimensions.

The generalization of these results can proceed in two directions:

(a) infinite dimensional (Hilbert space-valued) SDE's.

(*b*) Stochastic Partial Differential Equations (SPDE's) whose solutions are random fields.

Problem (*b*) poses difficulties of a different kind connected with the fact that one has to deal with multi-parameter processes and not with a single parameter (but infinite dimensional) process. In this paper, we consider a uniform approximation of the two parameter Wiener process for problem (*b*).

2. APPROXIMATION

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a complete probability space and let $(\mathcal{F}_{s,t})_{,0\ \ s\le S, 0\le t\le T,}$ be a family of sub-σ-fields of $\mathcal{F}$ satisfying

(1) if $s \le s'$ and $t \le t'$, then $\mathcal{F}_{s,t} \subset \mathcal{F}_{s',t'}$,

(2) $\mathcal{F}_{0,0}$ contains all null sets of $\mathcal{F}$,

(3) for each (s,t), $\mathcal{F}_{s,t} = \bigcap_{s<s',t<t'} \mathcal{F}_{s',t'}$,

(4) for each (s,t), $\mathcal{F}_{s,T}$ and $\mathcal{F}_{S,t}$ are conditionally independent given $\mathcal{F}_{s,t}$.

A two-parameter Wiener process (or Brownian sheet) is a sample continuous Gaussian process with mean zero and

$$E[W(s,t)W(s',t')] = \min(s,s')\min(t,t').$$

Here we take a family of sub-σ-fields of $\mathcal{F}$ as

$$\mathcal{F}_{s,t} = \sigma\{W(u,v) : u \le s, v \le t\} \vee \{P\text{ -null set of } \mathcal{F}\}\ .$$

We now define an approximation to the two-parameter Wiener process. Let ρ be a non-negative C^∞-function whose support is contained in $[0,1] \times [0,1]$ and let

$$\int_0^1 \int_0^1 \rho(u,v)\, du\, dv = 1\ .$$

Set

$$\rho_\delta(u,v) = \frac{1}{\delta^2}\rho(\frac{u}{\delta}, \frac{v}{\delta}) \text{ for } \delta > 0\ .$$

Then define an approximation $\{W_\delta(s,t)\}_{\delta>0}$ as follows:

$$W_\delta(s,t) = \int_0^\infty \int_0^\infty W(u,v)\rho_\delta(u-s, v-t)\, du\, dv. \tag{2.1}$$

This approximation is a straightforward extension of the mollifiers approximation to an one-parameter Wiener process. (2.1) can be written as

$$W_\delta(s,t) = \int_0^\delta \int_0^\delta W(u+s, v+t)\rho_\delta(u,v)\,du\,dv. \tag{2.2}$$

From the properties of ρ and (2.1), we have

$$\begin{aligned} \dot{W}_\delta(s,t) &:= \frac{\partial^2}{\partial t \partial s} W_\delta(s,t) \\ &= \frac{1}{\delta^2}\int_0^1 \int_0^1 W(s+\delta u, t+\delta v)\dot{\rho}(u,v)\,du\,dv. \end{aligned} \tag{2.3}$$

With regard to the mollifiers approximation to a two-parameter Wiener process, we have :

Proposition 1.

(a) For every $\omega \in \Omega$, $(s,t) \longmapsto W_\delta(s,t,\omega)$ is continuously differentiable.

(b) $W_\delta(0,0)$ is $\mathcal{F}_{\delta,\delta}$-measurable.

(c) $E[W_\delta(0,0)] = 0$.

(d) $E[| W_\delta(0,0) |^{2m}] \le c\delta^{2m}$ for $m = 1, 2, \ldots.$

(e) $E\left[\left(\int_{k\delta}^{(k+1)\delta} \int_{j\delta}^{(j+1)\delta} | \dot{W}_\delta(u,v) | \,du\,dv\right)^{2m}\right] = E\left[\left(\int_0^\delta \int_0^\delta | \dot{W}_\delta(u,v) | \,du\,dv\right)^{2m}\right]$ for $m = 1, 2, \ldots.$

(f) $E\left[\left(\int_0^\delta \int_0^\delta | \dot{W}_\delta(u,v) | \,du\,dv\right)^{2m}\right] \le c\delta^{2m}$ for $m = 1, 2, \ldots.$

Proof. (a) and **(b).** The properties (a) and (b) are immediate consequences of the definition of the mollifiers approximation.
(c). It follows from Fubini's theorem that

$$E[W_\delta(0,0)] = \int_0^\delta \int_0^\delta E[W(u,v)]\rho_\delta(u,v)\,du\,dv = 0.$$

(d). From (2.2), we observe that

$$\begin{aligned} | W_\delta(0,0) |^{2m} &= \left(\int_0^\delta \int_0^\delta W(u,v)\rho_\delta(u,v)\,du\,dv\right)^{2m} \\ &= \left(\int_0^1 \int_0^1 \frac{1}{\delta} W(\delta u, \delta v)\rho(u,v)\,du\,dv\right)^{2m} \delta^{2m} \end{aligned}$$

Hence, we get

$$E[| W_\delta(0,0) |^{2m}] = E\left[\left(\int_0^1 \int_0^1 W(u,v)\rho(u,v)\,du\,dv\right)^{2m}\right]\delta^{2m}.$$

(e). Frist we note that

$$\begin{aligned}
&W_\delta(s+j\delta, t+k\delta)\\
&= \int_0^1 \int_0^1 W(s+j\delta+\delta u, t+k\delta+\delta v)\rho(u,v)\,du\,dv\\
&= \int_0^1 \int_0^1 [W(s+j\delta+\delta u, t+k\delta+\delta v) - W(s+j\delta+\delta u, k\delta)\\
&\quad - W(j\delta, t+k\delta+\delta v) + W(j\delta, k\delta)]\rho(u,v)\,du\,dv\\
&\quad + \int_0^1 \int_0^1 W(s+j\delta+\delta u, k\delta)\rho(u,v)\,du\,dv\\
&\quad + \int_0^1 \int_0^1 W(j\delta, t+k\delta+\delta v)\rho(u,v)\,du\,dv\\
&\quad - \int_0^1 \int_0^1 W(j\delta, k\delta)\rho(u,v)\,du\,dv.
\end{aligned}$$

From this, we obtain

$$\begin{aligned}
&\dot{W}_\delta(s+j\delta, t+k\delta)\\
&= \frac{\partial^2}{\partial t \partial s}\int_0^1 \int_0^1 W(s+j\delta+\delta u, t+k\delta+\delta v)\rho(u,v)\,du\,dv\\
&= \frac{\partial^2}{\partial t \partial s}\int_t^{t+\delta} \int_s^{s+\delta} [W(j\delta+u, k\delta+v) - W(j\delta+u, k\delta) - W(j\delta, k\delta+v)\\
&\quad + W(j\delta, k\delta)]\frac{1}{\delta^2}\rho\left(\frac{u-s}{\delta}, \frac{v-t}{\delta}\right)du\,dv\\
&= \int_t^{t+\delta} \int_s^{s+\delta} [W(j\delta+u, k\delta+v) - W(j\delta+u, k\delta) - W(j\delta, k\delta+v)\\
&\quad + W(j\delta, k\delta)]\frac{1}{\delta^2}\frac{\partial^2}{\partial s \partial t}\rho\left(\frac{u-s}{\delta}, \frac{v-t}{\delta}\right)du\,dv\\
&= \frac{1}{\delta^2}\int_0^1 \int_0^1 [W(s+j\delta+\delta u, t+k\delta+\delta v) - W(s+j\delta+\delta u, k\delta)\\
&\quad - W(j\delta, t+k\delta+\delta v) + W(j\delta, k\delta)]\dot{\rho}(u,v)\,du\,dv.
\end{aligned} \tag{2.4}$$

According to the change of variables and (2.4), it follows that

$$E\left[\left(\int_{k\delta}^{(k+1)\delta} \int_{j\delta}^{(j+1)\delta} | \dot{W}_\delta(u,v) |\, du\,dv\right)^{2m}\right]$$

$$= E\left[\left(\int_0^\delta\int_0^\delta \mid \dot{W}_\delta(u+j\delta, v+k\delta)\mid du\,dv\right)^{2m}\right]$$

$$= E\left[\left(\int_0^\delta\int_0^\delta \mid \frac{1}{\delta^2}\int_0^1\int_0^1 W(u+\delta u', v+\delta v')\dot{\rho}(u',v')\,du'\,dv' \mid du\,dv)\right)^{2m}\right] \tag{2.5}$$

since $W(u+j\delta+\delta u', v+k\delta+\delta v') - W(u+j\delta+\delta u', k\delta) - W(j\delta, v+k\delta+\delta v') + W(j\delta, k\delta)$ and $W(u+\delta u', v+\delta v')$ have the same distribution. From (2.3), (2.5) is equal to

$$E\left[\left(\int_0^\delta\int_0^\delta \mid \dot{W}_\delta(u,v)\mid du\,dv\right)^{2m}\right],$$

and the assertion (e) follows.

(f). According to the change of variables and (2.3), we find that

$$E\left[\left(\int_0^\delta\int_0^\delta \mid \dot{W}_\delta(u,v)\mid du\,dv\right)^{2m}\right]$$
$$= E\left[\left(\int_0^1\int_0^1 \mid \dot{W}_\delta(\delta u,\delta v)\mid du\,dv\right)^{2m}\right]\delta^{4m}$$
$$= E\left[\left(\int_0^1\int_0^1 \mid \int_0^1\int_0^1 \frac{W(\delta(x+u),\delta(y+v))}{\delta}\dot{\rho}(u,v)\,du\,dv \mid dx\,dy\right)^{2m}\right]\delta^{2m}$$
$$= E\left[\left(\int_0^1\int_0^1 \mid \int_0^1\int_0^1 W(x+u,y+v)\dot{\rho}(u,v)\,du\,dv \mid dx\,dy\right)^{2m}\right]\delta^{2m},$$

since $\frac{W(\delta(x+u),\delta(y+v))}{\delta}$ and $W(x+u,y+v)$ have the same distribution. Hence the property (f) follows.

Remark 1. The following inequality will be needed to prove the main result.

$$E\left[\left(\int_0^{n\delta}\int_0^{n\delta} \mid \dot{W}_\delta(u,v)\mid du\,dv\right)^{2m}\right] \le cn^{4m}\delta^{2m} \tag{2.6}$$

for $m = 1, 2, \ldots$.

The proof of (2.6). From Hölder's inequality and (f), we get

$$E\left[\left(\int_0^{n\delta}\int_0^{n\delta} \mid \dot{W}_\delta(u,v)\mid du\,dv\right)^{2m}\right]$$

$$
\begin{aligned}
&= E\left[\left(\sum_{k=0}^{n-1}\sum_{j=0}^{n-1}\int_{k\delta}^{(k+1)\delta}\int_{j\delta}^{(j+1)\delta} \mid \dot{W}_\delta(u,v) \mid \, du\, dv\right)^{2m}\right] \\
&\leq n^{2(2m-1)} E\left[\sum_{k=0}^{n-1}\sum_{j=0}^{n-1}\left(\int_{k\delta}^{(k+1)\delta}\int_{j\delta}^{(j+1)\delta} \mid \dot{W}_\delta(u,v) \mid \, du\, dv\right)^{2m}\right] \\
&= n^{2(2m-1)}\sum_{k=0}^{n-1}\sum_{j=0}^{n-1} E\left[\left(\int_0^\delta\int_0^\delta \mid \dot{W}_\delta(u,v) \mid \, du\, dv\right)^{2m}\right] \quad \text{by } (e) \\
&= n^{4m} E\left[\left(\int_0^\delta\int_0^\delta \mid \dot{W}_\delta(u,v) \mid \, du\, dv\right)^{2m}\right] \\
&\leq c n^{4m}\delta^{2m}.
\end{aligned}
$$

3. THE MAIN THEOREM

Now we shall prove a uniform approximation result for the two-parameter Wiener process $(W(s,t))$. In order to prove this result, the maximal inequality in a two-parameter case is needed (see e.g. Theorem 1.2 in Cairoli and Walsh (1984)):

Let $\{M_{s,t}, 0 \leq s \leq S, 0 \leq t \leq T\}$ be a right-continuous martingale. Then

$$
E\left[\sup_{0\leq s\leq S, 0\leq t\leq T} \mid M_{s,t} \mid^p\right] \leq \left(\frac{p}{1-p}\right)^{2p} E[\mid M_{S,T} \mid^p], \quad p > 1.
$$

Theorem 1. For every $S \geq 0, T \geq 0$, we have

$$
\lim_{\delta\to 0} E\left[\sup_{0\leq s\leq S, 0\leq t\leq T}\left|W_\delta(s,t) - W(s,t)\right|^2\right] = 0
$$

where $W_\delta(s,t)$ is as given in (2.1).

Proof. Let us introduce the following notations:
Choose $n(\delta) : (0,1] \longmapsto Z_+$ such that $\lim_{\delta\to 0} n(\delta)^3\delta = 0$, and $\lim_{\delta\to 0} n(\delta) = \infty$, where Z_+ denotes the set of all non-negative integers. Write $\tilde{\delta} := n(\delta)\delta$. Then we define

$$
[s]^+(\tilde{\delta}) = (j+1)\tilde{\delta} \text{ and } [s]^-(\tilde{\delta}) = j\tilde{\delta} \text{ if } j\tilde{\delta} \leq s < (j+1)\tilde{\delta},
$$

and

$$[t]^+(\tilde{\delta}) = (k+1)\tilde{\delta} \text{ and } [t]^-(\tilde{\delta}) = k\tilde{\delta} \text{ if } k\tilde{\delta} \le t < (k+1)\tilde{\delta}.$$

Also set

$$m(s) = m(s)(\delta) = [s]^-(\tilde{\delta})/\tilde{\delta} \text{ and } m(t) = m(t)(\delta) = [t]^-(\tilde{\delta})/\tilde{\delta}.$$

With the above notations, let us write $W_\delta(s,t) - W(s,t)$ as the sum of the following 10 terms:

$$\begin{aligned}
& W_\delta(s,t) - W(s,t) \\
= & \Big\{W_\delta([s]^+(\tilde{\delta}), [t]^+(\tilde{\delta})) - W_\delta([s]^+(\tilde{\delta}), t) - W_\delta(s, [t]^+(\tilde{\delta})) + W_\delta(s,t)\Big\} \\
& -\{W_\delta([s]^+(\tilde{\delta}), [t]^+(\tilde{\delta})) - W_\delta([s]^+(\tilde{\delta}), [t]^-(\tilde{\delta})) - W_\delta([s]^-(\tilde{\delta}), [t]^+(\tilde{\delta})) \\
& + W_\delta([s]^-(\tilde{\delta}), [t]^-(\tilde{\delta}))\} \\
& + \Big\{W_\delta([s]^+(\tilde{\delta}), t) - W_\delta([s]^+(\tilde{\delta}), [t]^-(\tilde{\delta})) - W_\delta([s]^-(\tilde{\delta}), t) + W_\delta([s]^-(\tilde{\delta}), [t]^-(\tilde{\delta}))\Big\} \\
& + \Big\{W_\delta(s, [t]^+(\tilde{\delta})) - W_\delta([s]^-(\tilde{\delta}), [t]^+(\tilde{\delta})) - W_\delta(s, [t]^-(\tilde{\delta})) + W_\delta([s]^-(\tilde{\delta}), [t]^-(\tilde{\delta}))\Big\} \\
& - \Big\{W(s,t) - W([s]^-(\tilde{\delta}), t) - W(s, [t]^-(\tilde{\delta})) + W([s]^-(\tilde{\delta}), [t]^-(\tilde{\delta}))\Big\} \\
& + \Big\{W_\delta([s]^-(\tilde{\delta}), t) - W_\delta([s]^-(\tilde{\delta}), [t]^-(\tilde{\delta}))\Big\} \\
& + \Big\{W_\delta(s, [t]^-(\tilde{\delta})) - W_\delta([s]^-(\tilde{\delta}), [t]^-(\tilde{\delta}))\Big\} \\
& - \Big\{W([s]^-(\tilde{\delta}), t) - W([s]^-(\tilde{\delta}), [t]^-(\tilde{\delta}))\Big\} \\
& - \Big\{W(s, [t]^-(\tilde{\delta})) - W([s]^-(\tilde{\delta}), [t]^-(\tilde{\delta}))\Big\} \\
& + \Big\{W_\delta([s]^-(\tilde{\delta}), [t]^-(\tilde{\delta})) - W([s]^-(\tilde{\delta}), [t]^-(\tilde{\delta}))\Big\} \\
:= & \sum_{i=1}^{10} I_i(s,t), \text{ say }.
\end{aligned}$$

Now we shall show that $E\left[\sup_{0\le s\le S, 0\le t\le T} \left|I_i(s,t)\right|^2\right] \to 0$ as $\delta \to 0$ for $i = 1, \ldots, 10$.

Using the Cauchy-Schwarz inequality, (e) in Proposition 1 and (2.6), we get

$$\begin{aligned}
& E\left[\sup_{0\le s\le S, 0\le t\le T} \left|I_1(s,t)\right|^2\right] \\
= & E\left[\sup_{0\le s\le S, 0\le t\le T} \left|\int_t^{[t]^+(\tilde{\delta})} \int_s^{[s]^+(\tilde{\delta})} \dot{W}_\delta(u,v)\, du\, dv\right|^2\right]
\end{aligned}$$

$$
\begin{aligned}
&\le E\left[\sup_{0\le s\le S,0\le t\le T}\left(\int_{[t]^-(\tilde\delta)}^{[t]^+(\tilde\delta)}\int_{[s]^-(\tilde\delta)}^{[s]^+(\tilde\delta)} \mid \dot W_\delta(u,v)\mid du\,dv\right)^2\right]\\
&\le \left\{E\left[\max_{0\le j\le m(S),0\le k\le m(T)}\left(\int_{k\tilde\delta}^{(k+1)\tilde\delta}\int_{j\tilde\delta}^{(j+1)\tilde\delta} \mid \dot W_\delta(u,v)\mid du\,dv\right)^4\right]\right\}^{1/2}\\
&\le \left\{\sum_{k=0}^{m(T)}\sum_{j=0}^{m(S)} E\left[\left(\int_{k\tilde\delta}^{(k+1)\tilde\delta}\int_{j\tilde\delta}^{(j+1)\tilde\delta} \mid \dot W_\delta(u,v)\mid du\,dv\right)^4\right]\right\}^{1/2} \qquad (3.1)\\
&= \left\{(1+m(T))(1+m(S))E\left[\left(\int_0^{\tilde\delta}\int_0^{\tilde\delta} \mid \dot W_\delta(u,v)\mid du\,dv\right)^4\right]\right\}^{1/2} \qquad (3.2)\\
&\le K_1\left\{(1+m(T))(1+m(S))n(\delta)^8\delta^4\right\}^{1/2}\\
&\le K_2 n(\delta)^3\delta \to 0 \text{ as } \delta\to 0
\end{aligned}
$$

where (e) in Proposition 1 and (2.6) have been used in (3.1) and (3.2), respectively.

We can now proceed as in the estimation of $I_1(s,t)$ to prove that

$$E\left[\sup_{0\le s\le S,0\le t\le T}\left|I_2(s,t)\right|^2\right] \to 0 \text{ as } \delta\to 0,$$

$$E\left[\sup_{0\le s\le S,0\le t\le T}\left|I_3(s,t)\right|^2\right] \to 0 \text{ as } \delta\to 0,$$

$$E\left[\sup_{0\le s\le S,0\le t\le T}\left|I_4(s,t)\right|^2\right] \to 0 \text{ as } \delta\to 0.$$

From the Cauchy-Schwarz inequality and the two-parameter maximal inequality, we obtain

$$
\begin{aligned}
&E\left[\sup_{0\le s\le S,0\le t\le T}\left|I_5(s,t)\right|^2\right]\\
&= E\left[\sup_{0\le s\le S,0\le t\le T}\left|\int_{[t]^-(\tilde\delta)}^{t}\int_{[s]^-(\tilde\delta)}^{s} dW(u,v)\right|^2\right]\\
&\le \left\{E\left[\max_{0\le j\le m(S),0\le k\le m(T)}\sup_{0\le s,t\le\tilde\delta}\left|\int_{k\tilde\delta}^{t+k\tilde\delta}\int_{j\tilde\delta}^{s+j\tilde\delta} dW(u,v)\right|^4\right]\right\}^{1/2}\\
&\le \left\{\sum_{k=0}^{m(T)}\sum_{j=0}^{m(S)} E\left[\sup_{0\le s,t\le\tilde\delta}\left|\int_{k\tilde\delta}^{t+k\tilde\delta}\int_{j\tilde\delta}^{s+j\tilde\delta} dW(u,v)\right|^4\right]\right\}^{1/2}
\end{aligned}
$$

$$\le \left\{(1+m(T))(1+m(S))E\left[\sup_{0\le s,t\le\tilde\delta} \mid W(s,t)\mid^4\right]\right\}^{1/2} \quad (3.3)$$

$$\le \left\{\left(\frac{4}{3}\right)^8 (1+m(S))(1+m(T))E[\mid W(\tilde\delta,\tilde\delta)\mid^4]\right\}^{1/2} \quad (3.4)$$

$$= \left\{3\left(\frac{4}{3}\right)^8 (1+m(S))(1+m(T))\tilde\delta^4\right\}^{1/2}$$

$$\le K_3\tilde\delta \to 0 \text{ as } \delta \to 0$$

where we have applied the two-parameter maximal inequality to (3.3) for (3.4).

Now we shall estimate $I_6(s,t)$. First note that

$$W_\delta(s,t) = \int_0^1\int_0^1 W(\delta u + s, \delta v + t)\rho(u,v)\,du\,dv.$$

By the Cauchy-Schwarz inequality and the Doob's maximal inequality, we obtain

$$E\left[\sup_{0\le s\le S, 0\le t\le T}\left|I_6(s,t)\right|^2\right]$$

$$\le \left\{E\left[\sup_{0\le s\le S, 0\le t\le T}\left|I_6(s,t)\right|^8\right]\right\}^{1/4}$$

$$\le K_4\left\{E\left[\sup_{0\le s\le S, 0\le t\le T}\int_0^1\int_0^1 \left|W(\delta u + [s]^-(\tilde\delta), \delta v + t)\right.\right.\right.$$

$$\left.\left.\left. - W(\delta u + [s]^-(\tilde\delta), \delta v + [t]^-(\tilde\delta))\right|^8 du\,dv\right]\right\}^{1/4}$$

$$\le K_4\left\{E\left[\max_{0\le j\le m(S), 0\le k\le m(T)}\sup_{0\le t\le\tilde\delta}\int_0^1\int_0^1 \left|W(\delta u + j\tilde\delta, \delta v + t + k\tilde\delta)\right.\right.\right.$$

$$\left.\left.\left. - W(\delta u + j\tilde\delta, \delta v + k\tilde\delta)\right|^8 du\,dv\right]\right\}^{1/4}$$

$$\le K_4\left\{(1+m(T))\sum_{j=0}^{m(S)}\int_0^1\int_0^1 E\left[\sup_{0\le t\le\tilde\delta}\mid W(\delta u + j\tilde\delta, t)\mid^8\right] du\,dv\right\}^{1/4} \quad (3.5)$$

$$\le \left(\frac{8}{7}\right)^8 C_4\left\{(1+m(T))\sum_{j=0}^{m(S)}\int_0^1\int_0^1 E\left[\mid W(\delta u + j\tilde\delta, \tilde\delta)\mid^8\right] du\,dv\right\}^{1/4} \quad (3.6)$$

$$\le K_5\left\{(1+m(T))\sum_{j=0}^{m(S)}\int_0^1\int_0^1 (\delta u + j\tilde\delta)^4\tilde\delta^4\,du\,dv\right\}^{1/4}$$

$$\begin{aligned}
&\leq K_5\left\{(1+m(T))(1+m(S))(\delta+S)^4\tilde{\delta}^4\right\}^{1/4}\\
&\leq K_6\tilde{\delta}^{1/2}\to 0 \text{ as } \delta\to 0.
\end{aligned}$$

Here we have applied the Doob's maximal inequality to (3.5) for (3.6).

Similarly, as for $I_6(s,t)$, we can show that

$$E\left[\sup_{0\leq s\leq S, 0\leq t\leq T}\left|I_7(s,t)\right|^2\right]\to 0 \text{ as } \delta\to 0.$$

According to the Cauchy-Schwarz inequality and the Doob's maximal inequality, we find that

$$\begin{aligned}
&E\left[\sup_{0\leq s\leq S, 0\leq t\leq T}\left|I_8(s,t)\right|^2\right]\\
\leq\ &\left\{E\left[\sup_{0\leq s\leq S, 0\leq t\leq T}\left|W([s]^-(\tilde{\delta}),t)-W([s]^-(\tilde{\delta}),[t]^-(\tilde{\delta}))\right|^8\right]\right\}^{1/4}\\
=\ &\left\{E\left[\max_{0\leq j\leq m(S), 0\leq k\leq m(T)}\sup_{0\leq t\leq\tilde{\delta}}\left|W(j\tilde{\delta},t+k\tilde{\delta})-W(j\tilde{\delta},k\tilde{\delta})\right|^8\right]\right\}^{1/4}\\
=\ &\left\{\sum_{k=0}^{m(T)}\sum_{j=0}^{m(S)}E\left[\sup_{0\leq t\leq\tilde{\delta}}\left|W(j\tilde{\delta},t+k\tilde{\delta})-W(j\tilde{\delta},k\tilde{\delta})\right|^8\right]\right\}^{1/4}\\
=\ &\left\{(1+m(T))\sum_{j=0}^{m(S)}E\left[\sup_{0\leq t\leq\tilde{\delta}}\mid W(j\tilde{\delta},t)\mid^8\right]\right\}^{1/4} \qquad (3.7)\\
\leq\ &\left(\frac{8}{7}\right)^8\left\{(1+m(T))\sum_{j=0}^{m(S)}E\left[\mid W(j\tilde{\delta},\tilde{\delta})\mid^8\right]\right\}^{1/4} \qquad (3.8)\\
\leq\ &K_7\left\{(1+m(T))(1+m(S))\tilde{\delta}^4\right\}^{1/4}\\
\leq\ &K_8\tilde{\delta}^{1/2}\to 0 \text{ as } \delta\to 0.
\end{aligned}$$

where Doob's maximal inequality has been used in (3.7) for (3.8).

Similarly, as for $I_8(s,t)$, we can show that

$$E\left[\sup_{0\leq s\leq S, 0\leq t\leq T}\left|I_9(s,t)\right|^2\right]\to 0 \text{ as } \delta\to 0.$$

Before estimating $I_{10}(s,t)$, we first prove that

$$E\left[\left|W_\delta(j\tilde{\delta},k\tilde{\delta})-W(j\tilde{\delta},k\tilde{\delta})\right|^4\right]\leq c\delta^2. \qquad (3.9)$$

From (2.2) and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&E\left[\left|W_\delta(j\tilde{\delta},k\tilde{\delta})-W(j\tilde{\delta},k\tilde{\delta})\right|^4\right]\\
=\;& E\left[\left|\int_0^\delta\int_0^\delta\left\{W(u+j\tilde{\delta},v+k\tilde{\delta})-W(j\tilde{\delta},k\tilde{\delta})\right\}\rho_\delta(u,v)\,du\,dv\right|^4\right]\\
\le\;& \frac{K_9}{\delta^2}E\left[\int_0^\delta\int_0^\delta\left|W(u+j\tilde{\delta},v+k\tilde{\delta})-W(j\tilde{\delta},k\tilde{\delta})\right|^4du\,dv\right] \qquad (3.10)\\
\le\;& \frac{8K_9}{\delta^2}\int_0^\delta\int_0^\delta E\left[\left|W(u+j\tilde{\delta},v+k\tilde{\delta})-W(j\tilde{\delta},v+k\tilde{\delta})\right|^4\right]du\,dv\\
&+\frac{8K_9}{\delta^2}\int_0^\delta\int_0^\delta E\left[\left|W(j\tilde{\delta},v+k\tilde{\delta})-W(j\tilde{\delta},k\tilde{\delta})\right|^4\right]du\,dv \qquad (3.11)\\
=\;& \frac{8K_9}{\delta^2}\int_0^\delta\int_0^\delta\left\{E\left[\mid W(u,v+k\tilde{\delta})\mid^4\right]+E\left[\mid W(j\tilde{\delta},v)\mid^4\right]\right\}du\,dv\\
=\;& \frac{24K_9}{\delta^2}\int_0^\delta\int_0^\delta\left\{u^2(v+k\tilde{\delta})^2+v^2(j\tilde{\delta})^2\right\}du\,dv\\
=\;& \frac{K_{10}}{\delta^2}\left(\delta^6+k\tilde{\delta}\delta^5+(k\tilde{\delta})^2\delta^4+(j\tilde{\delta})^2\delta^4\right)\\
\le\;& K_{10}\left(\delta^4+T\delta^3+T^2\delta^2+S^2\delta^2\right)\\
\le\;& K_{11}\delta^2
\end{aligned}$$

where the inequality $(a+b)^4 \le 8(a^4+b^4)$ has been used in (3.10) for (3.11). Hence, the assertion (3.9) follows.

Finally we shall estimate $I_{10}(s,t)$. By the Cauchy-Schwarz inequality and (3.9),

$$\begin{aligned}
&E\left[\sup_{0\le s\le S,0\le t\le T}\left|I_{10}(s,t)\right|^2\right]\\
\le\;& \left\{E\left[\sup_{0\le s\le S,0\le t\le T}\left|W_\delta([s]^-(\tilde{\delta}),[t]^-(\tilde{\delta}))-W([s]^-(\tilde{\delta}),[t]^-(\tilde{\delta}))\right|^4\right]\right\}^{1/2}\\
=\;& \left\{E\left[\max_{0\le j\le m(S),0\le k\le m(T)}\left|W_\delta(j\tilde{\delta},k\tilde{\delta})-W(j\tilde{\delta},k\tilde{\delta})\right|^4\right]\right\}^{1/2}\\
=\;& \left\{\sum_{k=0}^{m(T)}\sum_{j=0}^{m(S)}E\left[\left|W_\delta(j\tilde{\delta},k\tilde{\delta})-W(j\tilde{\delta},k\tilde{\delta})\right|^4\right]\right\}^{1/2} \qquad (3.12)\\
\le\;& K_{12}\left\{(1+m(T))(1+m(S))\delta^2\right\}^{1/2}\\
\le\;& K_{13}\frac{1}{n(\delta)}\to 0 \text{ as } \delta\to 0.
\end{aligned}$$

Here we have applied (3.9) to (3.12). This completes the proof of Theorem 1.

Using this mollifiers approximation, we can investigate Wong-Zakai type approximation in the two-parameter case. From the one-parameter case, we can conjecture:

Conjecture. Let $a \in C_b^1(R^1)$ and $\sigma \in C_b^2(R^1)$. Suppose that $\{W_\delta(s,t,\omega)\}_{\delta>0}$ is given as in (2.1). Consider the following equations:

$$X_\delta(s,t) = \int_0^t \int_0^s a(X_\delta(u,v))\,du\,dv + \int_0^t \int_0^s \sigma(X_\delta(u,v))\,dW_\delta(u,v) \qquad (3.13)$$

and

$$X(s,t) = \int_0^t \int_0^s a(X(u,v))\,du\,dv + \int_0^t \int_0^s \sigma(X(u,v)) \circ dW(u,v). \qquad (3.14)$$

where $\circ$ denotes the Stratonovich integral (see Hajek (1982) for definition). Then for every $S, T \geq 0$,

$$\lim_{\delta \to 0} E\left[\sup_{0 \leq s \leq S, 0 \leq t \leq T} \left|X(s,t,w) - X_\delta(s,t,w)\right|^2\right] = 0,$$

where $(X_\delta(s,t,\omega))$ and $(X(s,t,\omega))$ are solutions to equation (3.13) and (3.14), respectively.

Remark 2. $\int_0^t \int_0^s \sigma(X(u,v)) \circ dW(u,v)$ is a special case of the Stratonovich integral defined and studied by Hajek (1982) and from his definition we have

$$\begin{aligned} \int_0^t \int_0^s \sigma(X(u,v)) \circ dW(u,v) &= \int_0^t \int_0^s \sigma(X(u,v))\,dW(u,v) \\ &+ \frac{1}{4} \int_0^t \int_0^s \sigma'(X(u,v))\sigma(X(u,v))\,du\,dv. \end{aligned}$$

Remark 3. Note that the existence and uniqueness of the solution to equation (3.14) follows from Theorem 4.1 in Hajek (1982): Let us fix $\omega \in \Omega'$ with $P(\Omega') = 1$. We define a mapping

$$F : C([0,S] \times [0,T] : R) \to C([0,S] \times [0,T] : R)$$

by

$$F(X_\delta)(s,t) = \int_0^t \int_0^s a(X_\delta(u,v))\,du\,dv + \int_0^t \int_0^s \sigma(X_\delta(u,v))\dot{W}_\delta(u,v)\,du\,dv. \qquad (3.15)$$

Then the proof of the existence and uniqueness of the solution to (3.13) is similar to that for Theorem 4.1 in Hajek (1982).

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