Journal of the Korean
Statistical Society
Vol. 26, No. 2, 1997

Some Basic and Asymptotic Properties in INMA(q) Processes†

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ABSTRACT

We propose an integer-valued MA(q) process with Poisson disturbance. Its various properties are discussed such as the joint distribution, time reversibility and regression. We derive the asymptotic distribution of autocovariance function and estimators of the parameters in the suggested model. We also consider the relationship between INMA(q) and $M/D/\infty$ processes.

Key Words : INMA(q) process; time reversibility; regression; autocovariance function; asymptotic distribution; $M/D/\infty$ process.

1. INTRODUCTION

In recent years, several articles have dealt with statistical data which are expressed in terms of counts taken sequentially in time and correlated. Such

†This paper is supported(in part) by the non-directed research fund, Korea research Foundation(1995).

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discrete variate data are often not appropriate for modelling with the familiar Gaussian ARMA models even as approximation.

Various models have been proposed for discrete valued stationary processes. Jacobs and Lewis (1978a, b and 1983) introduced so-called DARMA models. McKenzie (1986, 1988) attempted to construct some discrete ARMA models for dependent negative binomial and Poisson variate sequences by the use of binomial thinning. Al-Osh and Aly (1992) investigated various properties in a discrete AR(1) model with negative binomial marginals and in a generalized geometric AR(1) process. Alzaid and Al-Osh (1990) also introduced what they have called an integer-valued p-th order autoregressive (INAR(p)) model based on their INAR(1) and integer-valued moving average processes. Aly and Bouzar (1994) suggested some generalized integer-valued ARMA models. Recently, Park and Kim (1995) and Park and Oh (1995) derived some asymptotic properties in the INAR(1) process with Poisson and negative binomial marginals.

The purposes of this work are to introduce a model for a sequence of dependent discrete random variables with Poisson marginal distribution and to derive its asymptotic properties. The proposed model is the same as the models for INMA(q) processes suggested by Al-Osh and Alzaid(1988) in form but has a totally different assumption. With this different assumption, we can apply the results given in the following section to q-type $M/D/\infty$ queueing systems.

Section 2 will investigate their various properties such as the joint distribution, time reversability and regression. In section 3, we will find out the asymptotic behavior of the sample mean, sample autocovariance functions and estimators of parameters in the suggested model. Section 4 will study the relationship between the suggested model and $M/D/\infty$ queueing processes. To investigate the reliability of the parameter estimates, we will conduct a simulation study for INMA(2) model, in section 5.

Define an i.i.d.q-vector, $(Y_{i,j}^{(n)},\ j = 1, 2, \cdots, q)$, $i = 1, 2, \cdots$, $n = 0, \pm 1, \pm 2, \cdots$ of Bernoulli random variables having joint distribution for which

$$P[Y_{i,k_1}^{(n)} = 1, Y_{i,k_2}^{(n)} = 1, \cdots, Y_{i,k_j}^{(n)} = 1] = \beta_{k_j} \tag{1.1}$$

for $0 \leq k_1 \leq k_2 \cdots \leq k_j \leq q$ so that, in particular, $P[Y_{i,k}^{(n)} = 1] = \beta_k$. The joint distribution in (1.1) arises in the following situations. Suppose that customers arrive at a service facility with infinite servers in some random manner and that once served, they are assumed to leave the system immediately. Therefore, if a customer arrives at time n and is observed at time $n+j$ in the system, then he must be present in the system at times preceding time $n+j$ as well. Now let $[Y_{i,j}^{(n)}=1]$ be the event that the i-th customer among those arriving at time n is present in the system at time $n+j$. Then by the above, it is clear that the events $[Y_{i,k_j}^{(n)}=1]$ and $[Y_{i,k_1}^{(n)}=1, Y_{i,k_2}^{(n)}=1,\cdots,Y_{i,k_j}^{(n)}=1]$ are equivalent. Another example arises in a serious disease like AIDS and diabetes which are incurable for now. In this example, we can let $[Y_{i,j}^{(n)}=1]$ be the event that the i-th patient among new patients of AIDS or diabetes at time n will survive until time $n+j$. Note that by the joint distribution in (1.1) , $\beta_{k_j}\geq\beta_{k_l}$ if $k_j<k_l$ since $\{Y_{i,k_1}^{(n)}=1,\cdots,Y_{i,k_j}^{(n)}=1\}\supset\{Y_{i,k_1}^{(n)}=1,\cdots,Y_{i,k_l}^{(n)}=1\}$, (1.1) is the difference in assumption between our model and the model suggested by Alzaid and Al-Osh (1988) and that (1.1) enables us to apply our model to $M/D/\infty$ queueing processes, whereas the Alzaid and Al-Osh model is not applicable to $M/D/\infty$ queueing processes since their model assumed that

$$P[Y_{i,k_1}^{(n)}=1, Y_{i,k_2}^{(n)}=1,\cdots,Y_{i,k_j}^{(n)}=1]=\Pi_{t=1}^{j}\beta_{k_t-k_{t-1}},$$
$$0=k_0<k_1\leq k_2\cdots\leq k_j\leq q.$$

To construct a moving average model of order q, let $\{W_n\}$ be an i.i.d sequence of Poisson random variables with mean λ and be independent of $\{Y_{i,j}^{(n)},\ j=1,2,\cdots,q\}$. In the above examples, W_n is the number of customers arriving during the time interval $(n-1,n]$ or the number of new patients observed during the time interval $(n-1,n]$. We further require a thinning operation in order to describe our process. For $0\leq\beta_j\leq 1$, $1\leq j\leq q$, define

$$\beta_j * W_n=\sum_{i=1}^{W_n} Y_{i,j}^{(n)}$$

where the $Y_{i,j}^{(n)}$ are i.i.d.Bernoulli r.v.'s independent of W_n with $P[Y_{i,j}^{(n)}=1]=\beta_j$ for each j and n from (1.1). The star operation is referred to as thinning. Let $\mathcal{W}_n$ be a set of objects and W_n be the number of objects in $\mathcal{W}_n$. For example, $\mathcal{W}_n$ and W_n might be a waiting line and the number of persons in that waiting lines, respectively, in a queueing system at time n. Thus $\beta_j * W_n$, in the above example, can be interpreted as the number of customers in the system at time $n+j$ among a waiting line, i.e., $\mathcal{W}_n$ or survivors until time $n+j$ among those who have AIDS or diabetes at time n, i.e., $\mathcal{W}_n$. Now define the integer-valued moving average process of order q (INMA(q)) under the assumption (1.1) by

$$X_n=W_n+\beta_1 * W_{n-1}+\beta_2 * W_{n-2}+\cdots+\beta_q * W_{n-q},\ n=0,\pm1,\pm2,\cdots \quad (1.2)$$

where $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_q$. Note that β_j can be interpreted as the probability that a component of $\mathcal{W}_{n-j}$ will be a component of X_n. Model (1.2) is the process with $(q+1)$ unit times as the maximum life for a component of X_n. It can be easily shown that the maginal distribution of X_n in (1.2) is a Poisson distribution with mean $\mu = \lambda \sum_{i=0}^{q} \beta_i$,where $\beta_0 = 1$, by utilizing the probability generating function (p.g.f.) technique. Furthermore, it is apparent that the process is q-dependent, that is, X_{i_1} and X_{i_2} are independent if $|i_2 - i_1| > q$.

2. TIME REVERSIBILITY AND REGRESSION

The property of time reversibility may be a useful diagnostic for making a determination on adopting a model. In order to show that the INMA(q) process defined in (1.2) possesses the property of time reversibility, first consider the joint distribution of $X_n, X_{n-1}, \cdots, X_{n-l+1}$, $l \geq 2$ in terms of their joint p.g.f.. The joint p.g.f. of $X_n, X_{n-1}, \cdots, X_{n-l+1}$ can be shown as

$$\begin{aligned} &E(s_1^{X_n} s_2^{X_{n-1}} \cdots s_{l+1}^{X_{n-l}}) \\ &= \Pi_{j=1}^{min(q,l+1)} E_W \Pi_{i=1}^{W_n - j+1} E\left(s_1^{Y_{i,j-1}^{(n-j+1)}} s_2^{Y_{i,j-2}^{(n-j+1)}} \cdots s_j^{Y_{i,0}^{(n-j+1)}} \right) \\ &\times \Pi_{j_1=1}^{l-q+1} E_W \Pi_{i=1}^{W_n - q - j_1 + 1} E\left(s_{j_1}^{Y_{i,q}^{(n-q-j_1+1)}} s_{j+1}^{Y_{i,q-1}^{(n-q-j_1+1)}} \cdots s_{j_1+q}^{Y_{i,0}^{(n-q-j_1+1)}} \right) \\ &\times \Pi_{j_2=l-q+2}^{l+1} E_W \Pi_{i=1}^{W_n - q - j_2+1} \\ &\qquad \cdot E\left(s_{j_2}^{Y_{i,q}^{(n-q-j_2+1)}} s_{j_2+1}^{Y_{i,q-1}^{(n-q-j_2+1)}} \cdots s_{l+1}^{Y_{i,j_2-(l-q+1)}^{(n-q-j_2+1)}} \right) \end{aligned} \tag{2.1}$$

where E_W is the expectation w.r.t. W, the second term of (2.1) equals 1 if $l - q \leq 1$ and $s_k = 1$ if $k \leq 0$ in the third term. Suppose $l > q-2$ without loss of generality since if $l \leq q-2$ we can obtain the same result as the below.

Since the vector $(Y_{i,k_1}^{(n)}, \cdots, Y_{i,k_j}^{(n)})$, $0 \leq k_1 \leq k_2 \cdot \leq k_j \leq q$ is an i.i.d sequence for each j and n and the joint distribution of the vector depends on only lags $k_1, k_2, \cdots, k_j$, we can set $(Y_{i,k_1}^{(n)}, \cdots, Y_{i,k_j}^{(n)}) \equiv (Y_{k_1}, \cdots, Y_{k_j})$ for all i and n. Thus (2.1) can be written as

$$\begin{aligned}
&\Pi_{j=1}^{q} E_W \left[E(s_1^{Y_{j-1}} s_2^{Y_{j-1}} \cdots s_j^{Y_0})\right]^{W_n - j+1} \\
&\quad \times \Pi_{j_1=1}^{l-q+1} E_W \left[E(s_{j_1}^{Y_q} s_{j_1+1}^{Y_{q-1}} \cdots s_{j_1+q}^{Y_0})\right]^{W_n - q - j_1 + 1} \\
&\quad \times \Pi_{j_2=l-q+2}^{l+1} E_W \left[E(s_{j_2}^{Y_q} \cdots s_{l+1}^{Y_{j_2-(l-q+1)}})\right]^{W_n - l - j_2 + 1}.
\end{aligned} \tag{2.2}$$

Define $P_{i,j}[Y_{k_1} = 1, Y_{k_2} = 1, \cdots, Y_{k_m} = 1]$ to be the probability that $Y_{k_1} = 1, Y_{k_2} = 1, \cdots, Y_{k_m} = 1$, $0 \le i \le k_1 \le k_2 \cdots \le k_m \le j \le q$ and $Y_k = 0, k > k_m$ among $Y_i, Y_{i+1}, \cdots, Y_j$.

First, consider $1 \le j \le q$. The coefficient of $s_{j-k} s_{j-k+1} \cdots s_j$, $k \le j-1$ in the first term of (2.2) is

$$P_{0,j-1}[Y_0 = 1, \cdots, Y_k = 1] = \begin{cases} \beta_k - \beta_{k+1} & \text{if } k < j-1 \\ \beta_{j-1} & \text{if } k = j-1. \end{cases} \tag{2.3}$$

Note that the reverse order of $s_{j-k} s_{j-k+1} \cdots s_j$ is $\Pi_{t=0}^{k} s_{l-(j-t)+2}$ in the sense that the reverse order of $\Pi_{i=1}^{l+1} s_i$ is $\Pi_{i=-1}^{l-1} s_{l-i}$.

Now, consider the coefficient of $\Pi_{t=0}^{k} s_{l-(j-t)+2}$, $1 \le j \le q$, for $k < j-1$. This coefficient can be found only in the second term in (2.2). More precisely, there exists only one case having the terms $\Pi_{t=0}^{k} s_{l-(j-t)+2}$ in the second term of (2.2) since, in the second term of (2.2), no cases have $s_{l-(j-k)+2}$ if $j_1 < l-q-(j-k)+2$, and we also have undesirable elements, $s_{l-(j-k)+t}$, $3 \le t \le j-k+1$, $1 \le j \le q$, with probability 1 if $j_1 > l-q-(j-k)+2$. And we can not obtain $\Pi_{t=0}^{k} s_{l-(j-t)+2}$ in the third term of (2.2) since the undesirable term s_{l+1} is in $\Pi_{t=0}^{k} s_{l-(j-t)+2}$. Hence only when $j_1 = l-q-(j-k)+2$ in the second term of (2.2) the coefficient of $\Pi_{t=0}^{k} s_{l-(j-t)+2}$ exists and it is

$$P_{0,q}[Y_0 = 1, Y_1 = 1, \cdots, Y_k = 1] = \beta_k - \beta_{k+1}. \tag{2.4}$$

When $k = j-1$, we can find the reverse order, $\Pi_{t=0}^{j-1} s_{l-(j-t)+2}$ of $s_1 s_2 \cdots s_j$ in the last case of the second term and in the third term. Namely, only possible case to obtain $\Pi_{t=0}^{j-1} s_{l-(j-t)+2}$ in the second term is when $j_1 = l-q+1$ with probability $P_{0,q}[Y_0 = 1, Y_1 = 1, \cdots, Y_{j-1} = 1] = \beta_{j-1} - \beta_j$. In the third term of (2.2) when $j_2 = l-q+2$, we have $s_{l-q+2}^{Y_q} s_{l-q+3}^{Y_{q-1}} \cdots s_{l+1}^{Y_1}$. To have only $\Pi_{t=0}^{j-1} s_{l-(j-t)+2}$ in $\Pi_{i=2}^{q+1} s_{l-q+i}$, only $Y_1 = 1, Y_2 = 1, \cdots, Y_j = 1$ among $Y_1, \cdots, Y_q$ in which the probability is

$$P_{1,q}[Y_1 = 1, Y_2 = 1, \cdots, , Y_j = 1] = \beta_j - \beta_{j+1}.$$

Similarly, when $j_2 = l-q+u+1$, $2 \le u \le q-j$ we have $s_{l-q+u+1}^{Y_q} s_{l-q+u+2}^{Y_{q-1}} \cdots s_{l+1}^{Y_u}$. Thus with $P_{u,q}[Y_u = 1, Y_{u+1} = 1, \cdots, Y_{j+u-1} = 1] = \beta_{j+u} - \beta_{j+u+1}$, we have only $\Pi_{t=0}^{j-1} s_{l-(j-t)+2}$ in $\Pi_{i=u+1}^{q+1} s_{l-q+i}$. Note that when $j_2 = l-j+2$, i.e., $u = q+1-j$, we have $s_{l-j+2}^{Y_q} s_{l-j+3}^{Y_{q-1}} \cdots s_{l+1}^{Y_{q-j+1}}$ which is the last possible case to obtain $\Pi_{t=0}^{j-1} s_{l-(j-t)+2}$ with probability $P_{q+1-j,q}[Y_{q+1-j} = 1, \cdots, Y_q = 1] = \beta_q$. Therefore, the coefficient of $\Pi_{t=0}^{j-1} s_{l-(j-t)+2}$, $1 \le j \le q$ is

$$\sum_{u=0}^{q+1-j} P_{u,q}[Y_u = 1, Y_{u+1} = 1, \cdots, Y_{j+u-1} = 1] = \beta_{j-1}. \tag{2.5}$$

(2.3), (2.4) and (2.5) show that the coefficient of $\Pi_{t=0}^{k} s_{j-t}$, $1 \le j \le q$, $1 \le k \le j-1$, is the same as that of $\Pi_{t=0}^{k} s_{l-(j-t)+2}$.

Secondly, consider the coefficient of $s_{j-k} s_{j-k+1} \cdots s_j$, $q+1 \le j \le \left[\frac{l-q+2}{2}\right]$, $k \le j-1$ in (2.2). This coefficient can be obtained only in the second term of (2.2) and it is

$$P_{0,q}[Y_0 = 1, Y_1 = 1, \cdots, Y_k = 1] == \begin{cases} \beta_k - \beta_{k+1} & \text{if } k < j-1 \\ \beta_q & \text{if } k = q. \end{cases} \tag{2.6}$$

The corresponding reverse order to $\Pi_{t=0}^{k} s_{j-t}$ is $\Pi_{t=0}^{k} s_{l-(j-t)+2}$. Since the third term of (2.2) does no contain s_{l-j+2}, $q+1 \le j \le \left[\frac{l-q+2}{2}\right]$, the coefficient of $\Pi_{t=0}^{k} s_{l-(j-t)+2}$ exists only in the second term of (2.2). By the same arguments as for $1 \le j \le q$, there is only one case that is $s_{l-(j-k)+2-q}^{Y_q} s_{l-(j-k)+2-(q-1)}^{Y_{q-1}} \cdots s_{l-(j-k)+2}^{Y_0}$ to obtain the coefficient of $\Pi_{t=0}^{k} s_{l-(j-t)+2}$ which is

$$P_{0,q}[Y_0 = 1, Y_1 = 1, \cdots, Y_k = 1] = \begin{cases} \beta_k - \beta_{k+1} & \text{if } k < j-1 \\ \beta_q & \text{if } k = q. \end{cases} \tag{2.7}$$

Here, (2.6)=(2.7). Thus, by (2.3)-(2.7), the coefficient of $\Pi_{t=0}^{k} s_{j-t}$ for all $1 \le j \le l+1$, $k \le j-1$ is the same as that of $\Pi_{t=0}^{k} s_{l-(j-t)+2}$ since , for $\left[\frac{l-q+2}{2}\right] + 1 \le j \le l+1$, the procedure is simply a copy of the previous arguments.

Finally, since $\{W_n, n = 0, \pm 1, \pm 2, \cdots\}$ are i.i.d.Poisson sequence with parameter λ, by (2.2)-(2.7),

$$
\begin{aligned}
&E(s_1^{X_n} s_2^{X_{n-1}} \cdots s_{l+1}^{X_{n-l}}) \\
&= \exp[-\lambda\{\sum_{j=1}^{q}\Big(1 - E(s_1^{Y_{j-1}} s_2^{Y_{j-2}} \cdots s_j^{Y_0})\Big) + \sum_{j=1}^{l-q+1}\Big(1 - E(s_j^{Y_q} s_{j+1}^{Y_{q-1}} \cdots s_{j+q}^{Y_0})\Big) \\
&\quad + \sum_{j=l-q+2}^{l+1}\Big(1 - E(s_j^{Y_q} s_{j+1}^{Y_{q-1}} \cdots s_{l+1}^{Y_{j-(l-q+1)}})\Big)\}] \\
&= \exp[-\lambda\{\sum_{j=1}^{q}\Big(1 - E(s_{l+1}^{Y_{j-1}} s_l^{Y_{j-2}} \cdots s_{l-j+2}^{Y_0})\Big) \\
&\quad + \sum_{j=1}^{l-q+1}\Big(1 - E(s_{l-j+2}^{Y_q} s_{l-j+1}^{Y_{q-1}} \cdots s_{l-j-q+2}^{Y_0})\Big) \\
&\quad + \sum_{j=l-q+2}^{l+1}\Big(1 - E(s_{l-j+2}^{Y_q} s_{l-j+1}^{Y_{q-1}} \cdots s_1^{Y_{j-(l-q+1)}})\Big)\}] \\
&= E(s_{l+1}^{X_n} s_l^{X_{n-1}} \cdots s_1^{X_{n-l}}).
\end{aligned} \tag{2.8}
$$

(2.8) implies that INMA(q) process is time reversible as well as strictly stationary since the joint p.g.f. of $X_n, X_{n-1}, \cdots, X_{n-l+1}$ does not depend on the time n.

Another interesting property of the Poisson INMA(q) process is whether the regression of X_n on $X_{n-1}, \cdots, X_{n-q}$, $E(X_n|X_{n-1}, \cdots, X_{n-q})$, is linear in $X_{n-1}, \cdots, X_{n-q}$. This conditional expectation can be found by utilizing the joint p.g.f. of $X_n, X_{n-1}, \cdots, X_{n-q}$, which results in

$$
\begin{aligned}
&E(X_n|X_{n-1} = x_1, X_{n-2} = x_2, \cdots, X_{n-q} = x_q) \\
&= \lambda\{1 + \beta_1 \frac{p(x_1 - 1)}{p(x_1)} + \beta_2 \frac{p(x_1 - 1, x_2 - 1)}{p(x_1, x_2)} \\
&\quad + \cdots + \beta_q \frac{p(x_1 - 1, x_2 - 1, \cdots, x_q - 1)}{p(x_1, x_2, \cdots, x_q)}\},
\end{aligned} \tag{2.9}
$$

where $p(x_1, \cdots, x_q)$ is the joint probability mass function for the random variables $X_{n-1}, \cdots, X_{n-q}$. Thus one can see by (2.9) that the process possesses the linearity property in X_{n-1}, if $q = 1$, i.e. $E(X_n|X_{n-1}) = \lambda + \frac{\beta x}{1+\beta}$, but generally X_n does not have a linear regression on $x_{n-1}, \cdots, x_{n-q}$, for $q \geq 2$.

3. ASYMPTOTIC DISTRIBUTIONS OF SAMPLE AUTOCOVARIANCE FUNCTION

The estimator which we shall use for the autocovariance function $\gamma(p)$ from observations of $X_1, \ldots, X_n$ is

$$\hat{\gamma}(p) = \frac{1}{n}\sum_{t=1}^{n-p}(X_t - \bar{X}_n)(X_{t+p} - \bar{X}_n), \quad 0 \le p \le q,$$

where $\bar{X}_n = \frac{1}{n}\sum_{t=1}^{n} X_t$.

A straightforward calculation with assumption (2.1) and conditional expectation principle gives for $0 \le p \le q$,

$$\begin{aligned} E(X_n X_{n\pm p}) &= \sum_{i=0}^{q}\sum_{j=-p}^{q-p} E(\beta_i * W_{n-i} \cdot \beta_{j+p} * W_{n-j}) \\ &= E(W_1)\sum_{i=p}^{q}\beta_i + (Var(W_1) - E(W_1))\sum_{i=0}^{q-p}\beta_i\beta_{i+p} + \mu^2 \\ &= \lambda\sum_{i=p}^{q}\beta_i + \mu^2. \end{aligned} \tag{3.1}$$

Hence, (3.1) implies that $\gamma(\pm p) = \lambda\sum_{i=p}^{q}\beta_i$, $p \le q$ and $\gamma(\pm p) = 0$, $p > q$. Consequently, the INMA(q) process has only non-negative autocovariance and possesses the cut-off property which is the same as that of the usual continuous valued MA(q) process.

After some tedious calculations, we obtain that for $h, p, r \ge 0$,

$$\begin{aligned} &E(X_n X_{n+p} X_{n+p+h} X_{n+p+h+r}) - 3\mu^3 = \sum_{i=0}^{q}\sum_{j=-p}^{q-p}\sum_{k=-(p+h)}^{q-(p+h)}\sum_{l=-(p+h+r)}^{q-(p+h+r)} \\ &\qquad E(\beta_i * W_{n-i} \cdot \beta_{j+p} * W_{n-j} \cdot \beta_{k+p+h} * W_{n-k} \cdot \beta_{l+p+h+r} * W_{n-l}) \\ &\quad = \lambda\sum_{i=p+h+r}^{q}\beta_i + \gamma(p)\gamma(r) + \gamma(p+h)\gamma(h+r) + \gamma(p+h+r)\gamma(h) \\ &\qquad +\mu\sum_{j_1<j_2<j_3} E(X_{j_1}X_{j_2}X_{j_3}) - \mu^2\sum_{j_1<j_2} E(X_{j_1}X_{j_2}) \end{aligned} \tag{3.2}$$

where $j_1, j_2, j_3 = n, n+p, n+p+h, n+p+h+r$.

By utilizing (3.2), we have the following simple expression

$$\begin{aligned} &E((X_n - \mu)(X_{n+p} - \mu)(X_{n+p+h} - \mu)(X_{n+p+h+r} - \mu)) \\ &\quad = \lambda\sum_{i=p+h+r}^{q}\beta_i + \gamma(p)\gamma(r) + \gamma(p+h)\gamma(h+r) + \gamma(p+h+r)\gamma(h). \end{aligned} \tag{3.3}$$

Define $\tilde{\gamma}(h) = n^{-1}\sum_{t=1}^{n}(X_t - \mu)(X_{t+h} - \mu)$ and $X_{t+k}^* = X_{t+k} - \mu$. Then we have the following results.

Lemma 3.1. For $0 \le p_1 \le p_2 \le q$,

$$\begin{aligned} v_{p_1,p_2} &= \lim_{n\to\infty} nCov(\tilde{\gamma}(p_1), \tilde{\gamma}(p_2)) \\ &= \sum_{\xi=0}^{q}\{\gamma(\xi)\gamma(p_1-\xi-p_2) + \gamma(\xi+p_2)\gamma(p_1-\xi) \\ &+ \gamma(\xi)\gamma(\xi+p_1-p_2) + \gamma(\xi+p_1)\gamma(p_2-\xi)\} \\ &+ \lambda[(p_1-p_2)\sum_{i=p_1}^{q}\beta_i + \sum_{\xi=p_1-p_2}^{n-1}\sum_{i=\xi+p_2}^{q}\beta_i + \sum_{\xi=0}^{n-1}\sum_{i=\xi+p_1}^{q}\beta_i]. \end{aligned}$$

Proof. Observe that

$$\begin{aligned} &E(\tilde{\gamma}(p_1)\tilde{\gamma}(p_2)) \\ &= E\left(\frac{1}{n^2}\sum_{t=1}^{n}\sum_{s=1}^{n} X_t^* X_{t+p_1}^* X_s^* X_{s+p_2}^*\right) \\ &= \frac{1}{n^2}E(\sum_{\xi=0}^{p_1-p_2-1}(n-\xi)X_t^*X_{t+\xi}^*X_{t+\xi+p_2}^*X_{t+p_1}^* \\ &+ \sum_{\xi=p_1-p_2}^{p_1-1}(n-\xi)X_t^*X_{t+\xi}^*X_{t+p_1}^*X_{t+\xi+p_2}^* + \sum_{\xi=p_1}^{n-1}(n-\xi)X_t^*X_{t+p_1}^*X_{t+\xi}^*X_{t+\xi+p_2}^* \\ &+ \sum_{\xi=0}^{p_2-1}(n-\xi)X_t^*X_{t+\xi}^*X_{t+p_2}^*X_{t+\xi+p_1}^* + \sum_{\xi=p_2}^{n-1}(n-\xi)X_t^*X_{t+p_2}^*X_{t+\xi}^*X_{t+\xi+p_1}^* \quad (3.4) \end{aligned}$$

By (3.3), (3.4) can be rewritten as

$$\begin{aligned} &\gamma(p_1)\gamma(p_2) + \sum_{\xi=0}^{n-1}\frac{(n-\xi)}{n^2}[\gamma(\xi)\gamma(p_1-\xi-p_2) + \gamma(\xi+p_2)\gamma(p_1-\xi) \\ &+ \gamma(\xi)\gamma(\xi+p_1-p_2) + \gamma(\xi+p_1)\gamma(p_2-\xi)] \\ &+ \frac{\lambda}{n^2}[\sum_{\xi=0}^{p_1-p_2-1}(n-\xi)\sum_{i=p_1}^{q}\beta_i + \sum_{\xi=p_1-p_2}^{p_1-1}(n-\xi)\sum_{i=\xi+p_2}^{q}\beta_i \\ &+ \sum_{\xi=p_1}^{n-1}(n-\xi)\sum_{i=\xi+p_2}^{q}\beta_i + \sum_{\xi=0}^{p_2-1}(n-\xi)\sum_{i=\xi+p_1}^{q}\beta_i \\ &+ \sum_{\xi=p_2}^{n-1}(n-\xi)\sum_{i=\xi+p_1}^{q}\beta_i]. \end{aligned}$$

Now, subtracting $\gamma(p_1)\gamma(p_2)$ from $E(\tilde{\gamma}(p_1)\tilde{\gamma}(p_2))$ and multiplying it by n, we have the result by letting $n \to \infty$.

Theorem 3.2. Let $\{X_n\}$ be a sequence of INMA(q) process given in (1.2). Then

$$\sqrt{n}\begin{pmatrix} \tilde{\gamma}(0)-\gamma(0) \\ \tilde{\gamma}(1)-\gamma(1) \\ \vdots \\ \tilde{\gamma}(h)-\gamma(h) \end{pmatrix} \longrightarrow AN\left(\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix},\ V\right), \qquad h \le q$$

where $V = [\ v_{p_1,p_2}\ ]_{p_1,p_2=0,\cdots,h}$ and v_{p_1,p_2} is defined in Lemma 3.1.

Proof. Let $\underline{Y}'_n = (X^*_nX^*_n,\ X^*_nX^*_{n+1},\cdots,X^*_nX^*_{n+h})$. Then $\underline{Y}_n$ is a strictly stationary $(q+h)$ dependent sequence since $\{X_n\}$ is a strictly stationary q dependent sequence by (2.4). Observe that $\frac{1}{n}\sum_{t=1}^{n}\underline{Y}'_t = \Big(\tilde{\gamma}(0),\tilde{\gamma}(1),\cdots,\tilde{\gamma}(h)\Big)$ and the i-th diagonal element of V is positive for $i = 1,2,\cdots,q$ from Lemma 3.1. Hence V is a positive definite matrix so that

$$\lim_{n\to\infty} n^{-1}Var(\sum_{i=1}^{n} a'Y_t) = a'Va > 0 \text{ for all vectors } a \in R^{h+1}.$$

Therefore, by Theorem 6.4.2(Brockwell and Davis(1987)) and Cramer-Wold device, the result follows.

Next, we show that, under the assumptions of Theorem 3.2, $\tilde{\gamma}(h)$ and $\hat{\gamma}(h)$ have the same asymptotic distribution.

Theorem 3.3. If X_n is according to the model (2.2), then

$$\sqrt{n}\begin{pmatrix} \hat{\gamma}(0)-\gamma(0) \\ \hat{\gamma}(1)-\gamma(1) \\ \vdots \\ \hat{\gamma}(h)-\gamma(h) \end{pmatrix} \longrightarrow AN\left(\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix},\ V\right).$$

Proof. A simple algebra gives, for $0 \le h \le q$,

$$\begin{aligned}\sqrt{n}\Big(\tilde{\gamma}(h)-\hat{\gamma}(h)\Big) &= \sqrt{n}(\bar{X}_n-\mu)\Big[\frac{1}{n}\sum_{t=1}^{n-h}X_t + \frac{1}{n}\sum_{t=1}^{n-h}X_{t+h} - \Big(1-\frac{h}{n}\Big) \\ &\times\ (\bar{X}_n-\mu)\Big] + \frac{1}{\sqrt{n}}\sum_{t=n-h+1}^{n}(X_t-\mu)(X_{t+h}-\mu). \quad (3.5)\end{aligned}$$

The last term in (3.5) is $o_p(1)$, since $n^{-1/2}E\Big|\sum_{t=n-h+1}^{n}(X_t-\mu)(X_{t+h}-\mu)\Big| \le$

$n^{-1/2}h\gamma(0)$. Now, from (3.1)

$$\begin{aligned} E(\bar{X}_n^2) &= E(\frac{1}{n^2}\sum_{t=1}^{n} X_t \sum_{s=1}^{n} X_s) = \frac{1}{n^2}\Big(nE(X_t^2) + 2\sum_{\xi=1}^{n-1}(n-\xi)E(X_t X_{t+\xi})\Big) \\ &= \frac{1}{n}\Big(\mu + 2\lambda\sum_{\xi=1}^{n-1}(1-\frac{\xi}{n})\sum_{i=\xi}^{q}\beta_i\Big) + \mu^2. \end{aligned} \tag{3.6}$$

(3.6) implies that $\lim_{n\to\infty} nVar(\bar{X}_n) = \mu + 2\lambda\sum_{\xi=1}^{q}\sum_{i=\xi}^{q}\beta_i \neq 0$. Thus again by Theorem 6.4.2(Brockwell and Davis),

$$\sqrt{n}(\bar{X}_n - \mu) \longrightarrow AN\Big(0, \mu + 2\lambda\sum_{\xi=1}^{q}\sum_{i=\xi}^{q}\beta_i\Big)$$

so that the first term is $o_p(1)$. This completes the proof.

Theorem 3.4. Under the same assumptions in Theorem 3.3, natural moment estimators of λ and β_j, $j = 1, 2, \cdots, q$, are

$$\hat{\lambda} = \hat{\gamma}(0) - \hat{\gamma}(1),$$

$$\hat{\beta}_j = \frac{\hat{\gamma}(j) - \hat{\gamma}(j+1)}{\hat{\gamma}(0) - \hat{\gamma}(1)}.$$

And, we have

$$\sqrt{n}\begin{pmatrix} \hat{\lambda} - \lambda(0) \\ \hat{\beta}(1) - \beta(1) \\ \vdots \\ \hat{\beta}(q) - \beta(q) \end{pmatrix} \longrightarrow AN\left(\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad n^{-1}DVD'\right),$$

where V is defined by Theorem 3.2 and D matrix is given by

$$D = (\gamma^*(0))^{-2}\begin{bmatrix} (\gamma^*(0))^2 & -(\gamma^*(0))^2 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ -\gamma^*(1) & \gamma(0)-\gamma(2) & -1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ -\gamma^*(2) & \gamma^*(2) & 1 & -1 & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -\gamma^*(i) & \gamma^*(i) & \cdots & 0 & 1 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -\gamma(q) & \gamma(q) & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \end{bmatrix}$$

where $\gamma^*(j) = \gamma(j) - \gamma(j+1)$.

Proof. Moment estimators are obtained by solving $\gamma(j) = \lambda \sum_{i=j}^{q} \beta_i$. Let $g(\cdot)$ be the function from R^{q+1} into R^{q+1} defined by

$$g([\gamma(0), \gamma(1), \cdots, \gamma(q)]') = [\gamma^*(0), \frac{\gamma^*(1)}{\gamma^*(0)}, \cdots, \frac{\gamma(q)}{\gamma^*(0)}], \quad \gamma^*(0) > 0.$$

Then the result is immediate by Theorem 3.2.

4. RELATIONSHIP BETWEEN INMA(q) AND $M/D/\infty$ PROCESSES

Suppose that customers arrive at a service station in accordance with a Poisson process with rate ξ. Upon arrival, the customer is immediately served by one of an infinite number of possible servers and the service times are assumed to be a constant ν. Assume that we observe this system once every τ time units in which τ satisfies $\tau < \nu < 2\tau$. Let X_n be the queue length of such a system at time $n\tau$. Then X_n is known as an $M/D/\infty$ queueing process.

Let Z_n and Y_{n-1} be the number of the customers during the time interval $((n-1)\tau, n\tau]$ and the customers who arrived during the time interval $((n-2)\tau, (n-1)\tau]$ but was not served completely until time $n\tau$, respectively. Note $X_n = Z_n + Y_{n-1}$. $\{Z_n\}$ is an i.i.d Poisson sequence with parameter $\tau\xi$ due to the nature of a Poisson process, and Z_n and Y_{n-1} are independent since Y_{n-1} is a thinning process of Z_{n-1}. For a customer to be a member of Y_{n-1} among customers arriving at the system during the time interval $((n-2)\tau, (n-1)\tau]$, he should arrive at the system during time interval $(n\tau - \nu, (n-1)\tau]$. The probability that a customer will be such a member is $\frac{\nu - \tau}{\tau}$ since the distribution of the time at which the arrival occurs during the time interval $(n\tau - \nu, (n-1)\tau]$ given that one event has occured in $((n-2)\tau, (n-1)\tau]$ is uniform on $((n-2)\tau, (n-1)\tau]$ by the Theorem 3.2.(Ross(1985)). Hence the distribution of Y_{n-1} is Poisson $(\frac{\nu - \tau}{\tau}\tau\xi)$ due to the Poisson thinning processes. Consequently, if we set $\beta = \frac{\nu - \tau}{\tau}$, $\lambda = \tau\xi$ and q=1 in (1.2), INMA(1) processes represent $M/D/\infty$ queueing processes with constraint $\tau < \nu < 2\tau$ since $Y_{n-1} \stackrel{d}{=} \beta * Z_{n-1}$.

Next, consider an $M/D/\infty$ system having 2 kinds of jobs with constant service times of duration ν_1 and ν_2 with probabilties λ_1 and $1-\lambda_1$, respectively. Let K_{n-2} be the customers who arrived during the time interval $((n-3)\tau, (n-2)\tau]$ but was not served completely until time $n\tau$. Then we have $X_n = Z_n+Y_{n-1}+K_{n-2}$. By the similar approach as in the above, one can see that Z_n , Y_{n-1} and K_{n-2} are independent, and Z_n is Poisson$(\tau\xi)$, Y_{n-1} is Poisson$(\beta_1\tau\xi)$ and K_{n-2} is a Poisson$(\beta_2\tau\xi)$ where $\beta_1 = \dfrac{(\nu_1-\tau)\lambda_1+\tau(1-\lambda_1)}{\tau}$ and $\beta_2 = \dfrac{(\nu_2-2\tau)(1-\lambda_1)}{\tau}$. Since $Y_{n-1} \overset{d}{=} \beta_1 * Z_{n-1}$ and $K_{n-2} \overset{d}{=} \beta_2 * Z_{n-2}$, $X_n \overset{d}{=} Z_n + \beta_1 * Z_{n-1} + \beta_2 * Z_{n-2}$ which is the INMA(2) process.

Similarly, INMA(q) processes can also represent a generalized $M/D/\infty$ system by supposing that there are q kinds of jobs and each customer independently receives service i which has a constant service time ν_i, with probabilty λ_i, $i = 1, \cdots, q$ where $\sum_{i=1}^{q} \lambda_i = 1$. If the interval between observations from the process satisfies $\tau < \nu_1 < 2\tau < \nu_2 < 3\tau < \cdots < \nu_q < q\tau$, by similar argument as in the above, we can find

$$\beta_i = \frac{(\nu_i - i\tau)\lambda_i + \sum_{j=i+1}^{q} \lambda_i\tau}{\tau}. \tag{4.1}$$

After solving (4.1) about λ_i, one can easily find the estimators $\hat{\lambda}_i$ and the joint limiting distribution of $(\hat{\lambda}_1, \hat{\lambda}_2, \cdots, \hat{\lambda}_q)$ by using Theorem 3.4. For example, let's take $q = 2$. By Theorem 3.4, it can be shown that

$$\sqrt{n}\begin{pmatrix} \hat{\lambda}_1 - \lambda_1 \\ \hat{\lambda}_2 - \lambda_1 \end{pmatrix} \longrightarrow AN\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad WDVD'W'\right),$$

where $\hat{\lambda}_1 = \dfrac{(\nu_2-2\tau)\hat{\beta}_1\tau - \hat{\beta}_2\tau^2}{(\nu_1-\tau)(\nu_2-2\tau)}$, $\hat{\lambda}_2 = \dfrac{\hat{\beta}_2\tau}{(\nu_2-2\tau)}$, D and V are given in Theorem 3.4 and W is given as follows:

$$W = \begin{bmatrix} 0 & \dfrac{\tau}{\nu_2 - 2\tau} \\ \dfrac{\tau}{\nu_1-\tau} & \dfrac{\tau^2}{(\nu_1-\tau)(\nu_2-2\tau)} \end{bmatrix}.$$

5. SIMULATION STUDY

To investigate the reliablity of our estimates , $\hat{\lambda}$, $\hat{\beta}_1$ and $\hat{\beta}_2$ given in Theorem 3.4, a simulation study was conducted for the case in which X_n is according to the model INMA(2). The simulation is performed for samples of size n = 50, 100 and 500 for different values of the parameters λ =10; β_1 =0.3, 0.5, 0.7, 0.9 and β_2 =0.1, 0.3, 0.5, 0.7. For every possible combination of the parameters β_1 and β_2 with constraint $\beta_1 \geq \beta_2$, 500 replications are made on each sample. The average bias and mean square error(MSE) of parameter estimates are calculated. the MSE criterion is less meaningful than the bias criterion since $1 \geq \beta_1 \geq \beta_2 \geq 0$. Therefore, we consider only the average bias

.

Simulation results are tabulated in Tables 5.1 through 5.3 . In the simulation studies, it turns out from Tables 5.1 through 5.3 that for moderate small sample sizes n = 50, 100, $\hat{\lambda}, \hat{\beta}_1$ and $\hat{\beta}_2$ are unstable when β_1 and β_2 have high values , in particular, $\beta_1 = 0.9$ and $\beta_2 = 0.7$ while $\hat{\lambda}$, $\hat{\beta}_1$ and $\hat{\beta}_2$ are stable for the other values of β_1 and β_2. However, for a large sample size, i.e, n = 500, we can see that our estimators are sufficiently close to true parameters.

Table 5.1 Bias and MSE for n = 50

		$\hat{\lambda}$		$\hat{\beta}_1$		$\hat{\beta}_2$	
β_1	β_2	bias	MSE	bias	MSE	bias	MSE
0.3	0.1	-0.849	5.230	0.085	0.153	0.143	0.060
0.5	0.1	-0.504	6.883	0.071	0.266	0.163	0.007
	0.3	0.444	7.448	0.033	0.327	0.036	0.050
0.7	0.1	0.670	3.910	0.113	0.074	0.101	0.670
	0.3	0.641	9.589	0.022	0.400	0.045	0.061
	0.5	1.894	13.397	-0.110	0.279	-0.087	0.064
0.9	0.1	-0.172	5.546	0.139	0.999	0.088	0.028
	0.3	0.977	11.010	-0.002	0.497	0.025	0.055
	0.5	2.368	16.345	-0.204	0.308	-0.091	0.066
	0.7	3.740	25.523	-0.329	0.329	-0.248	0.111

Table 5.2 Bias and MSE for $n = 100$

		$\hat{\lambda}$		$\hat{\beta}_1$		$\hat{\beta}_2$	
β_1	β_2	bias	MSE	bias	MSE	bias	MSE
0.3	0.1	–0.321	2.596	0.013	0.050	0.078	0.023
0.5	0.1	–0.269	3.920	0.022	0.128	0.140	0.038
	0.3	0.338	5.429	0.022	0.147	0.033	0.044
0.7	0.1	–0.708	5.295	0.110	0.220	0.044	0.026
	0.3	0.206	5.881	0.053	0.024	–0.020	0.028
	0.5	–1.343	8.186	–0.089	0.151	–0.058	0.051
0.9	0.1	–0.748	5.667	0.158	0.343	–0.004	0.023
	0.3	1.560	7.002	0.006	0.284	0.028	0.045
	0.5	1.571	10.099	–0.122	0.218	–0.067	0.053
	0.7	2.510	14.132	–0.230	0.212	–0.174	0.076

Table 5.3 Bias and MSE for $n = 500$

		$\hat{\lambda}$		$\hat{\beta}_1$		$\hat{\beta}_2$	
β_1	β_2	bias	MSE	bias	MSE	bias	MSE
0.3	0.1	–0.002	0.475	–0.009	0.008	0.017	0.004
0.5	0.1	–0.009	0.625	–0.008	0.012	0.020	0.004
	0.3	0.125	1.151	–0.005	0.020	0.003	0.125
0.7	0.1	–0.068	08585	0.003	0.021	0.025	0.005
	0.3	0.120	1.509	–0.006	0.034	0.006	0.011
	0.5	0.253	2.039	–0.014	0.017	0.002	0.253
0.9	0.1	–0.176	1.143	0.025	0.034	0.034	0.008
	0.3	0.158	1.899	–0.003	0.052	0.010	0.014
	0.5	0.303	2.446	–0.021	0.061	0.004	0.021
	0.7	0.603	2.868	–0.064	0.060	–0.030	0.026

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