Journal of the Korean
Statistical Society
Vol. 26, No. 2, 1997

Effects of Order Misspecification on Unit Root Tests†

Dong-Wan Shin[1] and Yoon-Dong Lee[2]

ABSTRACT

Effects of order misspecification on statistical behavior of unit root tests are studied. We derive the limiting distributions of the Dickey-Fuller test statistics whose numerators are of the form $c\int W dW + \kappa$ where W is a standard Brownian motion on [0,1] and c is a real number. The term κ is a major consequence of order misspecification and its explicit expression is derived. Based on an analysis of κ, effects of order misspecification on unit root tests for AR(2), ARMA(1,1), and AR(3) models are investigated.

Key Words : Dickey-Fuller tests; Misspecification; Nonstationary time series; Unit root.

---

†This research was supported by NON-DIRECTED RESEARCH FUND, Korea Research Foundation, 1996.
[1]Department of Statistics, Ewha Womans University, Seodaemun, Seoul, 120-750, Korea
[2]Department of Statistics, Iowa State University, IA50010, USA.

1. INTRODUCTION

In time series analysis applications, model misspecification is a common problem because the true model of the time series is rarely known. One of the most frequent type of order misspecification happens in autoregressive modeling. In the real world, there are lots of nonstationary time series which are modeled as autoregressive processes. Nelson and Plosser(1982) showed that many macro economic time series can be modeled as autoregressive processes with one autoregressive unit root. Various tests have been developed for testing unit root by Dickey and Fuller(1979) and many others. The results are summarized in the books of Fuller(1996) and Hamilton(1994).

Order misspecification affects the statistical performance of the unit root tests. Due to misspecification, the unit root hypotheses may be rejected more or less often than the nominal level when the true process is a random walk. Also, misspecification can be a source of low power of unit root tests. In order to circumvent the problem due to order misspecification, Said and Dickey(1984) proposed a test based on a long lag autoregression. Phillips(1987) proposed a semiparametric test based on an AR(1) estimation. Pantula and Hall(1991) made a Monte-Carlo study showing that their instrumental variable based test, the test of Said and Dickey(1984) and the test of Phillips(1987) are all sensitive to order underspecification.

Consider a nonstationary autoregressive (AR) time series model defined by

$$y_t = \sum_{j=1}^{m} \alpha_j y_{t-j} + e_t \tag{1.1}$$

$t = 1, 2, ..., n$, where y_t are observations, m is a nonnegative integer either finite or infinite, e_t is an error sequence, and n is the number of observations. The characteristic equation $A(L) = 1 - \sum_{j=1}^{m} \alpha_j L^j = 0$ is assumed to have one unit root and other roots outside the unit circle. In order to isolate the unit root, we reparametrize model (1.1) into the two equations

$$y_t = y_{t-1} + z_t, \tag{1.2}$$

$$z_t = \sum_{j=1}^{m-1} \psi_j z_{t-j} + \epsilon_t. \tag{1.3}$$

Since m can be infinite, model (1.1) also contains ARMA processes. Conditions required for our study are given in Assumption 1 below.

Assumption 1. (i) *e_t is an identically distributed independent error sequence with mean zero and finite variance σ^2,*
(ii) *the equation $\psi(L) = \sum_{j=1}^{m-1} \psi_j L^j = 0$ has all roots outside the unit circle.*

2. LIMITING DISTRIBUTIONS OF TEST STATISTICS

In this section, we derive the limiting distributions of the usual unit root tests under order misspecification. Usually, for testing unit root purpose, people estimate the Dickey-Fuller regression

$$y_t = \hat{\rho} y_{t-1} + \hat{\beta}_1 z_{t-1} + \ldots + \hat{\beta}_{p-1} z_{t-p+1} + \hat{e}_t, \tag{2.1}$$

where $\hat{e}_t$ are the residuals. Order misspecification means $p \neq m$. Testing the unit root hypothesis is usually performed by the augmented Dickey-Fuller tests

$$n(\hat{\rho} - 1) \text{ and } \hat{\tau} = (\hat{\rho} - 1)/\ se(\hat{\rho}),$$

and their mean adjusted versions and trend adjusted versions, where $se(\hat{\rho})$ is the regression standard error of $\hat{\rho}$. In Theorem 1 below, limiting distributions of the test statistics are established.

Theorem 1. *Consider model (1.1) or equivalently model (1.2) — (1.3). Let Assumption 1 hold. Then*

$$n(\hat{\rho} - 1) \xrightarrow{d} (\sigma_z^2 G)^{-1} [\sigma_z^2 \xi (1 - \sum_{j=1}^{p-1} \beta_j) + \kappa] \tag{2.2}$$

and

$$\hat{\tau} \xrightarrow{d} (\sigma_z \sigma_v)^{-1} G^{-1/2} [\sigma_z^2 \xi (1 - \sum_{j=1}^{p-1} \beta_j) + \kappa] \tag{2.3}$$

where

$$\sigma_z^2 = \{1 - \sum_{j=1}^{m-1} \psi_j\}^{-2}\sigma^2,\ \sigma_v^2 = \gamma(0) - \sum_{j=1}^{p-1} \beta_j\gamma(j),$$

$$G = \int_0^1 W^2(s)ds,\ \xi = 2^{-1}\{W^2(1) - 1\},$$

$W(s)$ is a standard Brownian motion on $[0,\ 1]$,

$$\beta = (\beta_1,\ ...,\ \beta_{p-1})' = \Gamma_{p-1}^{-1}\gamma_{p-1},\ p \geq 2,$$

$$\kappa = \sum_{j=0}^{\infty}\{\gamma(j+1) - \beta_1\gamma(j) - ... - \beta_{p-1}\gamma(j-p+2)\},$$

$$\Gamma_k = E(Z_{t,k}Z_{t,k}'),\ \gamma_k = E(Z_{t,k}z_t) = (\gamma_1,\ ...,\ \gamma_k)',$$

$Z_{t,k} = (z_{t-1},\ ...,\ z_{t-k})'$, $k = 1,\ 2,\ ....$, $\gamma(j) = E(z_t z_{t+j})$, $j = 0,\ \pm 1,\ ...,$, *and* $\xrightarrow{d}$ *denotes convergence in distribution.*

When $p = 1$, $\kappa = \sum_{j=0}^{\infty}\gamma(j+1) = 2^{-1}(\sigma_z^2 - \gamma(0))$ and $\sigma_v^2 = \gamma(0)$. Therefore, the limiting distributions in (2.2) and (2.3) become

$$2^{-1}G^{-1}[W^2(1) - \gamma(0)/\ \sigma_z^2] \text{ and } \quad 2^{-1}\sigma_z\{\gamma(0)G\}^{-1/2}[W^2(1) - \gamma(0)/\ \sigma_z^2],$$

respectively, which are the limiting distributions of the unadjusted semiparametric tests of Phillips(1987).

When the order is correctly specified, $m = p$, $(\beta_1,\ ...,\ \beta_{p-1}) = (\psi_1,\ ...,\ \psi_{m-1})$, and $\sigma_v^2 = \sigma^2$ because of $\gamma(0) = \sum_{j=1}^{m-1}\psi_j\gamma(j) + \sigma^2$, which is obtained from multiplying z_t both sides of (1.3) and taking expectation. Therefore, $\gamma(j+1) = \psi_1\gamma(j) + ... + \psi_{p-1}\gamma(j-p+2)$, $j = 0,\ 1,\ ...,$ and

$$\kappa = \sum_{j=0}^{\infty}[\gamma(j+1) - \psi_1\gamma(j) - ... - \psi_{p-1}\gamma(j-p+2)] = 0$$

and

$$n(\hat{\rho} - 1) \xrightarrow{d} (1 - \psi_1 - ... - \psi_{p-1})G^{-1}\xi \quad \text{and} \quad \hat{\tau} \xrightarrow{d} G^{-1/2}\xi,$$

which are the same as the distributions of the augmented Dickey-Fuller tests.

When the order is underspecified, $\kappa \neq 0$ and the term κ corresponds to the unexplained autocorrelation in the $AR(p)$ fitting. Pantula and Hall(1991) made a Monte-Carlo study showing that their test based on instrumental variables, the test of Said and Dickey(1984), and the test of Phillips(1987) are robust to overspecifying the regression orders in the sense that the tests have empirical levels close to the nominal levels. However, the tests are sensitive to underspecification of orders. For example, when the true process is ARIMA(1,1,1), all the tests over-reject (reject more often than the nominal level) or under-reject the unit root hypothesis than the nominal level if estimated model is ARIMA(1,1,0), ARIMA(1,0,1), or ARIMA(2,0,0).

For overspecified model, $\kappa = 0$ and $n(\hat{\rho}-1)$ and $\hat{\tau}$ converge in distribution to $(1-\psi_1 - \dots -\psi_{p-1})G^{-1}\xi$ and $G^{-1/2}\xi$, respectively, which are the same as those based on the true model. Hence, the tests would be robust to the overspecification of orders as observed by Pantula and Hall. However, for underspecified model, $\kappa \neq 0$ and the limiting distribution of $n(\hat{\rho}-1)$ and $\hat{\tau}$ would be different from those when $\kappa = 0$. The unit root tests over-reject or under-reject the unit root hypothesis depending on the value of κ, σ_z^2, and σ_v^2. Next, this point is studied in more detail for some simple models.

3. ANALYSES OF SIMPLE MODELS

Assume that a second order autoregressive process

$$(1-L)(1-\psi L)y_t = e_t,\ |\psi| < 1, \tag{3.1}$$

is estimated by a first order autoregression. By Theorem 1, we have

$$\hat{\tau} \xrightarrow{d} G^{-1/2}[\xi + \psi(1-\psi)^{-1}] \tag{3.2}$$

because $p = 1$, $\gamma(j) = \sigma^2\psi^j/(1-\psi^2)$, $\sigma_z^2 = \sigma^2$, and hence, $\sigma_z^{-2}\kappa = \sigma^{-2}\sum_{j=0}^{\infty}\gamma(j+1) = \psi(1-\psi)^{-1}$. When $\psi > 0$, the distribution in (3.2) is positively skewed. This would make the unit root hypothesis accepted more frequently than the nominal level (under-reject). This also would cause low power of the test when the process is stationary. On the other hand, when $\psi < 0$, the distribution in (3.2) is negatively skewed. The test $\hat{\tau}$ tends to reject the unit root hypothesis more often than the nominal level in favor of stationarity of y_t (over-reject).

We next consider an ARMA(1,1) process with an AR unit root estimated by AR(1) model. Assume that an ARMA(1,1) process with one AR unit root

$$(1-L)y_t = (1-\theta L)e_t,\ |\theta| < 1, \tag{3.3}$$

is estimated by an AR(1) model. By Theorem 1, we have

$$\hat{\tau} \xrightarrow{d} G^{-1/2}[\xi - \theta(1-\theta)^{-2}].$$

When $\theta > 0$, the test over-rejects the unit root hypothesis. When $\theta < 0$, the test under-rejects the unit root hypothesis. Assume that a third order autoregressive process

$$(1-L)(1-\psi_1 L - \psi_2 L^2)y_t = e_t, \tag{3.4}$$

is estimated by a second order autoregression. We assume that all the roots of $1 - \psi_1 L - \psi_2 L^2 = 0$ lie outside the unit circle. By Theorem 1, we have

$$\hat{\tau} \xrightarrow{d} G^{-1/2}[\xi\{(1+\psi_2)/(1-\psi_2)\}^{1/2} + \psi_2(1-\psi_2^2)^{1/2}\{(1+\psi_2)(1-\psi_2+\psi_1)\}^{-1}].$$

When $\psi_2 > 0$, the test under-rejects the unit root hypothesis because $\kappa > 0$ and the distribution in (3.2) is positively skewed. When $\psi_2 < 0$, the test over-rejects the unit root

APPENDIX

Proof of Theorem 1. We first give proof for the theorem for $p \geq 2$. Note that (2.1) is equivalent to the ordinary least square regression

$$y_t = \hat{\alpha}_1 y_{t-1} + \hat{\alpha}_2 y_{t-1} + \ldots + \hat{\alpha}_p y_{t-p} + \hat{\epsilon}_t$$

through the relation

$$\hat{\rho} = \hat{\alpha}_1 + \ldots + \hat{\alpha}_p$$

and other relations between $(\hat{\alpha}_1, \ldots, \hat{\alpha}_p)$ and $(\hat{\beta}_1, \ldots, \hat{\beta}_{p-1})$. We have

$$\hat{\rho} = 1_p' \hat{\alpha}_p \text{ and } \hat{\alpha}_p = [\sum_{t=p+1}^{n} Y_{t,p} Y_{t,p}']^{-1} [\sum_{t=p+1}^{n} Y_{t,p} y_t],$$

where $Y_{t,k} = (y_{t-1}, \ldots, y_{t-k})'$. We define some notation. Let I_k be the $k \times k$ identity matrix, let 1_k be the $k \times 1$ matrix of ones, let $0_{k,\ell}$ be the $k \times \ell$ matrix of zeros, k, $\ell = 1, 2, \ldots$. Let $W_{t,k} = (y_{t-1}, Z'_{t,k-1})'$, $k = 2, \ldots$. Let $D_n = diag(n^{-1}, n^{-1/2} I_{p-1})$ and let

$$\Delta_n = D_n Q'_p \sum_{t=p+1}^{n} Y_{t,p} Y'_{t,p} Q_p D_n = D_n \sum_{t=p+1}^{n} W_{t,p} W'_{t,p} D_n, \; p = 1, 2, \ldots .$$

Then $Y_{t,p} = Q'^{-1}_p W_{t,p}$, where

$$Q_p = \begin{pmatrix} 1 & 1 & 0 & \ldots & 0 \\ 0 & -1 & 1 & \ldots & 0 \\ 0 & 0 & -1 & \ldots & 0 \\ .. & .. & .. & \ldots & \ldots \\ 0 & 0 & 0 & & -1 \end{pmatrix}$$

and

$$Q_p^{-1} = \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 0 & -1 & -1 & \ldots & -1 \\ 0 & 0 & -1 & \ldots & -1 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & -1 \end{pmatrix}$$

Observe that $n(\hat{\rho} - 1)$ is the first element of $D_n^{-1} Q_p^{-1} (\hat{\alpha}_p - \alpha_p(p))$.

We need some limiting results on $\hat{\alpha}_p$ from Shin and Lee(1997) given by

$$\hat{\alpha}_p \xrightarrow{p} \alpha_p(p) := Q_p \left\{ \begin{array}{c} 1 \\ \Gamma_{p-1}^{-1} \gamma_{p-1} \end{array} \right\}, p \geq 2, \qquad (A.1)$$

$$\Delta_n \xrightarrow{d} \Delta := \left\{ \begin{array}{cc} \sigma_z^2 G & 0_{1,p-1} \\ 0_{p-1,1} & \Gamma_{p-1} \end{array} \right\} \quad , \quad p \geq 2 \qquad (A.2)$$

and

$$n^{-1} \sum_{t=p+1}^{n} y_t z_{t-k} \xrightarrow{d} \sigma_z^2 \xi + \sum_{j=-k}^{\infty} \gamma(j), \; k = 0, \pm 1, .., \qquad (A.3)$$

which are the essential components for proving the theorem. Some simple algebra gives

$$D_n^{-1} Q_p^{-1} (\hat{\alpha}_p - \alpha_p(p)) = [D_n \sum_{t=p+1}^{n} W_{t,p} W'_{t,p} D_n]^{-1} [D_n \sum_{t=p+1}^{n} W_{t,p} v_t]$$

where

$$v_t = y_t - Y'_{t,p}\alpha_p(p) = y_t - W'_{t,p}\begin{Bmatrix} 1 \\ \Gamma_{p-1}^{-1}\gamma_{p-1} \end{Bmatrix} = z_t - Z'_{t,p-1}\beta.$$

By (A.3),

$$n^{-1}\sum_{t=p+1}^{n} y_{t-1}v_t \xrightarrow{d} \zeta, \qquad (A.4)$$

where $\zeta = \sigma_z^2\xi(1-\beta_1 - \ldots -\beta_{p-1}) + \kappa$. Since $Z_{t,p-1}$ and v_t are uncorrelated,

$$E\{\sum_{t=p+1}^{n} Z_{t,p-1}v_t\} = 0 \ , \ Var(\sum_{t=p+1}^{n} Z_{t,p-1}v_t) = O(n).$$

Hence

$$n^{-1/2}\sum_{t=p+1}^{n} Z_{t,p-1}v_t = O_p(1). \qquad (A.5)$$

Therefore, by (A.2), (A.4), and (A.5), the first component of

$$[D_n\sum_{t=p+1}^{n} W_{t,p}W'_{t,p}D_n]^{-1}D_n\sum_{t=p+1}^{n} W_{t,p}v_t$$

converges in distribution to (2.2). This establishes the limiting distribution of $n(\hat{\rho}-1)$.

We next derive the limiting distribution of $\hat{\tau} = (\hat{\rho}-1)/\ se(\hat{\rho})$. Note that

$$n\ se(\hat{\rho}) = \{\hat{\sigma}^2 n^2 1'_p(\sum_{t=p+1}^{n} Y_{t,p}Y'_{t,p})^{-1}1_p\}^{1/2},$$

where $\hat{\sigma}^2 = (n-p)^{-1}\sum_{t=p+1}^{n}(y_t - \hat{\alpha}'_p Y_{t,p})^2$. Observe that

$$(n1'_pQ_pD_n) = (1,0,...,0)'\ nD_n = (1,0,...,0)'.$$

By (A.2)

$$n^2 1'_p(\sum_{t=p+1}^{n} Y_{t,p}Y'_{t,p})^{-1}1_p = (n1'_pQ_pD_n)(D_nQ'_p\sum_{t=p+1}^{n} Y_{t,p}Y'_{t,p}Q_pD_n)^{-1}(D_nQ'_p1_pn)$$

$$\xrightarrow{d} (1, 0, ..., 0)\Delta^{-1}(1, 0, ..., 0)' = (\sigma_z^2 G)^{-1}.$$

By Theorem 3.4 of Phillips and Solo(1992)

$$D_n Q_p' \sum_{t=p+1}^{n} Y_{t,p} = O_p(1) \text{ and } D_n^{-1} Q_p^{-1}(\hat{\alpha}_p - \alpha_p(p)) = O_p(1). \quad (A.6)$$

Now by (A.2) and (A.6),

$$(n-p)\hat{\sigma}^2 = \sum_{t=p+1}^{n} [y_t - \alpha_p'(p)Y_{t,p} + (\alpha_p(p) - \hat{\alpha}_p)'Y_{t,p}]^2 = \sum_{t=p+1}^{n} v_t^2 + O_p(1).$$

Hence $\hat{\sigma}^2 = n^{-1}\sum_{t=p+1}^{n} v_t^2 + O_p(n^{-1}) \xrightarrow{d} \sigma_v^2$ and $n\ se(\hat{\rho})$ converges in distribution to $[(\sigma_z^2 G)^{-1}\sigma_v^2]^{1/2}$. Therefore, the limiting distribution of $\hat{\tau}$ for $p \geq 2$ follows.

Proof for $p = 1$ follows from the above arguments applied to the first components of vectors and (1,1) elements of matrices involved.

ACKNOWLEDGEMENTS

The authors greatly appreciate the two referees for their helpful comments.

REFERENCES

(1) Dickey, D. A. and Fuller, W. F. (1979). Distribution of the estimators for autoregressive time series with a unit root, *Journal of American Statistical Association*, **74**, 427-431.

(2) Fuller, W. A. (1996). *Introduction to Statistical Time Series Analysis*, John Wiley & Sons, 2nd edition New York.

(3) Hamilton, J. D. (1994). *Time series analysis*, Princeton university.

(4) Nelson, C. R. and Plosser, C. I.(1982). Trends and random walks in macroeconomic time series: some evidence and implications, *Journal of Monetary Economics*, **10**, 129-162.

(5) Pantula, S. G. and Hall, A. (1991). Testing for unit roots in autoregressive moving average models. An instrumental variable approach, *Journal of Econometrics*, **48**, 325-53.

(6) Phillips, P. C. B. (1987). Time series regression with a unit root. *Econometrica*, **55**, 277-302.

(7) Phillips, P. C. B. and Solo, V. (1992). Asymptotics for linear processes, *Annals of Statistics*, **20**, 971-1001.

(8) Said, S. E. and Dickey, D. A.. (1984). Testing for unit roots in autoregressive moving average models of unknown order, *Biometrika*, **71**, 599-607.

(9) Shin, D. W. and Lee, Y. D.(1997). A study on misspeicifed nonstationary autoregressive time series with a unit root, *Journal of Time Series Analysis*, Under press.