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Design of the Variable Sampling Rates $\bar{X}$-chart with Average Time to Signal Adjusted by the Sampling Cost [†]

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ABSTRACT

The variable sampling rates scheme is proposed by taking random sample size and sampling interval during the process. The performance of the scheme is measured in terms of the average time to signal adjusted by the sampling cost when the process is out of control. This measurement evaluates the effectiveness of the scheme in terms of the cost incurred due to nonconformities as well as sampling. The variable sampling rates scheme is shown to be effective especially for small and moderate shifts of the mean when compared to the standard scheme.

Key Words : Variable sampling rates; Markov chain; Variable sample size; Variable sampling interval.

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1. INTRODUCTION

In continuous production processes control charts are widely used in monitoring shifts in process parameters. A typical example of the control chart is the Shewhart $\bar{X}$ chart. Shewhart charts are shown to be less efficient in monitoring small shifts in the process parameters compared to large ones despite its easiness of application. Several studies have been done to improve the efficiency of the Shewhart chart by modifying the procedure. Examples are the use of warning lines by Page (1954) and run rules by Nelson (1984). These modifications have shown to improve the chart considerably.

A further modification of the Shewhart $\bar{X}$ chart by varying the sampling interval or the sample size was proposed to improve performance of the chart for small and moderate shifts. The variable sampling interval (VSI) chart scheme was proposed by Reynolds et al. (1988) and Runger and Pignatiello (1991). The variable sample size (VSS) chart scheme was also studied by Prabhu, Runger and Keats (1993), Costa (1994), and Park and Reynolds (1994).

In the VSI scheme the sampling interval between the current and the next samples is varied according to the current state of the chart statistic while the sample size is fixed. In the VSS scheme the size of the next sample is varied according to the current state of the chart statistic while the sampling interval is fixed. Both of the two schemes have shown to be more efficient than the classical Shewhart chart. A combined scheme of VSI and VSS was proposed for further modification by Prabhu, Montgomery and Runger (1994). The combined approach will be called as the variable sampling rates (VSR) scheme in this paper. They showed that the VSR $\bar{X}$ chart outperforms the VSI and VSS charts by comparing the average time necessary for an out-of-control signal.

The idea of the VSR chart is to combine the VSI and VSS schemes, that is to vary the next sample size and the sampling interval between the current and the next samples according to the current state of the chart statistic. If the current chart statistic falls near inside the control limits, we use large sample size and small sampling interval for the next sample to detect shifts more quickly if they occurred, otherwise we use small sample size and large sampling interval. The traditional approach to sampling for a control chart is to take the fixed sampling rate(FSR) in which we take a fixed sample size(FSS) with a fixed sampling interval(FSI) between samples.

In the classical control chart the performance of the chart is measured by the average run length (ARL) for comparing the efficiencies of control schemes. However the ARL is a valid measure only for cases where the sampling interval and the sample size remain the same. In the VSR scheme an alternative measure of performance to the ARL is the average time to signal (ATS) which represents the mean time until detection from the occurrence of an assignable cause. As long as the sampling cost remains fixed, large ATS when in control will increase profit due to conforming items and small ATS when out of control will reduce the loss due to nonconformities. Prabhu, Montgomery and Runger (1994) compared the performance of the VSR $\bar{X}$ chart with other schemes by the ATS.

The sampling cost depends on the number of samples and observations taken during the process. For a given period of time the short sampling interval as well as the large sample size increase the sampling cost. In evaluating the effectiveness of the VSR scheme, we need to consider the ATS together with the sampling cost. In this paper the design and the effectiveness of the VSR $\bar{X}$ chart are studied in terms of the ATS adjusted by the sampling cost by comparing it to the FSR scheme. Also the optimal chart parameters are selected for some given mean shifts.

2. DESIGN OF THE VSR $\bar{X}$-CHART

Consider a process in which the distribution of the observations is normal with mean μ and variance σ^2, and the objective is to detect shifts in μ from a target value μ_0. Suppose that random samples of variable size are taken at intervals of variable length during the process.

Let N_k and H_k be the k-th sample size and the sampling interval between (k-1)th and k-th samples, respectively, and let $\mathbf{X}'_k = (X_{k1}, \cdots, X_{kN_k})$ be the sample taken at k-th sampling point. Then the procedure of the VSR $\bar{X}$-chart is to compute the standardized sample mean $T_k = \sqrt{N_k}(\bar{X}_k - \mu_0)/\sigma$ at each sampling time k and plot on a control chart with control limit $\pm c$. If T_k falls outside the control limits, then an out-of-control signal is given.

In standard $\bar{X}$-charts, the sample size and the sampling interval (N_k, H_k) are fixed in advance, but in the VSR $\bar{X}$-chart, they are determined according to the value of the previous statistic T_{k-1}. In this paper, we consider only two number for both of the sample sizes and the sampling intervals for administrative convenience. Let n_s and n_l be the minimum and maximum sample sizes, and h_s and h_l be the minimum and maximum sampling intervals, respectively. The minimum sample size can usually be set as n_s=1. If we want a variance estimate the minimum sample size can be set as n_s=2.

The maximum sample size and minimum sampling interval can be restricted according to the process conditions such as the time necessary for obtaining and charting observations.

We partition the interval (-c,c) to determine the sample size and sampling interval for k-th sample as the following. For $k \geq 2$,

$$N_k = \begin{cases} n_1 & \text{if } |T_{k-1}| < c_S, \\ n_2 & \text{if } c_S \leq |T_{k-1}| < c, \end{cases} \tag{2.1}$$

$$H_k = \begin{cases} h_1 & \text{if } |T_{k-1}| < c_I, \\ h_2 & \text{if } c_I \leq |T_{k-1}| < c, \end{cases} \tag{2.2}$$

where $n_s \leq n_1 \leq n_2 \leq n_l, h_l \geq h_1 \geq h_2 \geq h_s$. The threshold limits to switch the sample size and the sampling interval are denoted by c_S and c_I, respectively. Actually we divide the interval (-c,c) into three regions for determining the sample size and sampling interval. Define $c_0 = 0,\ c_1 = min\{c_S, c_I\},\ c_2 = max\{c_S, c_I\},\ c_3 = c$. Then for $k \geq 2$,

$$(N_k, H_k) = \begin{cases} (n_1, h_1) & \text{if } 0 \leq |T_{k-1}| < c_1, \\ (n_*, h_*) & \text{if } c_1 \leq |T_{k-1}| < c_2, \\ (n_2, h_2) & \text{if } c_2 \leq |T_{k-1}| < c, \end{cases} \tag{2.3}$$

where

$$(n_*, h_*) = \begin{cases} (n_1, h_2) & \text{if } c_S \geq c_I, \\ (n_2, h_1) & \text{if } c_S < c_I. \end{cases}$$

Figure 1 shows two possible examples of the partition and the corresponding sampling rates.

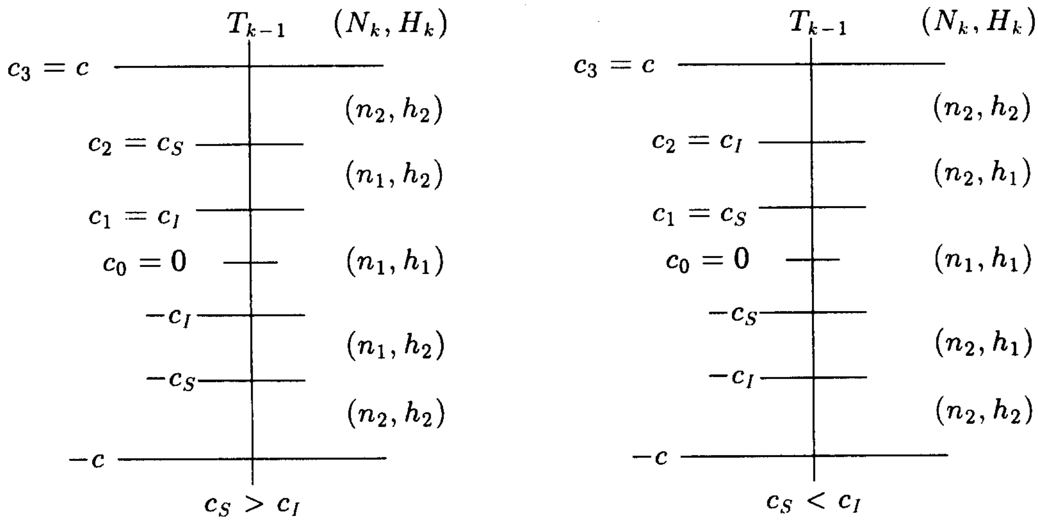

Figure 1 : The two possible partitions of (-c,c) and the corresponding (N_k, H_k)

Note that if $n_1 = n_2 = n_0$ then the VSR scheme reduces to the VSI scheme with the FSS n_0, if $h_1 = h_2 = h_0$, then the VSS scheme with the FSI h_0, and if $n_1 = n_2 = n_0$ and $h_1 = h_2 = h_0$ then the standard scheme with the FSS n_0 and the FSI h_0.

3. PROPERTIES OF THE VSR $\bar{X}$-CHART

The properties of the VSR $\bar{X}$-chart can be evaluated by the number of samples, the number of observations, and the length of time it takes to produce an out-of-control signal. Let the average number of samples to signal(ANSS) , the average number of observations to signal(ANOS), and the ATS be the expectation of the number of samples, the number of observations, and the time to signal it takes to produce an out-of-control signal, respectively.

The sequence of $\{N_k,\ H_k\}$ generates a Markov chain with four states corresponding to where T_{k-1} is plotted in the following four regions,

$$\begin{aligned} I_1 &= (-c_1,\ c_1), \\ I_2 &= (-c_2,\ -c_1] \cup [c_1,\ c_2), \\ I_3 &= (-c_3,\ -c_2] \cup [c_2,\ c_3), \\ I_4 &= (-\infty,\ -c_3] \cup [c_3,\ \infty). \end{aligned}$$

The region I_4 corresponds to an absorbing state and the transient state transition matrix is given by

$$\mathbf{Q} = [q_{ij}]_{3\times 3},$$

where $q_{ij} = P(c_{j-1} \leq |T_k| < c_j \mid c_{i-1} \leq |T_{k-1}| < c_i), i, j = 1, 2, 3.$

Let Z be a standard normal random variable and $\delta = (\mu - \mu_0)/\sigma$, then T_k is distributed as $Z + \sqrt{N_k}\delta$. Thus the probability q_{ij} can be expressed as, for $j = 1, 2, 3$,

$$\begin{aligned}
q_{1j} &= P\left(c_{j-1} \leq |Z + \sqrt{n_1}\delta| < c_j\right) \\
&= \Phi(c_j - \sqrt{n_1}\delta) - \Phi(c_{j-1} - \sqrt{n_1}\delta) + \Phi(-c_{j-1} - \sqrt{n_1}\delta) - \Phi(-c_j - \sqrt{n_1}\delta), \\
q_{2j} &= P\left(c_{j-1} \leq |Z + \sqrt{n_*}\delta| < c_j\right) \\
&= \Phi(c_j - \sqrt{n_*}\delta) - \Phi(c_{j-1} - \sqrt{n_*}\delta) + \Phi(-c_{j-1} - \sqrt{n_*}\delta) - \Phi(-c_j - \sqrt{n_*}\delta), \\
q_{3j} &= P\left(c_{j-1} \leq |Z + \sqrt{n_2}\delta| < c_j\right) \\
&= \Phi(c_j - \sqrt{n_2}\delta) - \Phi(c_{j-1} - \sqrt{n_2}\delta) + \Phi(-c_{j-1} - \sqrt{n_2}\delta) - \Phi(-c_j - \sqrt{n_2}\delta),
\end{aligned}$$

where $\Phi(\cdot)$ denotes the standard normal distribution function. Notice thatq_{2j} is equal to q_{1j} or q_{3j} according to $n_* = n_1$ or $n_* = n_2$,respectively.

Let S_0, O_0 and U_0 be the number of samples, number of ovservations taken when the process is in control, and the in-control period, respectively. Also let $\mathbf{s}_0$ be the probability vector of the starting state, then the ANSS, ANOS, and ATS when in control can be obtained by using Markov chain properties as follows.

$$E(S_0) = \mathbf{s}_0'[\mathbf{I} - \mathbf{Q}_0]^{-1}\mathbf{1}, \tag{3.1}$$

$$E(O_0) = \mathbf{s}_0'[\mathbf{I} - \mathbf{Q}_0]^{-1}\mathbf{n}, \tag{3.2}$$

$$E(U_0) = \mathbf{s}_0'[\mathbf{I} - \mathbf{Q}_0]^{-1}\mathbf{h}, \tag{3.3}$$

where $\mathbf{Q}_0$ is the transition matrix $\mathbf{Q}$ when in control, $\mathbf{1}' = (1,\ 1,\ 1)$, $\mathbf{n}' = (n_1,\ n_*,\ n_2)$, $\mathbf{h}' = (h_1,\ h_*,\ h_2)$, and $\mathbf{I}$ is 3×3 unit matrix. Note that rows of $\mathbf{Q}_0$ are all equal. In the beginning of the process we assume to use $N_1 = n_2$ and $H_1 = h_2$, i.e. $\mathbf{s}_0' = (0, 0, 1)$.

Let S_1, O_1 and U_1 be the number of samples to signal, number of observations to signal when the process is out-of-control, and the out-of-control period, respectively. Then the ANSS, ANOS, and ATS when out of control can be calculated by assuming that the process starts with $\mu = \mu_0 + \delta$. In most cases of practical interest it will be more realistic to assume that the process starts with $\mu = \mu_0$ and then the mean shifts to $\mu_0 + \delta$ at some random time in the future. The length of time from the start of the process to the time that an assignable cause occurs is called the in-control period. Also the length of time from an assignable cause occurs to the out-of-control signal is called the out-of-control period. Here we assume that the in-control period follows an exponential distribution with mean $1/\lambda$. Also we assume that the process is immediately reset to the starting state after each false alarm.

Let $\mathbf{s}_1$ be the probability vector of the state at the sampling time immediately before the shift, and τ be the time of occurrence of an assignable cause from the k-th sampling time when it occurs between sample k and k+1. Then by using the Markov chain properties we obtain the following expectations,

$$E(S_1) = \mathbf{s}_1'[\mathbf{I} - \mathbf{Q}_1]^{-1}\mathbf{1}, \tag{3.4}$$

$$E(O_1) = \mathbf{s}_1'[\mathbf{I} - \mathbf{Q}_1]^{-1}\mathbf{n}, \tag{3.5}$$

$$E(U_1) = \mathbf{s}_1'[\mathbf{I} - \mathbf{Q}_1]^{-1}\mathbf{h} - E(\tau), \tag{3.6}$$

where $\mathbf{Q}_1$ is the transition matrix $\mathbf{Q}$ when out of control. The probability vector $\mathbf{s}_1$ and $E(\tau)$ are obtained as follows [see Appendix].

Let $\beta = e^{-\lambda h_2}\,\{p_{h_1}\, e^{-\lambda(h_1 - h_2)} + 1 - p_{h_1}\}$ for $p_{h_1} = 2\{\Phi(c_I) - 0.5\}$, then for $\mathbf{s}_1' = (\mathbf{s}_1(1),\ \mathbf{s}_1(2),\ \mathbf{s}_1(3))$ and $q_{0j} = 2\{\Phi(c_j) - \Phi(c_{j-1})\}, j = 1, 2,$

$$\begin{aligned}
\mathbf{s}_1(1) &= q_{01} e^{-\lambda h_2}(1 - e^{-\lambda h_1})/(1 - \beta), \\
\mathbf{s}_1(2) &= \begin{cases} q_{02} e^{-\lambda h_2}(1 - e^{-\lambda h_2})/(1 - \beta) & \text{if } c_S \geq c_I, \\ q_{02} e^{-\lambda h_2}(1 - e^{-\lambda h_1})/(1 - \beta) & \text{if } c_S < c_I, \end{cases} \\
\mathbf{s}_1(3) &= 1 - e^{-\lambda h_2} + (1 - q_{01} - q_{02}) e^{-\lambda h_2}(1 - e^{-\lambda h_2})/(1 - \beta),
\end{aligned} \tag{3.7}$$

$$\begin{aligned}
E(\tau) = {} & \frac{1 - (1 + \lambda h_1)e^{-\lambda_1}}{\lambda(1 - e^{-\lambda h_1})} \frac{e^{-\lambda h_2}(1 - e^{-\lambda h_1})}{1 - \beta} p_{h_1} \\
& + \frac{1 - (1 + \lambda h_2)e^{-\lambda h_2}}{\lambda(1 - e^{-\lambda h_2})} \left[1 - \frac{e^{-\lambda h_2}(1 - e^{-\lambda h_1})}{1 - \beta} p_{h_1}\right].
\end{aligned} \tag{3.8}$$

The vector $\mathbf{s}_1$ and $E(\tau)$ depend on the parameter λ of the exponential distribution and the starting state probability vector $\mathbf{s}_0$. In order to remove the dependency on λ and $\mathbf{s}_0$, we assume that the shift occurs after a reasonably long time of the in-control state so that $\mathbf{s}_1$ and τ are in a steady state distribution. The steady state $\mathbf{s}_1$ and τ are obtained as follows by letting $\lambda \to 0$ and applying L'Hospital's rule.

$$\mathbf{s}_1(1) = \frac{q_{01} h_1}{p_{h_1} h_1 \ + \ (1 - p_{h_1}) h_2},$$
$$\mathbf{s}_1(2) = \begin{cases} \frac{q_{02} h_2}{p_{h_1} h_1 \ + \ (1 - p_{h_1}) h_2} & \text{if } c_S \geq c_I, \\ \frac{q_{02} h_1}{p_{h_1} h_1 \ + \ (1 - p_{h_1}) h_2} & \text{if } c_S < c_I, \end{cases} \tag{3.9}$$
$$\mathbf{s}_1(3) = \frac{(1 - q_{01} - q_{02}) h_2}{p_{h_1} h_1 \ + \ (1 - p_{h_1}) h_2},$$

$$E(\tau) = \frac{h_1}{2} \frac{p_{h_1} h_1}{p_{h_1} h_1 + (1 - p_{h_1}) h_2} + \frac{h_2}{2} \frac{(1 - p_{h_1}) h_2}{p_{h_1} h_1 + (1 - p_{h_1}) h_2}. \tag{3.10}$$

4. COMPARISON OF VSR TO FSR

In evaluating the performance of the VSR $\bar{X}$ chart, we compare it to the corresponding FSR $\bar{X}$ chart. The performance of the VSR chart is usually evaluated in terms of the ATS. Small ATS when out of control subject to a fixed ATS when in control ensures small loss due to nonconformities subject to a fixed profit due to conforming items. The ATS when out of control tends to decrease as the sample size increases and the sampling interval becomes short. However if excessively small sampling interval and large sample size are used during the short out-of-control period, there needs an increased sampling cost which may cancel out the advantage of the small ATS when out of control.

Let $R_1(\delta)$ be the loss per hour due to nonconformities produced during an out-of-control period with the mean shift δ, and a and b be the cost per sample and observation, respectively. Then the cost incurred during the out-of-control period can be expressed as

$$R_1(\delta) E(U_1) \ + \ a E(S_1) \ + \ b E(O_1). \tag{4.1}$$

Thus minimizing the cost incurred during the out-of-control period reduces to minimize the ATS adjusted by the sampling cost,

$$E(U_1) + \frac{a}{R_1(\delta)}E(S_1) + \frac{b}{R_1(\delta)}E(O_1). \tag{4.2}$$

In the VSR scheme with the control limit c and the threshold limits c_S and c_I, there are four chart parameters $\{n_1, n_2, h_1, h_2\}$. The optimal chart parameters are determined as values which minimize the adjusted ATS.

To compare the performance of the VSR $\bar{X}$-chart with the FSR $\bar{X}$-chart, we need to match the two schemes so that they have the same ANSS, ANOS, and ATS when $\delta = 0$. This matching can be accomplished as follows.

When the process is in control, the transition matrix $\mathbf{Q}$ reduces to

$$\mathbf{Q}_0 = \begin{bmatrix} q_{01} & q_{02} & q_{03} \\ q_{01} & q_{02} & q_{03} \\ q_{01} & q_{02} & q_{03} \end{bmatrix},$$

where $q_{03} = 2\{\Phi(c_3) - \Phi(c_2)\}$. Thus we can easily see that

$$[\mathbf{I}-\mathbf{Q}_0]^{-1} = \begin{bmatrix} 1-q_{02}-q_{03} & q_{02} & q_{03} \\ q_{01} & 1-q_{01}-q_{03} & q_{03} \\ q_{01} & q_{02} & 1-q_{01}-q_{02} \end{bmatrix} /(1-q_{01}-q_{02}-q_{03}).$$

Hence we have the followings.

$$E[S_0] = 1/(1-q_{01}-q_{02}-q_{03}), \tag{4.3}$$

$$E[O_0] = \frac{q_{01}n_1 + q_{02}n_* + (1-q_{01}-q_{02})n_2}{1-q_{01}-q_{02}-q_{03}}, \tag{4.4}$$

$$E[U_0] = \frac{q_{01}h_1 + q_{02}h_* + (1-q_{01}-q_{02})h_2}{1-q_{01}-q_{02}-q_{03}}. \tag{4.5}$$

From the expression of $E[S_0]$, we see that the control limits should be the same for the two schemes to make the ANSS's the same. Let n_0 and h_0 be the sample size and the sampling interval used for the FSR scheme, respectively. We have the same ANOS's when the process is in control if $n_0 = q_{01}n_1 + q_{02}n_* + (1-q_{01}-q_{02})n_2$, that is,

$$n_0 = \begin{cases} (q_{01}+q_{02})n_1 + (1-q_{01}-q_{02})n_2 & \text{if } c_S \geq c_I \\ q_{01}n_1 + (1-q_{01})n_2 & \text{if } c_S < c_I \end{cases}$$

$$= p_{n_1} n_1 + (1 - p_{n_1}) n_2, \tag{4.6}$$

where $p_{n_1} = 2\{\Phi(c_S) - 0.5\}$. We have the same ATS's when the process is in control if $h_0 = q_{01} h_1 + q_{02} h_* + (1 - q_{01} - q_{02}) h_2$,
that is,

$$h_0 = \begin{cases} (q_{01} + q_{02}) h_1 + (1 - q_{01} - q_{02}) h_2 & \text{if } c_S < c_I \\ q_{01} h_1 + (1 - q_{01}) h_2 & \text{if } c_S \geq c_I \end{cases}$$

$$= p_{h_1} h_1 + (1 - p_{h_1}) h_2. \tag{4.7}$$

For $n_1 < n < n_2$, we have from (4.6)

$$c_S = \Phi^{-1} \left[\frac{n_2 - n_0}{2(n_2 - n_1)} + \frac{1}{2} \right]. \tag{4.8}$$

Also, for $h_2 < h < h_1$, we have from (4.7)

$$c_I = \Phi^{-1} \left[\frac{h_0 - h_2}{2(h_1 - h_2)} + \frac{1}{2} \right]. \tag{4.9}$$

In the VSR scheme, we find the chart parameters $\{n_1, n_2, h_1, h_2\}$ which minimize the adjusted ATS. This nonlinear optimization problem was solved by the generalized reduced gradient procedure using finite difference approximations to the partial derivatives (see Lasdon et al.(1978) for details of this method). Because the sample sizes n_1 and n_2 are integers, we found the optimal set of chart parameters with the sample sizes fixed at each possible combinations of n_1 and n_2 with a range of values for $n_s \leq n_1 \leq n_2 \leq n_l$. We select

$$\delta = 0.5,\ 1.0,\ 1.5,\ 2.0,\ 3.0,\ 4.0,\ 5.0$$
$$n_s = 1, \quad n_l = 100$$
$$h_s = 0.1, \quad h_l = 10$$
$$\frac{R_1(\delta)}{b} = r\delta$$
$$r = 100,\ 1000,\ 10000$$
$$\frac{a}{b} = 0,\ 5,\ 10$$
$$c = 3, \ n_0 = 3,\ 5,\ 10, \quad h_0 = 1.$$

For the given parameters above, we obtain the optimal chart parameters and the adjusted ATS in Table 1 to Table 3. In each cell of the table the upper left number denotes the ATS by the VSR scheme, the upper right number in parentheses denotes the ATS by the FSR, the lower left integers in parentheses are $(n_1,\ n_2)$, and the lower right numbers in parentheses are $(h_1,\ h_2)$. The δ values not listed in the table indicate that the optimal design for the given δ is the FSR. When $n_1 = n_2$, it indicates that the optimal design is the VSI. For each set of $(n_0, r, \frac{a}{b})$, we see that the VSR scheme performs significantly better than the FSR for small and moderate shifts, and the predominance of the VSR decreases as the shift increases. The optimal chart parameters calculated are almost the same for different set of $(r, \frac{a}{b})$, whereas those are considerably different for n_0 and δ. The common parameter values are listed at the bottom of each column. In every case except for n_0=3 and δ=3, the optimal h_2 is selected as the minimum sampling interval 0.1.

We also see that the VSR is more useful for small n_0 than large one. For n_0=3, the optimal design is the VSR for $\delta \leq 1.5$, the VSI for δ=2 and 3, and the FSR for $\delta \geq 4$. For n_0=5, the optimal design is the VSR for $\delta \leq 1.5$, the VSI for $\delta = 2$, and the FSR for $\delta \geq 3.0$. For n_0=10, the optimal design is the VSR for $\delta \leq 1.0$, the VSI for $\delta = 1.5$, and the FSR for $\delta \geq 2.0$. No case shows that the optimal design is the VSS.

Table 1. Adjusted ATS and optimal chart parameters of the VSR scheme for n_0=3, c=3, h_0=1

$(r, \frac{a}{b})$	δ				
	0.5	1.0	1.5	2.0	3.0
(100,0)	15.038(63.828) (1,30)(1.69,0.1)	2.232(9.558) (2,10)(1.49,0.1)	1.017(2.466) (2,5)(1.19,0.1)	0.665(0.995) (3,3)(1.10,0.1)	0.523(0.524) (3,3)(1.004,0.83)
(100,5)	16.621(69.897) (1,30)(1.69,0.1)	2.427(10.046) (2,10)(1.49,0.1)	1.091(2.563) (2,5)(1.18,0.1)	0.702(1.032) (3,3)(1.10,0.1)	0.540(0.541) (3,3)(1.004,0.83)
(100,10)	18.205(75.966) (1,30)(1.69,0.1)	2.621(10.534) (2,10)(1.49,0.1)	1.164(2.660) (2,5)(1.18,0.1)	0.739(1.069) (3,3)(1.10,0.1)	0.556(0.558) (3,3)(1.004,0.83)
(1000,0)	13.545(60.551) (1,29)(1.70,0.1)	2.051(9.294) (2,9)(1.50,0.1)	0.967(2.414) (2,5)(1.19,0.1)	0.645(0.976) (3,3)(1.10,0.1)	0.514(0.515) (3,3)(1.004,0.83)
(1000,5)	13.704(61.158) (1,29)(1.70,0.1)	2.071(9.343) (2,9)(1.50,0.1)	0.974(2.424) (2,5)(1.19,0.1)	0.649(0.979) (3,3)(1.10,0.1)	0.515(0.517) (3,3)(1.004,0.83)
(1000,10)	13.863(61.765) (1,29)(1.70,0.1)	2.091(9.391) (2,10)(1.49,0.1)	0.982(2.433) (2,5)(1.19,0.1)	0.653(0.983) (3,3)(1.10,0.1)	0.517(0.519) (3,3)(1.004,0.83)
(10000,0)	13.395(60.224) (1,29)(1.70,0.1)	2.033(9.268) (2,9)(1.50,0.1)	0.962(2.409) (2,5)(1.19,0.1)	0.643(0.974) (3,3)(1.10,0.1)	0.513(0.514) (3,3)(1.004,0.83)
(10000,5)	13.411(60.284) (1,29)(1.70,0.1)	2.035(9.273) (2,9)(1.50,0.1)	0.963(2.409) (2,5)(1.19,0.1)	0.644(0.974) (3,3)(1.10,0.1)	0.513(0.515) (3,3)(1.004,0.83)
(10000,10)	13.426(60.345) (1,29)(1.70,0.1)	2.037(0.277) (2,9)(1.50,0.1)	0.964(2.411) (2,5)(1.19,0.1)	0.644(0.974) (3,3)(1.10,0.1)	0.513(0.515) (3,3)(1.004,0.83)
	(1,30)(1.70,0.1)	(2,10)(1.50,0.1)	(2,5)(1.20,0.1)	(3,3)(1.10,0.1)	(3,3)(1.004,0.83)

Table 2. Adjusted ATS and optimal chart parameters of the VSR scheme for n_0=5, c=3, h_0=1

$(r, \frac{a}{b})$	δ			
	0.5	1.0	1.5	2.0
(100,0)	8.28(36.24) (2,31)(1.64,0.1)	1.36(4.22) (4,12)(1.33,0.1)	0.72(1.12) (4,6)(1.10,0.1)	0.56(0.60)) (5,5)(1.03,0.1)
(100,5)	9.18(39.58) (2,31)(1.64,0.1)	1.49(4.44) (4,13)(1.33,0.1)	0.77(1.17) (4,6)(1.10,0.1)	0.58(0.63) (5,5)(1.03,0.1)
(100,10)	10.07(42.92) (2,32)(1.63,0.1)	1.62(4.67) (4,13)(1.33,0.1)	0.82(1.22) (4,6)(1.10,0.1)	0.61(0.66) (5,5)(1.03,0.1)
(1000,0)	6.78(33.23) (2,29)(1.65,0.1)	1.18(4.02) (4,12)(1.33,0.1)	0.67(1.07) (4,6)(1.10,0.1)	0.53(0.58) (5,5)(1.03,0.1)
(1000,5)	6.88(33.57) (2,29)(1.65,0.1)	1.20(4.04) (4,12)(1.33,0.1)	0.68(1.08) (4,6)(1.10,0.1)	0.54(0.58) (5,5)(1.03,0.1)
(1000,10)	6.97(33.90) (2,30)(1.64,0.1)	1.21(4.06) (4,13)(1.33,0.1)	0.68(1.08) (4,6)(1.10,0.1)	0.54(0.58) (5,5)(1.03,0.1)
(10000,0)	6.63(32.93) (2,29)(1.65,0.1)	1.17(4.00) (4,12)(1.33,0.1)	0.67(1.07) (4,6)(1.10,0.1)	0.53(0.58) (5,5)(1.03,0.1)
(10000,5)	6.64(32.97) (2,29)(1.65,0.1)	1.17(4.00) (4,12)(1.33,0.1)	0.67(1.07) (4,6)(1.10,0.1)	0.53(0.58) (5,5)(1.03,0.1)
(10000,10)	6.65(33.00) (2,29)(1.65,0.1)	1.17(4.00) (4,12)(1.33,0.1)	0.67(1.07) (4,6)(1.10,0.1)	0.53(0.58) (5,5)(1.03,0.1)
	(2,30)(1.65,0.1)	(4,12)(1.33,0.1)	(4,6)(1.10,0.1)	(5,5)(1.03,0.1)

Table 3. Adjusted ATS and optimal chart parameters of the VSR scheme for n_0=10, c=3, h_0=1

$(r, \frac{a}{b})$	δ		
	0.5	1.0	1.5
(100,0)	4.170(14.890) (7,38)(1.56,0.1)	0.881(1.449) (9,14)(1.12,0.1)	0.589(0.612) (10,10)(1.02,0.1)
(100,5)	4.633(16.173) (7,39)(1.56,0.1)	0.962(1.537) (9,14)(1.12,0.1)	0.624(0.647) (10,10)(1.02,0.1)
(100,10)	5.091(17.455) (6,39)(1.53,0.1)	1.042(1.626) (9,15)(1.11,0.1)	0.659(0.681) (10,10)(1.02,0.1)
(1000,0)	2.685(12.582) (7,36)(1.57,0.1)	0.717(1.289) (9,16)(1.11,0.1)	0.527(0.549) (10,10)(1.02,0.1)
(1000,5)	2.732(12.710) (7,36)(1.57,0.1)	0.725(1.298) (9,16)(1.11,0.1)	0.530(0.553) (10,10)(1.02,0.1)
(1000,10)	2.778(12.838) (7,37)(1.56,0.1)	0.733(1.307) (9,16)(1.11,0.1)	0.534(0.556) (10,10)(1.02,0.1)
(10000,0)	2.536(12.351) (7,36)(1.57,0.1)	0.700(1.273) (9,16)(1.11,0.1)	0.521(0.543) (10,10)(1.02,0.1)
(10000,5)	2.540(12.364) (7,36)(1.57,0.1)	0.701(1.274) (9,16)(1.11,0.1)	0.521(0.543) (10,10)(1.02,0.1)
(10000,10)	2.545(12.376) (7,36)(1.57,0.1)	0.702(1.275) (9,16)(1.11,0.1)	0.521(0.544) (10,10)(1.02,0.1)
	(7,37)(1.56,0.1)	(9,15)(1.11,0.1)	(10,10)(1.02,0.1)

5. SENSITIVITY ANALYSIS

It was shown in the previous section that the optimal chart parameters are considerably different for the amount of shift δ. We need to examine the performance of the VSR scheme for wide range of δ values when the chart parameters selected for a specific δ value are used. This was done by obtaining the adjusted ATS for various δ values by using the optimal chart parameters selected for a specific δ value and compare them with the optimal adjusted ATS. These values are listed in Table 4. In the table FSR indicates the adjusted ATS calculated by the FSR scheme, VSR indicates the adjusted ATS by using the optimal chart parameters for the shift δ, and VSR(δ) indicates the adjusted ATS by using the optimal chart parameters selected for the shift δ.

We see that the adjusted ATS's are considerably larger than the optimal one especially for δ values less than the specific δ for which the chart parameters are selected. All the ATS's are significantly smaller for small shift and moderately larger for large shift than the FSR scheme. This characteristic becomes more clear as the amount of shift becomes smaller for which the optimal chart parameters selected. It would be desirable to use chart parameters designed for small amount of shift such as δ=0.5 or 1.0. Then we can expect the overall performance of the VSR will be better than the FSR. In case where small shift may not be important to detect, we recommend to use chart parameters designed for the shift which is important to detect. Suppose that we want $E(S_0) = E(U_0) = 370.4$, $E(U_0) = 5E(S_0)$ and the mean shift $\delta = 1.0$ is important to detect. Then we choose c=3, n_0=5, h_0=1, and the chart parameters of the VSR scheme as those selected for δ=1.0 rather than δ=0.5, that is n_1=4, n_2=12, h_1=1.33, h_2=0.1 with threshold limits c_S=1.53 and c_l=1.11 calculated by (4.8) and (4.9), respectively .

6. CONCLUSIONS AND REMARKS

The VSR scheme is proposed in this paper to improve the sensitivity of the $\bar{X}$ chart for small and moderate amount of the shift in the mean. In evaluating the effectiveness of control charts, the ATS adjusted by the sampling cost is used instead of the ATS alone. The adjusted ATS takes into account the time to signal as well as the increased sampling cost. The use of the adjusted ATS makes it possible to choose appropriate sample size and sampling intervals, which was not possible in the statistical design of control charts. Also it was made possible to consider the statistical performance of control charts, which was not possible in the economic design of control charts. The VSR scheme is a combination of the two separate schemes, the VSS and the VSI. The numerical comparison shows that the VSI is more useful than the VSS. Thus when only one scheme is used for administrative convenience, the VSI is preferred.

Table 4. The adjusted ATS by using the optimal chart parameters determined for a specific δ

$n_0 = 3$

δ	0.5	1.0	1.5	2.0	3.0	4.0	5.0
FSR	61.158	9.343	2.424	0.979	0.517	0.502	0.502
VSR	13.711	2.071	0.975	0.649	0.515	0.502	0.502
VSR(0.5)	13.711	4.184	2.037	1.278	0.902	0.835	0.819
VSR(1.0)	26.662	2.071	1.071	0.838	0.739	0.727	0.726
VSR(1.5)	41.182	2.853	0.975	0.710	0.603	0.592	0.592
VSR(2.0)	54.581	6.041	1.247	0.649	0.549	0.547	0.547
VSR(3.0)	60.723	9.054	2.289	0.930	0.515	0.502	0.502

$n_0 = 5$

δ	0.5	1.0	1.5	2.0	3.0	4.0	5.0
FSR	33.569	4.040	1.077	0.581	0.503	0.503	0.502
VSR	6.875	1.197	0.677	0.536	0.515	0.503	0.503
VSR(0.5)	6.875	1.999	1.136	0.906	0.807	0.795	0.794
VSR(1.0)	13.552	1.197	0.752	0.672	0.652	0.651	0.650
VSR(1.5)	24.526	1.765	0.677	0.566	0.548	0.547	0.547
VSR(2.0)	30.714	2.841	0.745	0.536	0.517	0.516	0.516

$n_0 = 10$

δ	0.5	1.0	1.5	2.0	3.0	4.0	5.0
FSR	12.710	1.298	0.553	0.508	0.505	0.504	0.503
VSR	2.732	0.725	0.531	0.508	0.505	0.504	0.503
VSR(0.5)	2.732	0.930	0.783	0.759	0.756	0.755	0.754
VSR(1.0)	5.891	0.725	0.568	0.557	0.554	0.553	0.552
VSR(1.5)	11.072	0.927	0.531	0.517	0.514	0.513	0.512

APPENDIX

When the process is in-control, the statistic does not depend on the sample size. Thus we do not need to consider the sample size. The only thing we have to consider in obtaining $\mathbf{s}_1$ and $E(\tau)$ is the sampling interval. First we derive the distribution of S_0 and H_{S_0+1}.

$$\begin{aligned} P(S_0 = 0) &= P(U_0 < h_2) \\ &= 1 - e^{-\lambda h_2}. \end{aligned}$$

For $z \geq 1$,

$$\begin{aligned}
&P(S_0 = z, H_{z+1} = h_1) \\
&= \sum_{x=0}^{z-1} \{\text{take only } x \text{ number of } h_1 \text{ intervals and } z-1-x \text{ number of } h_2 \text{ intervals} \\
&\quad \text{from } H_2 \text{ to } H_z, H_{z+1} = h_1, \text{ and } h_1 x + h_2(z-x) < U_0 < h_1(x+1) + h_2(z-x)\} \\
&= p_{h_1} \sum_{x=0}^{z-1} \binom{z-1}{x} p_{h_1}^x (1-p_{h_1})^{z-1-x} \int_{h_1 x + h_2(z-x)}^{h_1(x+1)+h_2(z-x)} \lambda e^{-\lambda y} dy \\
&= p_{h_1} \sum_{x=0}^{z-1} \binom{z-1}{x} p_{h_1}^x (1-p_{h_1})^{z-1-x} \left[e^{-\lambda\{h_1 x + h_2(z-x)\}} - e^{-\lambda\{h_1(x+1)+h_2(z-x)\}} \right] \\
&= p_{h_1} e^{-\lambda h_2 z} (1 - e^{-\lambda h_1}) \sum_{x=0}^{z-1} \binom{z-1}{x} \left\{ p_{h_1} e^{-\lambda(h_1 - h_2)} \right\}^x (1-p_{h_1})^{z-1-x} \\
&= p_{h_1} e^{-\lambda h_2} (1 - e^{-\lambda h_1}) \left[e^{-\lambda h_2} \left\{ p_{h_1} e^{-\lambda(h_1-h_2)} + 1 - p_{h_1} \right\} \right]^{z-1} \\
&= p_{h_1} e^{-\lambda h_2} (1 - e^{-\lambda h_1}) \beta^{z-1}. \qquad (A.1)
\end{aligned}$$

Similarly,

$$P(S_0 = z, H_{z+1} = h_2) = (1 - p_{h_1}) e^{-\lambda h_2} (1 - e^{-\lambda h_2}) \beta^{z-1}. \qquad (A.2)$$

Let R_0 denote the state at the sampling time immediately before the shift (i.e. sampling time S_0), then $\mathbf{s}_1' = (P(R_0 = 1),\ P(R_0 = 2),\ P(R_0 = 3))$.

$$\begin{aligned}
P(S_0 = z, R_0 = 1) &= P(S_0 = z, H_{s_0+1} = h_1, N_{s_0+1} = n_1) \\
&= P(S_0 = z | H_{z+1} = h_1, N_{z+1} = n_1) P(H_{z+1} = h_1, N_{z+1} = n_1) \\
&= q_{01} P(S_0 = z | H_{z+1} = h_1) \\
&= q_{01} e^{-\lambda h_2} (1 - e^{-\lambda h_1}) \beta^{z-1}.
\end{aligned}$$

$$\begin{aligned} P(S_0 = z, R_0 = 2) &= P(S_0 = z | H_{z+1} = h_*, N_{z+1} = n_*) P(H_{z+1} = h_*, N_{z+1} = n_*) \\ &= q_{02} P(S_0 = z | H_{z+1} = h_*) \\ &= \begin{cases} q_{02} e^{-\lambda h_2}(1 - e^{-\lambda h_2})\beta^{z-1} & \text{if } c_S \geq c_I, \\ q_{02} e^{-\lambda h_2}(1 - e^{-\lambda h_1})\beta^{z-1} & \text{if } c_S < c_I. \end{cases} \end{aligned}$$

$$\begin{aligned} P(S_0 = z, R_0 = 3) &= P(S_0 = z | H_{z+1} = h_2, N_{z+1} = n_2) P(H_{z+1} = h_2, N_{z+1} = n_2) \\ &= (1 - q_{01} - q_{02}) P(S_0 = z | H_{z+1} = h_2) \\ &= (1 - q_{01} - q_{02}) e^{-\lambda h_2}(1 - e^{-\lambda h_2})\beta^{z-1}. \end{aligned}$$

Then

$$\begin{aligned} S_1(1) &= P(R_0 = 1) \\ &= P(S_0 = 0, R_0 = 1) + \sum_{z=1}^{\infty} P(S_0 = z, R_0 = 1) \\ &= 0 + \sum_{z=1}^{\infty} q_{01} e^{-\lambda h_2}(1 - e^{-\lambda h_1})\beta^{z-1} \\ &= q_{01} e^{-\lambda h_2}(1 - e^{-\lambda h_1})/(1 - \beta). \end{aligned}$$

$$\begin{aligned} S_1(2) &= P(R_0 = 2) \\ &= 0 + \sum_{z=1}^{\infty} P(S_0 = z, R_0 = 2) \\ &= \begin{cases} \sum_{z=1}^{\infty} q_{02} e^{-\lambda h_2}(1 - e^{-\lambda h_2})\beta^{z-1} & \text{if } c_S \geq c_I \\ \sum_{z=1}^{\infty} q_{02} e^{-\lambda h_2}(1 - e^{-\lambda h_1})\beta^{z-1} & \text{if } c_S < c_I \end{cases} \\ &= \begin{cases} q_{02} e^{-\lambda h_2}(1 - e^{-\lambda h_2})/(1 - \beta) & \text{if } c_S \geq c_I, \\ q_{02} e^{-\lambda h_2}(1 - e^{-\lambda h_1})/(1 - \beta) & \text{if } c_S < c_I. \end{cases} \end{aligned}$$

$$\begin{aligned} S_1(3) &= P(R_0 = 3) \\ &= P(S_0 = 0, R_0 = 3) + \sum_{z=1}^{\infty} P(S_0 = z, R_0 = 3) \end{aligned}$$

$$
\begin{aligned}
&= P(S_0 = 0) + \sum_{z=1}^{\infty}(1 - q_{01} - q_{02})e^{-\lambda h_2}(1 - e^{-\lambda h_2})\beta^{z-1} \\
&= 1 - e^{-\lambda h_2} + (1 - q_{01} - q_{02})e^{-\lambda h_2}(1 - e^{-\lambda h_2})/(1 - \beta).
\end{aligned}
$$

Next we consider the marginal probability of H_{S_0+1}.

$$
\begin{aligned}
P(H_{S_0+1} = h_1) &= P(S_0 = 0, H_{S_0+1} = h_1) + \sum_{z=1}^{\infty} P(S_0 = z, H_{S_0+1} = h_1) \\
&= 0 + \sum_{z=1}^{\infty} p_{h_1} e^{-\lambda h_2}(1 - e^{-\lambda h_1})\beta^{z-1} \\
&= \frac{p_{h_1} e^{-\lambda h_2}(1 - e^{-\lambda h_1})}{1 - \beta}, \qquad (A.3)
\end{aligned}
$$

$$
P(H_{S_0+1} = h_2) = \left[1 - \frac{e^{-\lambda h_2}(1 - e^{-\lambda h_1})}{1 - \beta} p_{h_1}\right]. \qquad (A.4)
$$

The conditional expectation $E(\tau | H_{S_0+1} = h)$ is obtained by Duncan(1971) as

$$
E(\tau | H_{S_0+1} = h) = \frac{1 - (1 + \lambda h)e^{-\lambda h}}{\lambda(1 - e^{-\lambda h})}.
$$

Thus by using the conditional expectation of τ and marginal probability of H_{S_0+1}, we have

$$
\begin{aligned}
E(\tau) = {} & \frac{1 - (1 + \lambda h_1)e^{-\lambda_1}}{\lambda(1 - e^{-\lambda h_1})} \frac{e^{-\lambda h_2}(1 - e^{-\lambda h_1})}{1 - \beta} p_{h_1} \\
& + \frac{1 - (1 + \lambda h_2)e^{-\lambda h_2}}{\lambda(1 - e^{-\lambda h_2})} \left[1 - \frac{e^{-\lambda h_2}(1 - e^{-\lambda h_1})}{1 - \beta} p_{h_1}\right].
\end{aligned}
$$

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