Journal of the Korean
Statistical Society
Vol. 26, No. 2, 1997

Breakdown Points of Direction Tests

Kyungmee Choi [1]

ABSTRACT

We briefly review three Raleigh type location tests based on direction vectors, which have been shown to be efficient when the distribution is unknown, skewed, or heavy-tailed. Then we calculate their test breakdown points and discuss the robustness of Randles multivariate sign test for one-sample.

Key Words : Level breakdown point; Power breakdown point; Sign test; Wilcoxon rank sum test.

1. INTRODUCTION

Multivariate multisample location tests for one sample or several samples have always been important to many statisticians and engineers especially when the distribution of data is unknown, skewed, or heavy-tailed. We briefly review these location tests and mainly consider three direction-based location tests for one- and two-sample cases to examine their robustness using the concepts of test breakdown points.

[1] Department of Mathematics, College of Science and Technology, Hong Ik University, ChoChiWon, ChoongNam, 339-800, KOREA.

Most tests developed in earlier days for the multivariate several-sample location problem were usually variablewise extensions of the univariate Wilcoxon rank sum test (Mann-Whitney test) or Kruskal-Wallis test (Friedman test). As affine-invariant tests, there are tests by Randles (1989) using interdirections of vectors, Hettmansperger and Oja (1994) and Hettmansperger, Nyblom, and Oja (1994) using simplexes, and Chaudhuri (1992) using direction vectors. All of these have been shown to be efficient and have been presumed to be robust against outliers.

The three multivariate location tests based on direction vectors to be considered here extend the sign test for the one-sample case and Wilcoxon test for the two-sample case by substituting direction vectors for signs in each test. Once we present them, we will examine their robustness against outliers using the concept of level and power breakdown points defined by He, Simpson, and Portnoy (1990). The first direction-based location test of our interest was developed by Raleigh for the one-sample case (Watson, 1983). Under $H_o : \theta_X = 0$ *and* X *has spherical symmetry* it is

$$V^* = p/m \sum_{i=1}^{m} \sum_{j=1}^{m} \frac{X_i^T}{\|X_i\|} \frac{X_j}{\|X_j\|} \xrightarrow{d} \chi_p^2, \tag{1.1}$$

where $\|\cdot\|$ is the Euclidean norm, and we can easily extend V^* to the several-sample case. That is the second direction-based location test of our interest. Here we focus on the two-sample test, we call it DRT (Choi, 1995), and define it as follows:

$$\begin{aligned} DRT \equiv \frac{mnp}{N} \Bigg(& \frac{1}{m^2} \sum_{i=1}^{m} \sum_{j=1}^{m} \frac{X_i^T}{\|X_i\|} \frac{X_j}{\|X_j\|} - \frac{2}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{X_i^T}{\|X_i\|} \frac{Y_j}{\|Y_j\|} \\ & + \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{Y_i^T}{\|Y_i\|} \frac{Y_j}{\|Y_j\|} \Bigg). \end{aligned} \tag{1.2}$$

Then it is asymptotically χ_p^2 under $H_o : \theta_X = \theta_Y$, X *and* Y *have spherical symmetry*. Assuming the spherical symmetry for data clouds in both V^* and DRT, direction vectors of X_i and Y_j are i.i.d. from $Uniform(\Omega)$, where Ω is the surface of the unit ball in R^p.

The third direction-based location test is defined using

$$W \equiv \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{X_i - Y_j}{\|X_i - Y_j\|}, \tag{1.3}$$

which is equivalent to Wilcoxon rank sum test when both populations are univariate. Then under the null hypothesis $H_o : \theta_X = \theta_Y$, $W^T \hat{\Sigma}_W^{-1} W \xrightarrow{d} \chi_p^2$

for any consistent and scale equivariant estimate of covariance matrix Σ_W and $N[\Sigma_W - (1/n + 1/m)\Phi_Z] \xrightarrow{p} 0$ from the usual U-statistic theory (Puri and Sen, 1971), where

$$\Phi_Z \equiv E\left[\frac{Z_1 - Z_2}{\|Z_1 - Z_2\|}\frac{(Z_1 - Z_3)^T}{\|Z_1 - Z_3\|}\right] \tag{1.4}$$

and Z_1, Z_2, and Z_3 are from the same distribution as X and Y under H_o. For any consistent and scale equivariant estimate $\hat{\Phi}_Z$ of Φ_Z we define

$$DWT \equiv NW^T \left[\left(\frac{N}{m} + \frac{N}{n}\right)\hat{\Phi}_Z\right]^{-1} W \tag{1.5}$$

and under H_o it asymptotically follows an χ_p^2 (Choi and Marden, 1997). Note that it is very much straightforward to show that V^* and DRT are orthogonal transformation invariant and DWT has the same property as long as we use a scale equivariant $\hat{\Phi}_Z$. So in order to check its performance it is enough to study distributions with diagonal scale matrix. And if Z_1, Z_2, and Z_3 are *i.i.d.* from an elliptically symmetric distribution with a diagonal scale matrix, then Φ_Z is also a diagonal matrix (Choi(1995) and Chaudhuri (1992)).

Randles (1989) calculated asymptotic relative efficiencies of V^* with respect to one-sample Hotelling's T^2 as a function of p and a parameter of the distribution heaviness in its tail. Choi (1995) shows that DRT has the same asymptotic efficiencies relative to two-sample Hotelling's statistic as that of V^* relative to one-sample Hotelling's. The asymptotic efficiencies of DWT are presented in the paper by Choi and Marden(1997).

To calculate their asymptotic efficiencies we involve the elliptically symmetric distribution used in Randles (1989), which has its probability density function as follows:

$$f(x - \theta) = k_p|\Sigma|^{-\frac{1}{2}} \exp\left(-[(x - \theta)^T\Sigma^{-1}(x - \theta)/c_p]^\nu\right) \tag{1.6}$$

for $x \in R^p$, where

$$c_p = \frac{p\Gamma\left(\frac{p}{2\nu}\right)}{\Gamma\left(\frac{p+2}{2\nu}\right)} \quad \text{and} \quad k_p = \frac{\nu\Gamma\left(\frac{p}{2}\right)}{\Gamma\left(\frac{p}{2\nu}\right)[\pi c_p]^{\frac{p}{2}}}. \tag{1.7}$$

In this density function, Σ denotes a scale matrix, R^p denotes the Euclidean p space, and ν is a tail shape parameter. If $\nu = 1$ it is a normal distribution, if $\nu < 1$ it is heavy-tailed, and if $\nu > 1$ it is light-tailed.

According to calculation results by Randles (1989), Choi (1995), and Choi and Marden(1997) assuming the scale matrix as the identity we see that when the distribution is heavy-tailed, all the three tests (V^*, DRT, DWT) are more efficient than corresponding one- or two-sample Hotelling's T^2 and when the distribution is light-tailed, they are a little bit less efficient. However Randles (1989) showed that when the distribution is light-tailed or normal the asymptotic relative efficiency of V^* with respect to T^2 increases and converges to 1, and when the distribution is heavy-tailed it decreases and converges to 1. Since V^* and DRT have the same asymptotic efficiencies, we can apply the same results to DRT. Choi and Marden (1997) show that the asymptotic efficiencies of DWT relative to two-sample Hotelling's statistic has similar increasing and decreasing tendencies as those of DRT.

In section 2, as measures of test robustness some definitions of test breakdown points are presented. In section 3, we calculate breakdown points of Raleigh's Statistic, and section 4 is about Raleigh type multivariate two-sample test DRT. Section 5 calculates those of DWT. Note that since V^*, DRT, and DWT are multivariate versions of one-sample, two-sample sign tests and Wilcoxon test, we might expect V^*, DRT and DWT to have the same breakdown points as those of one-sample, two-sample sign test and Wilcoxon test respectively. So, we will show that level and power breakdown points of both V^* and DRT are 1 and .5 respectively. The level breakdown point of DWT is 1 and its power breakdown point is .5 when two samples are jointly contaminated, and at least .2929 when two samples are separately contaminated. In section 6, there is a discussion on the robustness of Oja's spatial median and this leads us to conjecture the robustness of Randles' Sign test, which will be left for the future study.

2. LEVEL AND POWER BREAKDOWN POINTS

In this section, as measures of the stability of tests, resistance to rejection and acceptance (Ylvisaker, 1977) and level and power breakdown points (He, Simpson, and Portnoy, 1990) are explained. First, we use T($\cdot$) as a functional, that is, a mapping from the space of all distribution functions to R^1. Ylvisaker (1977)'s resistance to acceptance is a finite sample notion. For a sample $X_1, \cdots, X_N \in R^p$ Ylvisaker (1977) considers tests with critical regions $\{T(\hat{F}_N) > c_N\}$, where $\hat{F}_N$ is the empirical distribution, and defines resistance to acceptance (rejection) of the test as the smallest proportion m/N for which, no matter what $X_{m+1}, \cdots, X_N$ are, there are values $X_1, \cdots, X_m$

with $T(\hat{F}_N) \leq c_N$ ($T(\hat{F}_N) > c_N$). According to his work for the one sample Hotelling's T^2, resistance to acceptance is $1/N$ and resistance to rejection is $c_N/(1+c_N)$ or approximately $(1/N)\chi_p^2(\alpha)$.

Given a distribution F, let us define $Q_\epsilon(F) = \{(1-\epsilon)F + \epsilon G\}$ as a neighborhood of F and $T(Q_\epsilon(F))$ as the set $\{T(H), H \in Q_\epsilon(F)\}$. $Q_\epsilon(F)$ increases from $Q_0(F) = \{F\}$ to $Q_1(F) = \{\text{all distributions}\}$ as ϵ goes from 0 to 1. Then power breakdown point ϵ^* of T is defined as $\sup_\theta \inf_\epsilon \{\epsilon > 0 : T(F_{\theta_0}) \in T(Q_\epsilon(F_\theta))$ for some $\theta_0 \in \Theta_0\}$. The level breakdown point ϵ^{**} is defined as $\sup_\theta \inf_\epsilon \{\epsilon > 0 : T(F_\theta) \in T(Q_\epsilon(F_{\theta_0}))$ for some $\theta_0 \in \Theta_0\}$. He, *et al.* (1990) show that for a univariate sample location test, the breakdown function of the sign test uniformly dominate that of the Wilcoxon rank sum test (W), and the sign test (S) is the uniformly most robust M test. By pointing out that the power breakdown points of the sign test and Wilcoxon test reproduce breakdown points of the median and Hodges-Lehmann estimator respectively, they show that at the normal location model the breakdown points are given by $\epsilon^*(S) = .5, \epsilon^*(W) = 1 - \sqrt{1/2} = .2929$, and $\epsilon^{**}(S) = \epsilon^{**}(W) = 1$.

Let us define for the two sample case that two samples are jointly contaminated if (X, Y) is from $(1-\epsilon)F_\theta + \epsilon G$, where θ is zero under the null hypothesis and G is an arbitrary distribution. On the other hand we want to say that two samples are separately contaminated if X is from $(1-\epsilon)F + \epsilon G$ and Y is from $(1-\epsilon)F_\theta + \epsilon G^*$, where G and G^* are arbitrary distributions. To discuss the breakdown points of V^*, DRT, and DWT we define $V_f \equiv X/\|X\|$, $R_f \equiv X/\|X\| - Y/\|Y\|$, and $W_f \equiv (X-Y)/\|X-Y\|$ and we will use them to define the corresponding functionals.

3. BREAKDOWN POINT OF RALEIGH'S STATISTIC

To discuss the breakdown points of V^* we use $V_f = X/\|X\|$ and the corresponding functional $E[V_f]$. The next two theorems provide the level and power breakdown points of V^* and they are the same as those of the univariate sign test.

Theorem 1. Under the contaminated null hypothesis the level breakdown point of V^* is $\epsilon^{**}(V^*) = 1$.

Proof. Note that $\sup_F \|V_f\| = 1$ from above. Now let us take everything under the ϵ-contaminated null hypothesis. Since $E_{F_\theta}[V_f] = 0$ under the null hypothesis, $E_{(1-\epsilon)F_\theta+\epsilon G}[V_f] = \epsilon E_G[V_f]$, whose magnitude is always bounded by ϵ. Therefore the level breakdown point of V^* is 1.

Theorem 2. The power breakdown point of V^* is $\epsilon^*(V^*) = 1/2$.

Proof. For $0 \leq \epsilon \leq 1$, $F_\theta(x) = F(x-\theta)$, and an arbitrary distribution G, let X be from $(1-\epsilon)F_\theta + \epsilon G$. Then

$$E[V_f]^T E[V_f] = \|E_{(1-\epsilon)F_\theta+\epsilon G}\left[\frac{X}{\|X\|}\right]\|^2 \tag{3.2}$$

$$= \|(1-\epsilon)E_{F_\theta}\left[\frac{X}{\|X\|}\right] + \epsilon E_G\left[\frac{X}{\|X\|}\right]\|^2. \tag{3.3}$$

Here let $V_\theta \equiv E_{F_\theta}[X/\|X\|]$ and $\alpha \equiv E_G[X/\|X\|]$. Since G could be any distribution, we put $G = -F_\theta$. So $\|E_{(1-\epsilon)F_\theta+\epsilon G}[X/\|X\|]\|^2 = (1-2\epsilon)^2\|V_\theta\|^2$. Then $\epsilon = 1/2$ makes this quantity zero. Therefore $\epsilon^* \leq 1/2$. On the other hand, suppose $\epsilon^* < 1/2$. Then $\|(1-\epsilon)V_\theta + \epsilon\alpha\| \geq (1-\epsilon)\|V_\theta\| - \epsilon\|\alpha\|$. Note that the norms of V_θ and α are bounded by 0 and 1. Then for θ big enough there exists $0 < \delta < (1-2\epsilon)/(1-\epsilon)$ such that $(1-\epsilon)\|V_\theta\| - \epsilon\|\alpha\| \geq (1-\epsilon)(1-\delta) - \epsilon > 0$, which means $\|(1-\epsilon)V_\theta + \epsilon\alpha\|$ is always positive. Therefore the power breakdown point ϵ^* is $1/2$.

4. BREAKDOWN POINTS OF DRT

The functional for DRT can be written as the expected value of $R_f = X/\|X\| - Y/\|Y\|$. DRT is the two-sample multivariate sign test, so we expect that DRT should have the same level and power breakdown point as the univariate sign test.

Theorem 3. Under both jointly and separately contaminated null hypotheses the level breakdown point of DRT is $\epsilon^{**}(R^*) = 1$.

Proof. It is similar to that of Theorem 1.

Theorem 4. The power breakdown point of DRT is $\epsilon^*(DRT) = 1/2$.

Proof. A proof for joint contamination is similar to that for separate contamination. So here we stick to the case of separate contamination, which gives us a clearer view. For $0 \leq \epsilon \leq 1$ and $F_\theta(x) = F(x-\theta)$, let X be from $(1-\epsilon)F + \epsilon G$ and Y be from $(1-\epsilon)F_\theta + \epsilon G^*$, where G and G^* are arbitrary distributions. Then

$$
\begin{aligned}
&E[R_f]^T E[R_f] \\
&= \|E_{(1-\epsilon)F+\epsilon G,(1-\epsilon)F_\theta+\epsilon G^*}\left[\frac{X}{\|X\|}-\frac{Y}{\|Y\|}\right]\|^2 \quad (4.1)\\
&= \|(1-\epsilon)E_F\left[\frac{X}{\|X\|}\right]+\epsilon E_G\left[\frac{X}{\|X\|}\right] \\
&\quad -(1-\epsilon)E_{F_\theta}\left[\frac{Y}{\|Y\|}\right]-\epsilon E_{G^*}\left[\frac{Y}{\|Y\|}\right]\|^2 \quad (4.2)\\
&= \|(1-\epsilon)E_{F,F_\theta}\left[\frac{X}{\|X\|}-\frac{Y}{\|Y\|}\right]+\epsilon E_{G,G^*}\left[\frac{X}{\|X\|}-\frac{Y}{\|Y\|}\right]\|^2. \quad (4.3)
\end{aligned}
$$

Let us define $R_\theta \equiv E_{F,F_\theta}[R_f]$ and $\alpha \equiv E_{G,G^*}[R_f]$. Taking $G = F_\theta$ and $G^* = F$ we can make $\alpha = -R_\theta$. So $\|E_{(1-\epsilon)F+\epsilon G,(1-\epsilon)F_\theta+\epsilon G^*}\ [X/\|X\| - Y/\|Y\|]\ \|^2 = (1-2\epsilon)^2\|R_\theta\|^2$. And $\epsilon = 1/2$ makes this zero so that $\epsilon^* \leq 1/2$. Now suppose $\epsilon^* < 1/2$. Then $\|(1-\epsilon)R_\theta + \epsilon\alpha\| \geq (1-\epsilon)\|R_\theta\| - \epsilon\|\alpha\|$. Note that norms of both R_θ and α are bounded by 0 and 2. Then for θ big enough there exists $0 < \delta < 2(1-2\epsilon)/(1-\epsilon)$ such that $(1-\epsilon)\|R_\theta\| - \epsilon\|\alpha\| \geq (1-\epsilon)(2-\delta) - 2\epsilon > 0$ which means $\|(1-\epsilon)R_\theta + \epsilon\alpha\|$ is always positive. Therefore the power breakdown point ϵ^* is 1/2.

5. BREAKDOWN POINTS OF DWT

Note that $N[\Sigma_W - (1/m + 1/n)\Phi_Z] \xrightarrow{p} 0$ (see section 1) and $W_f = (X - Y)/\|X - Y\|$. To calculate the level breakdown point of DWT we use $E[W_f]^T \Sigma_{W_f}^{-1} E[W_f]$ as the functional of DWT and to calculate the power breakdown point of DWT we use $E[W_f]^T \Phi_Z^{-1} E[W_f]$ as its corresponding functional. The following Lemma 1 shows that DWT can be arbitrarily large as $\|\theta\|$ goes to ∞. Theorem 5 shows that when $\|\theta\|$ is bounded under the ϵ-contaminated null hypothesis, DWT does not break down. Lemma 2 shows that $E[W_f]$ and $E[W_f]^T \Phi_Z^{-1} E[W_f]$ have a common power breakdown point. The last two theorems discuss the power breakdown point of DWT when X and Y are contaminated separately and jointly, respectively.

Lemma 1. Under any contamination Σ_{W_f} is positive definite as long as $Cov_{F_\theta}\ (W_f)$ is positive definite. Moreover if $\|\theta\|$ goes to infinity then at least along some path Σ_{W_f} tends to zero so that $\sup_\theta E[W_f]^T \Sigma_{W_f}^{-1} E[W_f] = \infty$.

Proof. Assume that $Cov_{F_\theta}(W_f)$ is positive definite for given θ such that $\|\theta\| < \infty$. In this proof we focus on joint contamination for simplicity. So, let the underlying distribution be $(1-\epsilon)F_\theta + \epsilon G$, where F_θ is the joint distribution of (X, Y) from the null or alternative hypothesis and G is an arbitrary distribution. Then we have

$$
\begin{aligned}
&Cov_{(1-\epsilon)F_\theta+\epsilon G}(W_f) \\
&\quad = E_{(1-\epsilon)F_\theta+\epsilon G}[W_f W_f^T] - E_{(1-\epsilon)F_\theta+\epsilon G}[W_f] E_{(1-\epsilon)F_\theta+\epsilon G}[W_f^T] && (5.1)\\
&\quad = (1-\epsilon)Cov_{F_\theta}(W_f) + (1-\epsilon)E_{F_\theta}[W_f]E_{F_\theta}[W_f^T] + \epsilon Cov_G(W_f) \\
&\qquad + \epsilon E_G[W_f]E_G[W_f^T] - (1-\epsilon)^2 E_{F_\theta}[W_f]E_{F_\theta}[W_f^T] \\
&\qquad - \epsilon(1-\epsilon)E_{F_\theta}[W_f]E_G[W_f^T] \\
&\qquad - \epsilon(1-\epsilon)E_G[W_f]E_{F_\theta}[W_f^T] - \epsilon^2 E_G[W_f]E_G[W_f^T] && (5.2)\\
&\quad = (1-\epsilon)Cov_{F_\theta}(W_f) + \epsilon Cov_G(W_f) + \epsilon(1-\epsilon) && (5.3)\\
&\qquad \times \Big(E_{F_\theta}[W_f]E_{F_\theta}[W_f^T] - E_{F_\theta}[W_f]E_G[W_f^T] \\
&\qquad - E_G[W_f]E_{F_\theta}[W_f^T] + E_G[W_f]E_G[W_f^T]\Big) \\
&\quad = (1-\epsilon)Cov_{F_\theta}(W_f) + \epsilon Cov_G(W_f) + \epsilon(1-\epsilon) \\
&\qquad \times (E_{F_\theta}[W_f] - E_G[W_f])(E_{F_\theta}[W_f] - E_G[W_f])^T && (5.4)\\
&\quad \geq (1-\epsilon)Cov_{F_\theta}(W_f). && (5.5)
\end{aligned}
$$

These equalities and inequalities show that the Σ_{W_f} is positive definite under the contaminated null or alternative hypotheses since the last term is positive definite. Now suppose $\|\theta\|$ goes to ∞. When $p=1$, $E_{F_\theta}[W_f] = E_{F_\theta}[sign(X-Y)]$ is bounded for any F_θ. And $Var_{F_\theta}(W_f) = 1 - [E_{F_\theta}(sign(X-Y))]^2$ approaches to zero as $|\theta|$ goes to ∞. When $p \geq 2$, $\Sigma_{W_f} = E_{F_\theta}[W_f W_f^T]$ $-$ $E_{F_\theta}[W_f]$ $E_{F_\theta}[W_f]^T$. Note that under H_a, as $k \to \infty$ we can find a sequence of $\theta_{(k)}$ such that $\|\theta_{(k)}\| \to \infty$ at least along some path. Then $W_f = (X - Y + \theta_{(k)}) / \|X - Y + \theta_{(k)}\|$ converges to $L \equiv \lim_{k\to\infty} \theta_{(k)} / \|\theta_{(k)}\|$ for each X and Y. Since $\lim_{k\to\infty} E_{F_{\theta_{(k)}}}[W_f W_f^T] = E_{F_0}[\lim_{k\to\infty} (X - Y + \theta_{(k)}) / \|X - Y + \theta_{(k)}\| \cdot (X - Y + \theta_{(k)})^T / \|X - Y + \theta_{(k)}\|] = LL^T$ and $\lim_{k\to\infty} E_{F_{\theta_{(k)}}}[W_f] = E_{F_0}[\lim_{k\to\infty} (X - Y + \theta_{(k)}) / \|X - Y + \theta_{(k)}\|] = L$, $\Sigma_{W_f} \to LL^T - LL^T = 0$. Therefore $E[W_f]^T \Sigma_{W_f}^{-1} E[W_f] \to \infty$ as $\|\theta_{(k)}\| \to \infty$.

Theorem 5. Under both separate and joint contamination, the level breakdown point of DWT, $\epsilon^{**}(DWT) = 1$.

Proof. Under the ϵ-contaminated H_o, both W_f and $\Sigma_{W_f}^{-1}$ are bounded. Therefore DWT cannot be arbitrarily large with $\epsilon < 1$.

Lemma 2 implies that unless $\|E[W_f]\| \to 0$, $E[W_f]^T \Phi_Z^{-1} E[W_f]$ does not vanish to zero. On the other hand, if $E[W_f]$ goes to zero, then $E[W_f]^T \Phi_Z^{-1} E[W_f]$ goes to zero since Φ_Z is bounded. Therefore the breakdown point of $\|E[W_f]\|$ is that of $E[W_f]^T \Phi_Z^{-1} E[W_f]$. Next two theorems present two different power breakdown points of DWT, one for separately contaminated distribution and another for jointly contaminated distribution.

Lemma 2. $E[W_f^T]\Phi_Z^{-1}E[W_f] \geq \|E[W_f]\|^2$, where $\Phi_Z = E[(Z_1 - Z_2)/\|Z_1 - Z_2\| \cdot (Z_1 - Z_3)^T \|Z_1 - Z_3\|]$ for i.i.d. Z_1, Z_2 and Z_3 from an elliptically symmetric distribution with a diagonal scale matrix.

Proof. Note that $E[W_f]^T \Phi_Z^{-1} E[W_f] \geq 1/\lambda_{max}(\Phi_Z)\|E[W_f]\|^2 \geq \|E[W_f]\|^2$ since the maximum eigenvalue of Φ_p, that is, $\lambda_{max}(\Phi_Z)$ is positive and bounded by 1 from above. These inequalities hold because when Z_1, Z_2,and Z_3 are *i.i.d.* from an elliptically symmetric distribution with a diagonal scale matrix, Φ_Z is also a diagonal with all the elements less than 1 (Choi and Marden, 1997).

Theorem 6. Suppose two samples are contaminated separately. Then the power breakdown point of DWT is at least $1 - \sqrt{1/2}$.

Proof. Let $W_\theta \equiv E_{F,F_\theta}[W_f]$, $\alpha_1 \equiv E_{G,F_\theta}[W_f]$, $\alpha_2 \equiv E_{F,G^*}[W_f]$, $\beta \equiv E_{G,G^*}[W_f]$, Then

$$\begin{aligned} &\|E_{(1-\epsilon)F+\epsilon G,(1-\epsilon)F_\theta+\epsilon G^*}[W_f]\| \\ &\quad = \|(1-\epsilon)^2 W_\theta + \epsilon(1-\epsilon)\alpha_1 + \epsilon(1-\epsilon)\alpha_2 + \epsilon^2\beta\| \quad (5.6)\\ &\quad \geq (1-\epsilon)^2\|W_\theta\| - \epsilon(1-\epsilon)\|\alpha_1\| - \epsilon(1-\epsilon)\|\alpha_2\| - \epsilon^2\|\beta\|. \quad (5.7)\end{aligned}$$

If $\epsilon < 1 - \sqrt{1/2}$ or $\epsilon_o \equiv 2(1-\epsilon)^2 - 1 > 0$, then choose $\|\theta\|$ big enough so that $\|W_\theta\| > 1 - \delta$ for some $0 < \delta < \epsilon_o/(1-\epsilon)^2 < 1$ so that $\|E_{(1-\epsilon)F+\epsilon G,(1-\epsilon)F_\theta+\epsilon G^*}[W_f]\| \geq (1-\delta)(1-\epsilon)^2 - 2\epsilon(1-\epsilon) - \epsilon^2 > 0$. Thus the power breakdown point of DWT is at least $1 - \sqrt{1/2}$.

Theorem 7. When two samples are jointly contaminated, the power breakdown point of DWT is $1/2$.

Proof. Let $W_\theta \equiv E_{F_\theta}[W_f]$ and $\alpha \equiv E_G[W_f]$. Then

$$\begin{aligned}\|E_{(1-\epsilon)F_\theta+\epsilon G}[W_f]\| &= \|(1-\epsilon)E_{F_\theta}[W_f] + \epsilon E_G[W_f]\| \quad (5.8)\\ &= \|(1-\epsilon)W_\theta + \epsilon\alpha\|. \quad (5.9)\end{aligned}$$

For the least favorable case, we can take α to be the opposite direction of U_θ. That is, $\alpha = -W_\theta$ by taking $X_G = -X_{F_\theta}$ and $Y_G = -Y_{F_\theta} - \theta$. Then we have $((1-\epsilon) - \epsilon)\|W_\theta\| = 0$, or $(1-2\epsilon) = 0$ so that $\epsilon^* \leq 1/2$. But if $\epsilon < 1/2$,

$\|E_{(1-\epsilon)F_\theta+\epsilon G}[(X-Y)/\|X-Y\|]\| \geq (1-\epsilon)\|W_\theta\| - \epsilon\|\alpha\| \geq (1-\epsilon)(1-\delta)-\epsilon > 0$ for sufficiently large $\|\theta\|$ such that $\|W_\theta\| \geq 1-\delta$ and $0 < \delta < (1-2\epsilon)/(1-\epsilon) < 1$. Thus $\epsilon^* = 1/2$.

6. DISCUSSION AND CONCLUSIONS

Based on Raleigh's V^*, Randles (1989) developed a multivariate sign test V_m for the one-sample case introducing interdirections between vectors. Assuming that observations are *i.i.d.* from an elliptically symmetric distributions with pdf (eq. ??) he represents $X_1, \cdots, X_m$ with elliptical directions and lengths of vectors. Let $U_1, \cdots, U_m$ be *i.i.d.* from Uniform(Ω), where Ω is the surface of the unit ball in R^p, and $R_1, \cdots, R_m$ be any positive random variables in R^1. Let D be any non-singular matrix. Note that $E[U_i] = 0$ and $Cov(U_i) = I_p/p$ for $i = 1, \cdots, m$. Then under H_o, $X_i = R_i D U_i$ for $i = 1, \cdots, m$ and

$$V_m = p/m \sum_{i=1}^{m} \sum_{j=1}^{m} \cos(\pi \hat{p}_{ij}) \tag{6.1}$$

where

$$\hat{p}_{ij} = \frac{H_{ij} + d_m}{C(m,p)} \tag{6.2}$$

if $i \neq j$, and 0 otherwise. Here $C(m,p)$ is "m choose p" and H_{ij} is the interdirection defined as the number of hyperplanes formed by the origin and other points (not X_i nor X_j) such that X_i and X_j are on the opposite sides of the hyperplanes formed. The d_m is a correction factor defined as follows:

$$d_m = \frac{C(m,p-1) - C(m-2,p-1)}{2}. \tag{6.3}$$

Then since $\hat{p}_{ij}$ is a consistent estimate of $\arccos(U_i^T U_j)/\pi$, V^* and V_m are asymptotically equivalent. So, V_m is asymptotically as efficient as V^* relative to one-sample Hotelling's T^2. Moreover under elliptically symmetric distributions V_m is distribution-free since it uses only discretized (or ranked) direction vectors. Thus it seems that Randles multivariate sign test for the one-sample case is very robust against outliers and we might rush to conclude that Randles' sign test is as robust as Raleigh's. However this conjecture is full of doubt.

It is not quite easy to calculate the power and level breakdown points of Randles multivariate sign test V_m since it employs a bunch of hyperplanes formed by $p-1$ vectors and the origin. Note that V_m and a spatial median defined by Oja (Oja 1983, Small 1990) share one common thing, that is, employing hyperplanes (or simplexes). Even though this property provides very good aspects to both of them, it lowers down their breakdown points. Niinimaa, Oja, and Tableman (1990) points out that employing a simplex formed by p data points and the global location estimate leads the Oja's spatial median to have a very low breakdown point, $2/(m+2)$, where m is the sample size. So we conjecture that the Randles multivariate sign test is very vulnerable to outliers.

We can apply the same idea of involving interdirections to Raleigh type statistic for the several-sample case. For instance, in the two-sample test DRT ($eq.$1.2) all the products of direction vectors of three summations can be replaced by cos (interdirection between $X - \hat{\theta}$ and $Y - \hat{\theta}$), where $\hat{\theta}$ is the global location estimate of two populations. Note that this statistic should involve the global location estimate in addition to a bunch of hyperplanes. So, it is not easy to find its corresponding functional and its test breakdown points. Picking a robust global location estimate is another difficulty that Randles type tests confront for the several-sample problem.

ACKNOWLEDGEMENT

The author thanks professor Xuming He and professor John I.Marden at University Illinois at Urbana-Champaign for their helpful advices, comments, and corrections.

REFERENCES

(1) Chaudhuri, P. (1992), Multivariate location estimate using extension of R-Estimates through U-Statistics type approach, *Annals of Statistics*, 20, no.2, 897-916.

(2) Choi, K. (1995), Nonparametric Multivariate Multisample Tests for the Location Problem and Multivariate Regression Based on Directions of Data, Ph.D Dissertation, University of Illinois at Urbana - Champaign.

(3) Choi, K. and Marden, J. I. (1997), An approach to multivariate rank transform tests in multivariate analysis of variance, *An accepted manuscript by Journal of American Statistical Association.*

(4) He, X., Simpson, D. G. and Portnoy, S. L. (1990), Breakdown robustness of tests, *Journal of American Statistical Association*, **85**, 446-452.

(5) Hettmansperger, T. P. and Oja, H. (1994), Affine invariant multivariate multisample sign tests, *Journal of Royal Statistical Society B*, **56**, 235-249.

(6) Hettmansperger, T. P., Nyblom, J. and Oja, H. (1994), Affine invariant multivariate one-sample sign tests, *Journal of Royal Statistical Society B*, **56**, 221-234.

(7) Niinimaa, A., Oja, H. and Tableman, M. (1990), The finite-sample breakdown point of the Oja bivariate median and of the corresponding half-samples version, *Statistical Probability Letter*, **10**, 325-328.

(8) Oja, H. (1983), Descriptive statistics for multivariate distributions, *Statistical Probability Letter*, **1**, 327-332.

(9) Puri, M. L. and Sen, P. K. (1971), *"Nonparametric Methods in Multivariate Analysis"*, Wiley, New York.

(10) Randles, R. H. (1989), A distribution-free multivariate sign test based on interdirections, *Journal of American Statistical Association*, **84**, 1045-1050.

(11) Small, C. G. (1990), A Survey of Multidimensional medians, *International Statistical Review*, **58**, 263-277.

(12) Watson, G. S. (1983), *Statistics on Spheres*, Wiley-Interscience, New York. item[(13)] Ylvisaker, D. (1977), Test resistance, *Journal of American Statistical Association*, **72**, 551-556.