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The Admissibility of Some Nonparametric Tests[†]

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ABSTRACT

It is demonstrated that many standard nonparametric test such as the Mann-Whitney-Wilcoxon test, the Fisher-Yates test, the Savage test and the median test are admissible for a two-sample nonparametric testing problem. The admissibility of the Kruskal-Wallis test is demonstrated for a nonparametric one-way layout testing problem.

Key Words : Admissibility; Linear rank tests; Nonparametric testing.

1. INTRODUCTION

Although some optional properties of the standard two-sample nonparametric test are known, see Ferguson (1967) and Lehmann (1986), their admissibility has been an open question (see, for example, Ferguson,1967, p252). In this note we prove the admissibility of certain linear rank tests which include the Mann-Whitney-Wilcoxon test, the Fisher-Yates test, the Savage test and the median test. Recently the admissibility of certain estimators have been

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demonstrated for various nonparametric problems, see for example Cohen and Kuo (1985), Brown (1988), Meeden et al (1985) and Meeden et al (1989). In all these cases the admissibility for the nonparametric problem follows easily once admissibility had been demonstrated for a related multinomial problem. As we shall see the same strategy works for testing problems as well. We note that a two sample linear rank test can be considered as a test for a certain two sample multinomial testing problem. For this multinomial testing problem its admissibility follows from the results of Matthes and Truax (1967) from which its admissibility for nonparametric problem follows.

2. THE RESULT

We consider a two-sample problem where X = (X_1 , ... , X_m) and Y = (Y_1 , ... , Y_n) are independent random samples from unknown distributions, F and G respectively. For the nonparametric problem, $\gamma = (F, G)$ is assumed to belong to Γ, the cartesian product of the set of all possible distribution functions on the Real numbers with itself. We consider testing the hypothesis H: $\gamma \in \Gamma_o$ against K: $\gamma \notin \Gamma_o$ where $\Gamma_o = \{(F, G) : F(x) = G(x) \text{ for all } x\}$. Note a test ϕ is said to be admissible if there does not exist a test ϕ' such that $E_\gamma \phi' \leq E_\gamma \phi$ for $\gamma \in \Gamma_o$ and $E_\gamma \phi' \geq E_\gamma \phi$ for $\gamma \notin \Gamma_o$ with at least one of the two inequallities being strict for at least one value γ.

For this two-sample problem $\left(X_{(1)}, \ldots, X_{(m)}\right)$ and $\left(Y_{(1)}, \ldots, Y_{(n)}\right)$, the two vectors of order statistics for the two samples, are a complete and sufficient statistic. Let $(R_1, \ldots, R_m, R_{m+1}, \ldots, R_{m+n})$ be the set of ranks, *i.e.* $R_1 \leq R_2 \leq \ldots \leq R_m$ are the ranks of the order statistic of the X_i's in the total sample of $N = m + n$ observations and $R_{m+1} \leq R_{m+2} \leq \ldots \leq R_{m+n}$ are the ranks of the order statistic of the Y_j's. We consider a linear rank statistic $L = \sum_{i=1}^{N} c_i a(R_i)$ where $a(1), \ldots, a(N)$ and $c_1, \ldots, c_N$ are sets of N constants such that the numbers within each set are not all the same. The $a(i)$'s are called the score and the c_i's the regression constants. We can generate various statistics by choosing the $a(i)$'s and c_i's appropriately. For example if $c_i = 0$ for $i = 1, \ldots, m$ and 1 for $i = m + 1, \ldots, N$ and $a(i) = i$ for $i = 1, \ldots, N$ then L becomes $\sum_{j=1}^{m} R_{j+m}$ which is the well-known Mann-Whitney-Wilcoxon statistic or rank sum statist.

Next we consider the appropriate two-sample multinomial testing problem. We suppose that the X_i's and Y_j's can only take on k distinct values which without loss of generality we assume to be $1, 2 \ldots, k$. Let $P(X_1 = i) = p_i$ and $P(Y_1 = i) = q_i$ for $i = 1, \ldots, k$ where $0 \leq p_i \leq 1, 0 \leq q_i \leq 1$, and $\sum_{i=1}^{k} p_i = 1$ and $\sum_{i=1}^{k} q_i = 1$. We assume the probability vectors p$=(p_1, \ldots, p_k)$ and q$=(q_1, \ldots, q_k)$ are completely unknown and we wish to test H: $p = q$ against K: $p \neq q$. For $i = 1, \ldots, k$, let t_i be the number of X_j's $= i$ and s_i be the number of Y_j's $= i$. Now the t_i's and s_i's are jointly complete and sufficient for p and q and joint probability function is proportional to $\left(\prod_{i=1}^{k} p_i^{t_i}\right)\left(\prod_{i=1}^{k} q_i^{s_i}\right)$ which can be rewritten as

$$f_{\omega,\theta}(t,u) = c(\omega,\theta;t,u)e^{t\omega+u\theta}$$

where for $j = 1, \ldots, k-1$, $\theta_j = \log(q_j/q_k)$, $\omega_j = \log\left((p_j/q_j)\,/\,(p_k/q_k)\right)$, $u_j = t_j + s_j$, $t\omega = \sum_{j=1}^{k-1} t_j\omega_j$ and $u\theta = \sum_{j=1}^{k-1} u_j\theta_j$. Since p $=$ q if and only if $\omega = 0$, our testing problem can be reformulated to testing H: $\omega = 0$ against K: $\omega \neq 0$ where θ is a vector of nuisance parameters. The following lemma gives a sufficient condition for tests based on a linear rank statistic to be admissible.

Lemma 1. For testing H: $\omega = 0$ against K: $\omega \neq 0$, a test which rejects H if and only if $\sum_{i=1}^{m+n} c_i a(R_i)$ is greater than k_1 or less than k_2 where $k_1 > k_2$ is admissible if the c_i's are constant over each sample, *i.e.* if $c_1 = c_2 = \cdots = c_m$ and $c_{m+1} = c_{m+2} = \cdots = c_{m+n}$.

Proof. Suppose the two samples contains u_1 1's and u_2 2's and so on. Then each $1, 2 \ldots, k$ in the sample has rank

$$\frac{u_1+1}{2}, \quad u_1 + \frac{u_2+1}{2}, \quad u_1 + u_2 + \frac{u_3+1}{2}, \ldots, u_1 + \cdots + u_{k-1} + \frac{u_k+1}{2}$$

respectively. Thus L becomes

$$\begin{aligned}
L &= a\left(\frac{u_1+1}{2}\right)\left[\sum_{i=1}^{t_1} c_i + \sum_{i=m+1}^{m+s_1} c_i\right] \\
&\quad + a\left(u_1 + \frac{u_2+1}{2}\right)\left[\sum_{i=t_1+1}^{t_1+t_2} c_i + \sum_{i=m+s_1+1}^{m+s_1+s_2} c_i\right] \\
&\quad\quad \vdots \\
&\quad + a\left(u_1 + \cdots + u_{k-1} + \frac{u_k+1}{2}\right) \\
&\quad\quad \times \left[\sum_{i=t_1+\cdots+t_{k-1}+1}^{m} c_i + \sum_{m+s_1+\cdots+s_{k-1}+1}^{N} c_i\right] \\
&= a\left(\frac{u_1+1}{2}\right)\left[t_1c_1 + (u_1 - t_1)c_{m+1})\right]
\end{aligned}$$

$$
\begin{aligned}
&+a\left(u_1+\frac{u_2+1}{2}\right)[t_2c_1+(u_2-t_2)c_{m+1}]\\
&\qquad\vdots\\
&+a\left(u_1+\cdots+u_{k-1}+\frac{u_k+1}{2}\right)\\
&\qquad\times\left[\left(m-\sum_{i=1}^{k-1}t_i\right)c_1+\left(N-\sum_{i=1}^{k-1}u_i-\sum_{i=1}^{k-1}t_i\right)c_{m+1}\right] \quad (2.1)
\end{aligned}
$$

Note that for each fixed $u_1, \cdots, u_{k-1}$, the a's are constant and L is a linear function of $t_1, \cdots, t_{k-1}$. Because we reject the null hypothesis if L is too large or too small, the acceptance region of the test is convex in $t_1, \cdots, t_{k-1}$, for each set of fixed u_i's. Hence the linear rank test has convex acceptance section and is admissible by the results of Matthes and Truax (1967).

Let ϕ_o denote a linear rank test which satisfies the conditions of the lemma. Note that neither the size of k nor the assumed values $1, 2, \ldots, k$ play any role in the proof, *i.e.* ϕ_o will be admissible for all such two-sample multinomial problem. The admissibility for the two-sample multinomial problem follows easily from this fact.

Theorem 1. Let ϕ_o be a test based on a linear rank statistic which satisfies the conditions of the lemma, then ϕ_o is admissible for the two-sample nonparametric problem.

Proof. We will assume ϕ_o is not admissible for the nonparametric problem and get a contradiction.

If ϕ_o is not admissible for the nonparametric problem, then there exists a test ϕ such that ϕ dominates ϕ_o. Hence there exist $x_1, \ldots, x_m, y_1, \ldots, y_n$ such that

$$\phi_o(x_1, \ldots, x_m, y_1, \ldots, y_n) \neq \phi(x_1, \ldots, x_m, y_1, \ldots, y_n)$$

Let $\alpha_1, \ldots, \alpha_k$ be the k-distinct values which appear in the set $\{x_1, \ldots, x_m, y_1, \ldots, y_n\}$ and let $\Gamma(\alpha_1, \ldots, \alpha_k)$ denote all distribution functions which concentrate all their mass on $\alpha_1, \ldots, \alpha_k$. We now consider the testing problem H: $F = G$ against K: $F \neq G$ where $F, G \in \Gamma(\alpha_1, \ldots, \alpha_k)$. In this case $x_1, \ldots, x_m$ and $y_1, \ldots, y_n$ are the outcomes for the random samples from *multinomial* $(1, p_1, \ldots, p_k)$ and *multinomial* $(1, q_1, \ldots, q_k)$ where $p_i = Pr(X = \alpha_i)$ and $q_i = Pr(Y = \alpha_i)$ for $i = 1, 2, \ldots, k$. Note that $\Gamma(\alpha_1, \ldots, \alpha_k)$ is equivalent to the $(k-1)$-dimensional simplex

$$\Gamma^1 = \left\{p = (p_1, \cdots, p_k); 0 \leq p_i \leq 1 \quad \text{for } i = 1, \ldots, k \quad \text{and } \textstyle\sum_{i=1}^k p_i = 1\right\}.$$

and each $p \in \Gamma$ determines a unique F, say F_p.

Without loss of generality we may assume ϕ is also a function of the complete sufficient statistic $(t_1, \ldots, t_{k-1})$ and $(s_1, \ldots, s_{k-1})$. Since ϕ_o is admissible for the multinomial problem it must be the case that

$$E_{(\mathrm{p},\mathrm{q})}\phi_o = E_{(\mathrm{p},\mathrm{q})}\phi \quad \text{for all p, q} \in \Gamma^1.$$

Hence by completeness $P_{(\mathrm{p},\mathrm{q})}(\phi_o = \phi) = 1$ for all p, q $\in \Gamma^1$ which is a contradiction.

Remark 1. As we have noted the Mann-Whitney-Wilcoxon test satisfies the conditions of the lemma and hence admissible for the two-sample nonparametric testing problem. It is easy to check that the Fisher-Yates test, the two-sample median test and the Savage test also satisfy the conditions of the lemma and hence admissible for the two-sample nonparametric problem.

We conclude by noting that this technique extends easily to multi-sample problems. In particular we will demonstrate the admissibility of the Kruskal-Wallis test for the one-way layout problem.

Suppose that $X_{11}, \cdots, X_{1n_1}, X_{21}, \ldots, X_{v1}, \ldots, X_{vn_v}$ are v independent random samples from $F_1, F_2, \ldots, F_v$, respectively. We will consider a test H: $F_1 = \ldots = F_v$ against K: $F_i \neq F_j$ for at least one $i \neq j$. Let R_{ij} be the rank of X_{ij} in the combined samples and let $R_{i.} = \sum_{j=1}^{n_i} R_{ij}$ and $\overline{R}_{i.} = R_{i.}/n_i$. Then under the null hypothesis, it can be shown that $E(\overline{R}_{i.}) = (N+1)/2$ and $Var(\overline{R}_{i.}) = (N-n_i)(N+1)/12n_i$, where $N = \sum_{i=1}^{v} n_i$. Thus the difference $\overline{R}_{i.} - (N+1)/2$ represent the departure of $\overline{R}_{i.}$ from its expected value and we could reject the null hypothesis if the accumulated departure is too large. This suggests a test statistic of the form

$$W = \sum_{i=1}^{v} c_i \left\{ \frac{\overline{R}_{i.} - \frac{N+1}{2}}{\sqrt{var(\overline{R}_{i.})}} \right\}^2$$

where $c_1, \ldots, c_v$ are constants which are chosen so that W has a convenient distribution. One such statistic is the Kruskal-Wallis statistic. Kruskal and Wallis (1952) chose $c_i = 1 - n_i/N$ so that the limiting distribution of W would be chi-square with $v-1$ degrees of freedom.

As before, we begin with the multinomial problem. For convenience we will only consider the case $v = 3$.

Let $X_{11}, \ldots, X_{1n_1}, X_{21}, \ldots, X_{2n_2}$, and $X_{31}, \ldots, X_{3n_3}$ be independent random samples from *multinomial*$(1, \mathrm{p}_1)$, *multinomial* $(1, \mathrm{p}_2)$, and *multinomial* $(1, \mathrm{p}_3)$, respectively, where $\mathrm{p}_i = (p_{i1}, \ldots, p_{ik})$, $i = 1, 2, 3$. We will test the null hypothesis H : $p_1 = p_2 = p_3$ against the alternative hypothesis K : $p_i \neq p_j$ for at least one $i \neq j$.

The joint probability distribution of the X_{ij}'s is

$$f(x_{11}, \dots, x_{3n_3}; p_1, p_2, p_3) = c(n_1, n_2, n_3) p_{1k}^{n_1} p_{2k}^{n_2} p_{3k}^{n_3} \times \exp\left\{ \sum_{i=1}^{3} \sum_{l=1}^{k-1} t_{il} \log \frac{p_{il}}{p_{ik}} \right\}.$$

where t_{il} is the number of observations in the l-th category in the i-th sample. Since the exponent term can be written as

$$\sum_{i=1}^{3} \sum_{l=1}^{k-1} t_{il} \log \frac{p_{il}}{p_{ik}} = \sum_{i=1}^{2} \sum_{i=1}^{k-1} \left\{ t_{il} \left(\log \frac{p_{il}}{p_{ik}} - \log \frac{p_{3l}}{p_{3k}} \right) \right\} + \sum_{l=1}^{k-1} (t_{1l} + t_{2l} + t_{3l}) \log \frac{p_{3l}}{p_{3k}},$$

we can rewrite the probability function as follows.

$$f(t, u; \omega, \theta) = c_o(n_1, n_2, n_3) \exp\{t\omega + u\theta\}$$

where $t\omega = \sum_{i=1}^{2} \sum_{l=1}^{k-1} t_{il}\omega_{il}$, $\omega_{il} = \log((p_{il}/p_3)/(p_{1k}/p_{3k}))$, $u_l = t_{1l} + t_{2l} + t_{3l}$, $\theta_l = \log(p_{3l}/p_{3k})$ and $u\theta = \sum_{l=1}^{k-1} u_l \theta_l$.

Now the original testing problem is equivalent to H : $\omega = 0$ against K : $\omega \neq 0$ and by Matthes and Truax (1967) we will be done if we can show that W is a convex function of $t_{11}, \dots, t_{1k-1}, t_{21}, \dots, t_{2k-1}$ for each fixed $u_1, \dots, u_{k-1}$, because the acceptance region of the test is given by $W < c$ for some constant c.

Note that

$$W = \frac{12}{N(N+1)} \sum_{i=1}^{3} n_i \left(\overline{R}_{i\cdot} - \frac{N+1}{2} \right)^2 = \frac{12}{N(N+1)} \sum_{i=1}^{3} \frac{R_{i\cdot}^2}{n_i} - 3(N+1)$$

where $N = n_1 + n_2 + n_3$. Thus if we show that $\sum_{i=1}^{3} R_{i\cdot}^2/n_i$ is a convex function of $t_{11}, \dots, t_{1k-1}, t_{21}, \dots, t_{2k-1}$ for each fixed $u_l, \dots, u_{k-1}$, then we are done. Since each $1, 2, \dots, k$ has rank $(u_1+1)/2, u_1 + (u_2+1)/2, u_1 + u_2 + (u_3+1)/2, \dots, \sum_{l=1}^{k-1} u_l + (u_k+1)/2$, respectively,

$$R_{i\cdot} = t_{i1}\left(\frac{u_1+1}{2}\right) + \dots + t_{ik}\left(u_1 + \dots + u_{k-1} + \frac{u_k+1}{2}\right) \quad \text{for } i = 1, 2$$

and

$$R_{3\cdot} = (u_1 - t_{11} - t_{21})\left(\frac{u_1+1}{2}\right) + \ldots + (u_k - t_{1k} - t_{2k})\left(u_1 + \ldots + u_{k-1} + \frac{u_k+1}{2}\right)$$

Clearly, for fixed $u_1, \ldots, u_{k-1}$, $R_{1\cdot}$ is a linear function of $t_{11}, \ldots, t_{1k-1}$, $R_{2\cdot}$ is a linear function of $t_{21}, \ldots, t_{2k-1}$ and $R_{3\cdot}$ is a linear function of $t_{11}, \ldots, t_{1k-1}$ and $t_{21}, \ldots, t_{2k-1}$. *i.e.* the $R_{i\cdot}$'s are convex functions. Hence by a standard argument, see for example, Roberts, A. W. and Varberg, D. E. (1973) pp. 15–16, we see that $\sum_{i=1}^{3} R_{i\cdot}^2/n_i$ is a convex function and we are done.

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