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An Empirical Central Limit Theorem for the Kaplan-Meier Integral Process on $[0, \infty)^\dagger$

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ABSTRACT

In this paper we investigate weak convergence of the integral processes whose index set is the non-compact infinite time interval. Our first goal is to develop the empirical central limit theorem as random elements of $D[0, \infty)$ for an integral process which is constructed from *iid* variables. In developing the weak convergence as random elements of $D[0, \infty)$, we will use a result of Ossiander(4) whose proof heavily depends on the total boundedness of the index set. Our next goal is to establish the empirical central limit theorem for the Kaplan-Meier integral process as random elements of $D[0, \infty)$. In achieving the goal, we will use the above *iid* result, a representation of Stute(6) on the Kaplan-Meier integral, and a lemma on the uniform order of convergence. The first result, in some sense, generalizes the result of empirical central limit theorem of Pollard(5) where the process is regarded as random elements of $D[-\infty, \infty]$ and the sample paths of limiting Gaussian process may jump. The second result generalizes the first result to random censorship model. The later also generalizes one dimensional central limit theorem of Stute(6) to a process version. These results may be used in the nonparametric statistical inference.

Key Words : Infinite time scale process; Empirical CLT; Random censorship model; Kaplan-Meier integral process.

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1. INTRODUCTION

Weak convergence of the integral processes whose index set is the non-compact infinite time interval is investigated in the context of empirical central limit theorems. Beginning with an empirical process, we first discuss the differences between compactness and non-compactness of index sets on which the process is defined. Our first goal of the present work is to develop the empirical central limit theorem as random elements (*r.e.*) of $D[0,\infty)$ for an integral process which is constructed from the independent and identically distributed(*iid*) random variables. Using the weak convergence result of Ossiander(4), for example, whose result heavily depends on the total boundedness of the index set of the process, we establish the weak convergence as *r.e.* of $D[0,\infty)$. In this work we use the uniform metric on compacta as the underlying metric of $D[0,\infty)$. Our next goal of the paper is to establish the empirical central limit theorem for the Kaplan-Meier integral process as *r.e.* of $D[0,\infty)$. In achieving the second goal, we will use the *iid* result, a representation of Stute(6) on the Kaplan-Meier integral, and a lemma on the uniform order of convergence. For the completeness of the proof of the second result, we provide the proof of the lemma in Appendix.

Proposition 1 of the present paper, in some sense, generalizes the result of empirical central limit theorem of Pollard(5) where the process is regarded as *r.e.* of $D[-\infty,\infty]$ and the sample paths of limiting Gaussian process may jump. Theorem 1 is a generalization of Proposition 1 in the sense of dealing with censored data. That is, when there is no censoring, Theorem 1 boils down to Proposition 1. Theorem 1 is also a generalization of the one dimensional central limit theorem of Stute(6). Projecting to a point, Theorem 1 boils down to the central limit theorem under random censorship.

These results may be used in the nonparametric statistical inference such as in constructing confidence bands of a survival distribution or in testing where the test statistic takes a continuous functional form of the Kaplan-Meier integral process. See, for example, Gill(2).

Let X be a random variable defined on a probability space $(\Omega,\mathcal{F},P)$ with distribution function F. Consider a sequence $\{X_i : i\geq 1\}$ of independent copies of X. Define

$$S_n(t)=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}[\{X_i\leq t\}-F(t)]. \tag{1.1}$$

Weak convergence of the *non-compact* infinite time scale process such as S_n is of interest. Recall that a *cadlag* function is meant by a real-valued function defined on a subset of real line that is right continuous with left limit. One may think of S_n as *r.e.* of $D[-\infty,\infty]$, the space of *cadlag* functions on the *compact* interval of extended real line $[-\infty,\infty]$ by setting $S_n(-\infty) = S_n(\infty) = 0$. However, it sometimes more natural to think of S_n as *r.e.* of $D(-\infty,\infty)$, the space of *cadlag* functions on the *non-compact* interval of real line $(-\infty,\infty)$. See, for example, Pollard(5).

Throughout the paper, having applications in survival analysis in mind, the weak convergence of the main processes will be treated as *r.e.* of $D[0,\infty)$, the space of all real-valued *cadlag* functions on $[0,\infty)$, not on $D(-\infty,\infty)$. Events are identified with their indicator functions when there is no risk of ambiguity. So, for example, the events $\{X_i \le t\}$ in the summand of the Eq.(1.1) means the indicator functions of the events $\{X_i \le t\}$.

Consider a measurable function $\varphi : R \to R$ such that $\int \varphi^2 dF < \infty$. Since $X_1, \dots, X_n, \dots$ are *iid* random variables, we see that $\varphi(X_1), \dots, \varphi(X_n), \dots$ are also *iid* random variables. Then the process defined by

$$S_n(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} [\varphi(X_i)\{X_i \le t\} - \int \varphi(x)\{x \le t\} dF(x)] \text{ for } t \in [0,\infty) \quad (1.2)$$

is more flexible than that defined in the Eq.(1.1) in many applications. By considering $\varphi = 1$, we notice that the Eq.(1.2) reduces to the Eq.(1.1).

2. THE MAIN RESULTS

Our first goal of the present paper is to establish the weak convergence of the process S_n defined in the Eq.(1.2) to a Gaussian process as *r.e.* of $D[0,\infty)$.

Let X be a random variable defined on a probability space $(\Omega, \mathcal{F}, P)$ with distribution function F. Consider a sequence $\{X_i : i \ge 1\}$ of independent copies of X. Let $\varphi : R \to R$ be a measurable function such that $\int \varphi^2 dF < \infty$. Consider the process $\{S_n\}$ given by the Eq.(1.2). By introducing the usual empirical distribution function defined by $F_n(x) = \frac{1}{n}\sum_{i=1}^{n}\{X_i \le x\}$ for $x \in R$, we may simplify $S_n(t)$ as

$$S_n(t) = n^{1/2} \int \varphi(x)\{x \le t\} d(F_n - F)(x).$$

We are interested in the weak convergence of the *integral process* $\{S_n(t) : t \in [0,\infty)\}$ as *r.e.* of $D[0,\infty)$, to a Gaussian process whose sample paths are continuous. We use the following definition of convergence in distribution in developing the weak convergence of the processes as *r.e.* of $D[0,\infty)$. We first need to specify the metric we will use.

Definition 1. (Pollard(5)) A sequence of functions $\{x_n\}$ in $D[0,\infty)$ converges uniformly on compacta to a function x if

$$\sup_{t\le k}|x_n(t)-x(t)| \to 0 \text{ as } n\to\infty \text{ for each fixed } k.$$

Equivalently, $d(x_n,x)\to 0$, where

$$\begin{aligned} d(x_n,x) &= \sum_{k=1}^{\infty} 2^{-k}\min[1, d_k(x_n,x)] \\ d_k(x_n,x) &= \sup_{t\le k}|x_n(t)-x(t)|. \end{aligned}$$

We equip the space $D[0,\infty)$ with the metric d and the projection σ-field $\mathcal{P}$. See, for example, Pollard(5).

Definition 2. A sequence $\{Y_n\}$ of *r.e.* of $D[0,\infty)$ converges in distribution to a random element Y, denoted $Y_n \Rightarrow Y$, if

$$Eg(Y_n)\to Eg(Y) \text{ for each } g\in C(D[0,\infty))$$

where $C(D[0,\infty))$ is the set of all bounded, continuous, $\mathcal{P}$-measurable functions from $D[0,\infty)$ into R.

Write, for each $t\in[0,\infty)$,

$$\xi(t)=\varphi(X)\{X\le t\}-\int\varphi(x)\{x\le t\}dF(x).$$

Let $\{Z(t): t\in[0,\infty)\}$ be the mean zero Gaussian process with

$$Cov(Z(s),Z(t)) = Cov(\xi(s),\xi(t)). \tag{2.1}$$

We are ready to state one of the main results which can be regarded as the empirical central limit theorem for the integral process constructed from *iid* random variables.

Proposition 1. Suppose that $\int\varphi^2 dF<\infty$. Then

$$S_n\Rightarrow Z \text{ as } r.e. \text{ of } D[0,\infty).$$

The process $\{Z(t) : t \in [0, \infty)\}$ is the mean zero Gaussian with continuous sample paths and the covariance structure is given by the Eq.(2.1).

Remark 1. The result of Proposition 1, in the sense of considering the process whose index set is non-compact, extends the empirical central limit theorem of Pollard, see Theorem 5.11 of Pollard(5), which states that the empirical process $\{S_n\}$ where, as in the Eq.(1.1),

$$S_n(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} [\{X_i \le t\} - F(t)]$$

converges in distribution to a Gaussian process as *r.e.* of $D[-\infty, \infty]$. It is worth noting that the limiting Gaussian process need not have continuous sample paths.

In the proof of Proposition 1, we will make use of a result on the weak convergence of an empirical process whose index set is a totally bounded metric space. We first look at the process $\{S_n|_{[0,1]}\}$, the restriction of S_n to $[0, 1]$, that is given by

$$S_n|_{[0,1]}(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \xi_i(t) \text{ for } t \in [0, 1],$$

where, for each fixed $t \in [0, 1]$, $\{\xi_i(t)\}$ are independent copies of $\xi(t)$. Then the restricted process $\{S_n|_{[0,1]}\}$ can be regarded as *r.e.* of $D[0, 1]$. Notice that, for each fixed $t \in [0, 1]$, $\xi(t)$ has the mean zero and finite variance. We will use the following Lemma in the proof of Proposition 1.

Lemma 1. Suppose that $\int \varphi^2 dF < \infty$. Then

$$S_n|_{[0,1]} \Rightarrow Z|_{[0,1]} \text{ as } r.e. \text{ of } D[0, 1].$$

The limiting process $\{Z(t) : t \in [0, 1]\}$ is the mean zero Gaussian with continuous sample paths and the covariance structure is given by the Eq.(2.1).

Proof of Lemma 1. The result is well known because the underlying index set of the process is compact, in particular totally bounded. See, for example, Theorem 3.1 of Ossiander(4).

Proof of Proposition 1. By Lemma 1, we notice that for each fixed k, $S_n|_{[0,k]} \Rightarrow Z|_{[0,k]}$ as *r.e.* of $D[0, k]$, where the limiting process $\{Z(t)|_{[0,k]} : t \in [0, k]\}$ is the mean zero Gaussian with continuous sample paths and the covariance structure is given by the Eq.(2.1). Now the d-continuity of sample paths of the Gaussian process $\{Z(t) : t \in [0, \infty)\}$ follows from d_k-continuity of sample paths of $\{Z(t)|_{[0,k]} : t \in [0,k]\}$ for all k. Now use Theorem 5.23(p. 108) in Pollard(5), by noticing that the continuity of sample paths of the limiting random element Z, to conclude that $S_n \Rightarrow Z$ as *r.e.* of $D[0,\infty)$. The proof of Proposition 1 is completed.

Our second goal is to establish the weak convergence of the Kaplan-Meier integral process on $[0,\infty)$ which is constructed from the *incomplete data* of random censorship model. Again let $\varphi : R \to R$ be a measurable function such that $\int \varphi^2 dF < \infty$. Consider the random censorship model where one observes the incomplete data $\{Z_i, \delta_i\}$. The $\{Z_i\}$ are independent copies of Z whose distribution is H. The $\{Z_i, \delta_i\}$ are obtained by the equations $Z_i = \min(X_i, Y_i)$ and $\delta_i = \{X_i \le Y_i\}$ where the $\{Y_i\}$ are independent copies of the censoring random variable Y with distribution G which is also assumed to be independent of F, the distribution of *iid* random variables X_i of original interest in a statistical inference. Let $F\{a\} = F(a) - F(a-)$ denote the jump size of F at a and let A be the set of all atoms of H which is an empty set when H is continuous. Let $\tau_H = \inf\{x : H(x) = 1\}$ denotes the least upper bound of the support of H. Notice that the τ_H is not necessarily finite which provides one of the reason why we need the theory of weak convergence of the *infinite time scale* stochastic processes. Consider a subdistribution function $\tilde{F}$ that is defined by

$$\tilde{F}(x) = \begin{cases} F(x), & \text{if } x < \tau_H, \\ F(\tau_H-) + \{\tau_H \in A\}F\{\tau_H\}, & \text{if } x \ge \tau_H. \end{cases}$$

Consider the Kaplan-Meier integral process $\{U_n(t) : t \in [0,\infty)\}$ defined by

$$U_n(t) = n^{1/2} \int \varphi(x)\{x \le t\} d(\hat{F}_n - \tilde{F})(x) \text{ for } t \in [0,\infty), \tag{2.2}$$

where $\hat{F}_n$ is the usual Kaplan-Meier estimator constructed from the random censorship model. See, for example, Kaplan and Meier(3). The process $\{U_n(t) : t \in [0,\infty)\}$ will be the proper extension of the process given in the Proposition 1 to the random censorship model. As mentioned before our goal is to consider the weak convergence of the Kaplan-Meier integral process U_n to a Gaussian process as *r.e.* of $D[0,\infty)$ under the minimal assumptions due to Stute(6). In order to describe the assumptions, we need to consider the following subdistributions functions

$$\begin{aligned} \tilde{H}^0(z) &= P(Z \le z, \delta = 0) = \int_{-\infty}^{z} (1 - F(y))G(dy), \text{ and} \\ \tilde{H}^1(z) &= P(Z \le z, \delta = 1) = \int_{-\infty}^{z} (1 - G(y-))F(dy), \ z \in R. \end{aligned}$$

Define

$$\begin{aligned}\gamma(x) &= \exp\{\int_{-\infty}^{x-} \frac{\tilde{H}^0(dz)}{1-H(z)}\}, \text{ and} \\ C(x) &= \int_{-\infty}^{x-} \frac{G(dy)}{[1-H(y)][1-G(y)]}.\end{aligned}$$

The following two assumptions, which are trivially satisfied when there is no censoring occurred, will be imposed on Theorem 1.

$$\int \varphi^2(x)\gamma^2(x)\tilde{H}^1(dx) = \int [\varphi(Z)\gamma(Z)\delta]^2 dP < \infty, \text{ and} \tag{2.3}$$

$$\int |\varphi(x)| C^{1/2}(x)\tilde{F}(dx) < \infty. \tag{2.4}$$

Before stating Theorem 1, we need more notations:

$$\begin{aligned}\gamma_1^t(x) &= \frac{1}{1-H(x)} \int \{x < w\}\varphi_t(w)\gamma(w)\tilde{H}^1(dw), \\ \gamma_2^t(x) &= \int\int \{v < x, v < w\} \frac{\varphi_t(w)\gamma(w)}{[1-H(v)]^2} \tilde{H}^0(dv)\tilde{H}^1(dw),\end{aligned}$$

where, for notational simplicity, $\varphi(\cdot)1_{(-\infty,t]}(\cdot)$ is denoted by $\varphi_t(\cdot)$. Write, for each fixed $t \in [0,\infty)$,

$$\xi(t) = \varphi_t(Z)\gamma(Z)\delta - \int \varphi_t d\tilde{F} + \gamma_1^t(Z)(1-\delta) - \gamma_2^t(Z).$$

Let $\{W(t) : t \in [0,\infty)\}$ be the mean zero Gaussian process with

$$Cov(W(s), W(t)) = Cov(\xi(s), \xi(t)). \tag{2.5}$$

We are now ready to state Theorem 1 which can be regarded as the empirical central limit theorem for the Kaplan-Meier integral process that are constructed from incomplete data in the random censorship model.

Theorem 1. Assume that (2.3) and (2.4). Then

$$U_n \Rightarrow W \text{ as } r.e. \text{ of } D[0,\infty).$$

The process $\{W(t) : t \in [0,\infty)\}$ is the mean zero Gaussian with continuous sample paths and the covariance structure is given by the Eq.(2.5).

Remark 2. The central limit theorem for a Kaplan-Meier integral of Stute(6), which states that under the same assumptions of Theorem 1, for each fixed $t \in [0, \infty)$,

$$n^{1/2} \int \varphi_t d(\hat{F}_n - \tilde{F}) \to \ N(0, \sigma^2(t)) \text{ in distribution},$$

where $\sigma^2(t) = Var\{\varphi_t(Z)\gamma(Z)\delta + \gamma_1^t(Z)(1-\delta) - \gamma_2^t(Z)\}$, will be one dimensional version of Theorem 1. To see this, for each fixed t, apply the usual Continuous Mapping Theorem with the continuous mapping π_t defined by $\pi_t(x) = x(t)$ on Theorem 1.

In the proof of Theorem 1, we will use the following Proposition 1 of Stute(6), Proposition 1 of the empirical central limit theorem for the integral process of *iid* variables, and Lemma 2 whose proof is provided in Appendix.

Proposition 2. Assume that (2.3) and (2.4). Then for each fixed $t \in [0, \infty)$ we have

$$\int \varphi_t d\hat{F}_n = \frac{1}{n}\sum_{i=1}^{n} \varphi_t(Z_i)\gamma(Z_i)\delta_i + \frac{1}{n}\sum_{i=1}^{n} \gamma_1^t(Z_i)(1-\delta_i) - \frac{1}{n}\sum_{i=1}^{n} \gamma_2^t(Z_i) + R_n(t), \tag{2.6}$$

where $\sqrt{n}|R_n(t)| \to^P 0$.

Lemma 2. Under the assumption (2.3) and (2.4),

$$\limsup_{n\to\infty} P\{\sup_{t\in R} \sqrt{n}|R_n(t)| > \epsilon\} = 0 \text{ for every } \epsilon > 0.$$

Remark 3. The summands of the first sum in the Eq.(2.6) have expectation $\int \varphi_t d\tilde{F}$, while the summands of the second and the third sum have identical expectations. That is, by the Eq.(2.6), we have the representation,

$$n^{1/2} \int \varphi_t(w) d(\hat{F}_n - \tilde{F})(w) = n^{-1/2} \sum_{i=1}^{n} \xi_i(t) + n^{1/2} R_n(t). \tag{2.7}$$

where $\xi_i(t) = \varphi_t(Z_i)\gamma(Z_i)\delta_i - \int \varphi_t d\tilde{F} + \gamma_1^t(Z_i)(1-\delta_i) - \gamma_2^t(Z_i)$ and, for each fixed t, the $\xi(t)$'s are *iid* with the mean zero and the variance $\sigma^2(t) = Var\{\varphi_t(Z)\gamma(Z)\delta + \gamma_1^t(Z)(1-\delta) - \gamma_2^t(Z)\}$.

Proof of Theorem 1. By the Eq.(2.7), we have the representation

$$U_n(t) = n^{1/2} \int \varphi_t(w) d(\hat{F}_n - \tilde{F})(w) = V_n(t) + n^{1/2} R_n(t),$$

where $V_n(t) = n^{-1/2}\sum_{i=1}^{n}\xi_i(t)$. Apply Proposition 1 to conclude that $V_n \Rightarrow W$ as *r.e.* of $D[0,\infty)$ and the limiting process $\{W(t) : t \in [0,\infty)\}$ is the mean zero Gaussian with continuous sample paths and the covariance structure is given by the Eq.(2.5). Now use Lemma 2 to complete the proof of Theorem 1.

APPENDIX

In this appendix we make our efforts to complete the proof of Lemma 2 which gives the uniform order of convergence of the remainder $R_n(t)$, an essential fact in deriving the empirical central limit theorem for the Kaplan-Meier integral process. For the purpose we examine the representations of R_n of Stute(6). We begin with stating ingredients which are need to examine the remainder R_n.

Let H_n, $\tilde{H}_n^0$, and $\tilde{H}_n^1$ be the empirical (sub-) distribution functions of H, $\tilde{H}^0$, and $\tilde{H}^1$, respectively. In order to describe the specific form of the remainder terms $R_n(t)$ we need the following form of $\int \varphi_t d\hat{F}_n$. Lemma 2.1 of Stute(6) states that, under continuity of H,

$$\int \varphi_t d\hat{F}_n = \int \varphi_t(w)\exp\{n\int_{-\infty}^{w-}\ln[1+\frac{1}{n(1-H_n(z))}]\tilde{H}_n^0(dz)\}\tilde{H}_n^1(dw). \tag{1}$$

The exponential term in the Eq.(1) can be expended as

$$\begin{aligned}
&\exp\{\int_{-\infty}^{Z_i-}\frac{\tilde{H}^0(dz)}{1-H(z)}\}[1+n\int_{-\infty}^{Z_i-}\ln[1+\frac{1}{n(1-H_n(z))}]\tilde{H}_n^0(dz) \\
&-\int_{-\infty}^{Z_i-}\frac{\tilde{H}^0(dz)}{1-H(z)}]+\frac{1}{2}e^{\Delta_i}\{n\int_{-\infty}^{Z_i-}\ln[1+\frac{1}{n(1-H_n(z))}]\tilde{H}_n^0(dz) \\
&-\int_{-\infty}^{Z_i-}\frac{\tilde{H}^0(dz)}{1-H(z)}\}^2,
\end{aligned}$$

where Δ_i is between the two terms in brackets. Write

$$\begin{aligned}
A_{in} &:= n\int_{-\infty}^{Z_i-}\ln[1+\frac{1}{n(1-H_n(z))}]\tilde{H}_n^0(dz)-\int_{-\infty}^{Z_i-}\frac{\tilde{H}^0(dz)}{1-H(z)} \\
&:= B_{in}+C_{in},
\end{aligned}$$

with

$$B_{in} := n\int_{-\infty}^{Z_i-} \ln[1+\frac{1}{n(1-H_n(z))}]\tilde{H}_n^0(dz) - \int_{-\infty}^{Z_i-}\frac{\tilde{H}_n^0(dz)}{1-H_n(z)},$$

and

$$C_{in} := n\int_{-\infty}^{Z_i-}\frac{\tilde{H}_n^0(dz)}{1-H_n(z)} - \int_{-\infty}^{Z_i-}\frac{\tilde{H}^0(dz)}{1-H(z)}.$$

Denote for simplicity

$$\begin{aligned}
H_n(u,v,w) &= H_n(u)\tilde{H}_n^0(v)\tilde{H}_n^1(w),\\
\tilde{H}_n(u,v,w) &= H_n(u)\tilde{H}^0(v)\tilde{H}^1(w) + H(u)\tilde{H}_n^0(v)\tilde{H}^1(w)\\
&\quad + H(u)\tilde{H}^0(v)\tilde{H}_n^1(w) - 2H(u)\tilde{H}^0(v)\tilde{H}^1(w),\\
H_n(v,w) &= \tilde{H}_n^0(v)\tilde{H}_n^1(w), \text{ and}\\
\tilde{H}_n(v,w) &= \tilde{H}^0(v)\tilde{H}_n^1(w) + \tilde{H}_n^0(v)\tilde{H}^1(w) - \tilde{H}^0(v)\tilde{H}_n^1(w).
\end{aligned}$$

Then the $R_n(t)$ is given by

$$R_n(t) = S_{n1}(t) + S_{n2}(t) + R_{n1}(t) - R_{n2}(t) + 2R_{n3}(t),$$

where

$$\begin{aligned}
S_{n1}(t) &= \frac{1}{n}\sum_{i=1}^n \varphi_t(Z_i)\gamma(Z_i)\delta_i B_{in},\\
S_{n2}(t) &= \frac{1}{n}\sum_{i=1}^n \frac{1}{2}|\varphi_t(Z_i)|\delta_i e^{\Delta_i}(B_{in}+C_{in})^2,\\
R_{n1}(t) &= \int\int \varphi_t(w)\frac{\gamma(w)\{z<w\}(H_n(z)-H(z))^2}{(1-H(z))(1-H_n(z))}\tilde{H}_n^0(dz)\tilde{H}_n^1(dw),\\
R_{n2}(t) &= \int\int\int \varphi_t(w)\frac{\gamma(w)\{v<u, v<w\}}{(1-H(v))^2}H_n(du,dv,dw)\\
&\quad - \int\int\int \varphi_t(w)\frac{\gamma(w)\{v<u,v<w\}}{(1-H(v))^2}\tilde{H}_n(du,dv,dw), \text{ and}\\
R_{n3}(t) &= \int\int \varphi_t(w)\frac{\gamma(w)\{v<w\}}{1-H(v)}H_n(dv,dw)\\
&\quad - \int\int \varphi_t(w)\frac{\gamma(w)\{v<w\}}{1-H(v)}\tilde{H}_n(dv,dw).
\end{aligned}$$

Remark 4. An important observation in a derivation of the uniform order of convergence of $R_n(t)$ is that A_{in}, B_{in}, C_{in}, Δ_i, and γ do not depend on t.

Let us begin with the following temporary assumption which is originally due to Stute(6).

$$\varphi(x) = 0 \text{ for all } x > T, \text{ for some } T < \tau_H. \quad (2)$$

Remark 5. Using the assumptions (2.3) and (2.4), the temporary assumption (2) can readily be removed without any harm, as was done in Stute(6).

Remark 6. The fact that, under the assumption (2.3) and (2.4) and the temporary assumption (2),

$$\sup_{t\in R}|S_{n1}(t)|, \ \sup_{t\in R}|S_{n2}(t)|, \text{ and } \sup_{t\in R}|R_{n1}(t)|$$

are $O(\frac{\ln n}{n})$,with probability 1, is a direct consequence of Lemma (2.6), Lemma (2.5), and Lemma (2.7) of Stute(6).

The uniform order of convergence of $R_{n2}(t)$ and $R_{n3}(t)$ need more work.

Lemma 3. Under the assumption (2.3) and (2.4) and the temporary assumption (2),

$$\sqrt{n}\cdot\sup_{t\in R}|R_{n2}(t)| = o(1) \text{ with probability } 1.$$

Proof. Let T be such that $\varphi(x) = 0$ for all $x > T$, for some $T < \tau_H$. Notice that $\sup_{w\le T}\ \gamma(w) < \infty$, and $\sup_{w\le T}\sup_{v\in(-\infty,w)}\frac{1}{(1-H(v))^2} < \infty$. Putting $M := \sup_{w\le T}\gamma(w)\ \sup_{w\le T}\ \sup_{v\in(-\infty,w)}\frac{1}{(1-H(v))^2}$, we get

$$\begin{aligned}
|R_{n2}(t)| &= |\int\int\int \varphi_t(w)\frac{\gamma(w)\{v<u,v<w\}}{(1-H(v))^2}H_n(du,dv,dw) \\
&\quad -\int\int\int \varphi_t(w)\frac{\gamma(w)\{v<u,v<w\}}{(1-H(v))^2}\tilde{H}_n(du,dv,dw)| \\
&\le \int\int\int |\varphi(w)|\frac{\gamma(w)\{v<u,v<w\}}{(1-H(v))^2}|H_n(du,dv,dw) \\
&\quad -\tilde{H}_n(du,dv,dw)| \\
&= \int_{-\infty}^{T}|\varphi(w)|\gamma(w)\int_{-\infty}^{w}\frac{1}{(1-H(v))^2}\int_v^\infty |H_n(du,dv,dw) \\
&\quad -\tilde{H}_n(du,dv,dw)| \\
&\le M\cdot\int_{-\infty}^{T}|\varphi(w)|\int_{-\infty}^{w}\int_v^\infty |H_n(du,dv,dw)-\tilde{H}_n(du,dv,dw)|.
\end{aligned}$$

By noticing that $H_n(\infty, dv, dw) = H_n(dv, dw)$ and $\tilde{H}_n(\infty, dv, dw) = \tilde{H}_n(dv, dw)$, we have

$$\begin{aligned} &\int_v^\infty |H_n(du, dv, dw) - \tilde{H}_n(du, dv, dw)| \\ &\quad = |H_n(dv, dw) - \tilde{H}_n(dv, dw)| - |H_n(v, dv, dw) - \tilde{H}_n(v, dv, dw)| \\ &\quad \leq |H_n(dv, dw) - \tilde{H}_n(dv, dw)| + |H_n(v, dv, dw) - \tilde{H}_n(v, dv, dw)|. \end{aligned}$$

By noticing also that the empirical (sub-) distributions H_n, $\tilde{H}_n^0$, and $\tilde{H}_n^1$, and the (sub-) distributions H, $\tilde{H}^0$, and $\tilde{H}^1$ vanish at $-\infty$, we have

$$\int_{-\infty}^w |H_n(dv, dw) - \tilde{H}_n(dv, dw)| = |H_n(w, dw) - \tilde{H}_n(w, dw)|, \text{ and}$$
$$\int_{-\infty}^w |H_n(v, dv, dw) - \tilde{H}_n(v, dv, dw)| = |H_n(v, w, dw) - \tilde{H}_n(v, w, dw)|.$$

Now use the simple algebraic identities $b_n c_n - bc_n - b_n c + bc = (b_n - b)(c_n - c)$, and

$$\begin{aligned} &a_n b_n c_n - a_n bc - ab_n c - abc + 2abc \\ &\quad = a_n(b_n - b)(c_n - c) + (a_n - a)b(c_n - c) + (a_n - a)(b_n - b)c \end{aligned}$$

to factor $H_n(w, dw) - \tilde{H}_n(w, dw)$ and $H_n(v, w, dw) - \tilde{H}_n(v, w, dw)$ as $[\tilde{H}_n^0(w) - \tilde{H}^0(w)] \cdot [\tilde{H}_n^1(dw) - \tilde{H}^1(dw)]$, and

$$\begin{aligned} H_n(v) \cdot [\tilde{H}_n^0 - \tilde{H}^0](w) \cdot [\tilde{H}_n^1 - \tilde{H}^1](dw) + [H_n - H](v) \cdot \tilde{H}^0(w) \cdot [\tilde{H}_n^1 - \tilde{H}^1](dw) \\ + [H_n - H](v) \cdot [\tilde{H}_n^0 - \tilde{H}^0](w) \cdot \tilde{H}^1(dw), \end{aligned}$$

respectively. Therefore, we finally have $\sqrt{n} \cdot \sup_{t \in R} |R_{n2}(t)|$ is bounded by the sum of I and II where

$$\begin{aligned} I :=& \quad M \cdot \sqrt{n} \cdot \sup_{w \in R} |\tilde{H}_n^0 - \tilde{H}^0|(w) \cdot \int_{-\infty}^T |\varphi(w)||\tilde{H}_n^1 - \tilde{H}^1|(dw), \text{ and} \\ II :=& \quad M \cdot \sup_{v \in R} H_n(v) \cdot \sqrt{n} \sup_{w \in R} [\tilde{H}_n^0 - \tilde{H}^0](w) \cdot \int_{-\infty}^T |\varphi(w)||\tilde{H}_n^1 - \tilde{H}^1|(dw) \\ &+ \quad M \cdot \sqrt{n} \sup_{v \in R} [H_n - H](v) \cdot \sup_{w \in R} \tilde{H}^0(w) \cdot \int_{-\infty}^T |\varphi(w)||\tilde{H}_n^1 - \tilde{H}^1|(dw) \\ &+ \quad M \cdot \sqrt{n} \sup_{v \in R} [H_n - H](v) \cdot \sup_{w \in R} |\tilde{H}_n^0 - \tilde{H}^0|(w) \cdot \int_{-\infty}^T |\varphi(w)| \tilde{H}^1(dw). \end{aligned}$$

Claim 1. The term I, with probability 1, is $o(1)$.

Proof of Claim 1. Notice that the sequence of random variables $\sqrt{n}$ $\sup_{w \in R} |\tilde{H}_n^0 - \tilde{H}^0|(w)$ converges in distribution. See, for example, Breiman(1). Hence the sequence $\sqrt{n}\, \sup_{w \in R} |\tilde{H}_n^0 - \tilde{H}^0|(w)$ is tight(or mass preserving). See, for example, Corollary 8. 11 of Breiman(1). Notice also that $\int_{-\infty}^{T} |\varphi(w)||\tilde{H}_n^1 - \tilde{H}^1|(dw) = o(1)$ as follows from the uniform convergence of $\tilde{H}_n^1$ to $\tilde{H}^1$. The proof of Claim 1 is completed.

The same reasoning as in Claim 1 applies to conclude that the second term II, with probability 1, is $o(1)$. This completes the proof of Lemma 3.

Lemma 4. Under the assumption (2.3) and (2.4) and the temporary assumption (2),

$$\sqrt{n} \cdot \sup_{t \in R} |R_{n3}(t)| = o(1) \text{ with probability } 1.$$

Proof. Similar argument as in the last Lemma applies.

Proof of Lemma 2. The result directly follows from Remark 6, the last two Lemmas, and Remark 5.

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