The Korean Communications
in Statistics Vol. 4, No. 1, 1997
pp. 101-109

Some Tests for Variance Changes in Time Series with a Unit Root

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Abstract

For the detection of variance changes in the nonstationary time series with a unit root two types of test statistics are proposed, of which one is based on the cumulative sum of squares and the other is based on the likelihood ratio test. The properties of the cusum type test statistic are derived and the performance of two tests in small samples are compared through Monte Carlo study. It is observed that the test based on the cumulative sum of squares can detect a small change in the variance faster than the one based on the likelihood ratio.

1. Introduction

Many economic time series are known as nonstationary time series with a unit root. Since these series are observed for a long period they reflect the changes of policy and economic system and these changes often result in structural changes such as level shifts, slope changes and variance changes. If the impacts of those changes are overlooked in model building, they may cause an invalid inference and provide an inaccurate forecasting. It should be noted that the parameter constancy is a necessary condition for the accurate forecasting and the practice of policy evaluation.

In this paper we consider the variance change problems in the nonstationary time series models. Wichern, Miller and Hsu(1976) first considered the variance change problem in AR(1) models using F-type test. Based on the likelihood ratio test Ryu and Cho(1987) also proposed a detection procedure for multiple variance change points in AR(1) models. Tsay(1988) used the intervention analysis approach to detect outliers, level shifts and variance changes in ARMA models. McCullouch and Tsay(1993) suggested a Bayesian approach for mean and variance shifts in AR models. Incla'n and Tiao(1994) used the iterated cumulative sum of squares algorithm to detect multiple variance change points in a sequence of independent observations. Their approach was further extended by Kim(1996) to the ARCH

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type models.

But most of the detection procedures were developed under the stationary assumptions. If we know that there is a variance change in the nonstationary time series with a unit root we may be able to use one of the existing tests after the transformation of the series into a stationary one by taking a first difference. The use of the existing tests for stationary series is limited since it is possible to use when the series has no level or the series is to be centered.

In this paper we propose test statistics for the detection of variance changes in the nonstationary time series with a unit root and possibly a drift. It is shown that the cusum type test statsitic can be used regardless of the stationarity of the series

This paper is organized as follows. In Section 2, we propose the test statistic for the detection of variance changes based on the cumulative sum of squares(CUSUM) and based on the likelihood ratio test(LRT) when the series contains a unit root. In Section 3, the empirical distributions of the two test statistics are obtained and the performance of two tests are compared through Monte Carlo study. In final section we apply the test to the Series B of Box and Jenkins(1987).

2. Test Statistics

Consider the following AR(1) model with a drift

$$\begin{aligned} Y_t &= \delta + \phi Y_{t-1} + a_t, \quad t = 1, \ldots, k \\ Y_t &= \delta + \phi Y_{t-1} + b_t, \quad t = k+1, \ldots, T, \end{aligned} \tag{1}$$

where a_t and b_t are series of independent normal random variables with means 0 and variance σ_a^2 and σ_b^2 respectively. We consider the test of $H_0: \sigma_a^2 = \sigma_b^2$ against $H_1: \sigma_a^2 \neq \sigma_b^2$ in time series with a unit root, i.e., $\phi = 1$. Thus the null hypothesis is that there is no variance change while the alternative is that there is a variance change at $t = k+1$. The rejection of the null model is taken as an evidence of variance change.

For the detection of variance changes Wichern, Miller and Hsu(1976) and Ryu and Cho(1987) used the likelihood ratio test for a stationary time series model without a drift, i.e., $\phi < 1$ and $\delta = 0$ in model (1) and Incla'n and Tiao(1994) used the cumulative sum of squares for a sequence of independent observations, i.e., $\phi = 0$ and $\delta = 0$ in model (1). In this paper we propose two types of test statistics, one based on the likelihood ratio and the other based on the cumulative sum of squares, for the detection of variance changes in the time series with a unit root and possibly a drift, i.e., $\phi = 1$ and $\delta \neq 0$.

Assuming zero initial value $Y_0=0$ and $\phi=1$ in (1) Y_t is represented as the partial sum of errors so that $Var(Y_t)=t\sigma^2$ is changing over time under H_0. It should be noted that we are interested in the change of the error variance instead of the variance of Y_t. Under H_0 (1) can be rewritten as

$$Y_t=\delta t+Y_0+\zeta_t, \tag{2}$$

where $\zeta_t=a_1+a_2+\cdots+a_t$ since $a_t=b_t$. δ is a coefficient of regressor t when we regress Y_t on t in model (2). Without loss of generality the estimator of δ is $\hat{\delta}=\sum_{t=1}^{n} tY_t/\sum_{t=1}^{n} t^2$.

2.1 Test based on the cumulative sum of squares

Define the partial sum of $\hat{\xi}_t$ as $S_k=\sum_{t=1}^{k}\hat{\xi}_t$ where $\hat{\xi}_t=(Y_t-Y_{t-1}-\hat{\delta})^2$. Replacing Y_t by $\delta t+Y_0+\zeta_t$ we have

$$\begin{aligned} S_k=\sum_{t=1}^{k}\hat{\xi}_t &= \sum_{t=1}^{k}(\zeta_t-\zeta_{t-1}-\sum_{t=1}^{n} t\zeta_t/\sum_{t=1}^{n} t^2)^2 \\ &= \sum_{t=1}^{k}(a_t-\sum_{t=1}^{n} t\zeta_t/\sum_{t=1}^{n} t^2)^2 \\ &= \sum_{t=1}^{k} a_t^2-2(\sum_{t=1}^{k} a_t)(\sum_{t=1}^{n} t\zeta_t/\sum_{t=1}^{n} t^2)+k(\sum_{t=1}^{n} t\zeta_t/\sum_{t=1}^{n} t^2)^2 \\ &= O_p(n)+O_p(1)+O_p(n^{-1}) \end{aligned} \tag{3}$$

since $\sum a_t=O_p(\sqrt{n})$, $\sum a_t^2=O_p(n)$ and $\sum t\zeta_t=O_p(n^{5/2})$. (See Bilingsley, 1968)

Define the test statistic for variance change in the nonstationary time series as follows:

$$\text{CUSUMD}=\sqrt{n}\max_{1\le k\le n}|D_k| \tag{4}$$

where $D_k=S_k/S_n-k/n$. If we premultiply D_k by $\sum_{t=1}^{n}\hat{\xi}_t/\sqrt{n}$, we obtain

$$\begin{aligned} \left(\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\hat{\xi}_t\right)D_k &= \frac{1}{\sqrt{n}}\sum_{t=1}^{k}\hat{\xi}_t-\frac{k}{n}\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\hat{\xi}_t \\ &\xrightarrow{D} \sigma(W(r)-rW(1)) \end{aligned} \tag{5}$$

where $W(r)$ is a standard Brownian motion. Expression of limiting distribution in (5) is known as $B(r)=W(r)-rW(1)$ where $B(r)$ is a Brownian bridge. So the limiting distribution of test statistic CUSUMD defined in (4) is

$$\text{CUSUMD} \xrightarrow{D} \sup_{0\le r\le 1}|B(r)|$$

since the left-hand side of (5) is written by $\sqrt{n}(\sum \hat{\xi}_t/n)D_k=(\sum \hat{\xi}_t/\sqrt{n})D_k$. This result is the same as that of Incla'n and Tiao(1994) and Kim(1996).

Now we consider the model without drift δ. Suppose $\delta=0$ in (1), then the expession (3) becomes

$$\begin{aligned} S_k = \sum_{t=1}^{k} \hat{\xi}_t = & \sum_{t=1}^{k}(a_t - \sum_{t=1}^{n} tY_t/\sum_{t=1}^{n} t^2)^2 \\ = & \ O_p(\sqrt{n}) \end{aligned}$$

since $Y_t=\zeta_t$ under $\delta=0$. It is not too difficult to show that the limiting distribution of the $(\sum_{t=1}^{n} \xi_t/\sqrt{n})D_k$ is not changed whether $\delta=0$ or not. This implies that the proposed test statistic can be used whether the nonstationary series has nonzero drift or not.

Suppose we know that the series Y_t contains a unit root and obtain W_t by taking a first difference of Y_t, i.e., $W_t=Y_t-Y_{t-1}$. If $\delta=0$ under H_0, $\sum W_t^2=\sum a_t^2=O_p(n)$. Hence if we let $S_k=\sum W_t^2$, we obtain the same test statistics proposed by Incla'n and Tiao(1994). While if $\delta\neq 0$, $\sum W_t^2=\sum(\delta+a_t)^2=O_p(n)$ and the test statistic expressed by the function of $n^{-1}\sum W_t^2$ has a standard limiting distribution. Hence we can not use the result of Incla'n and Tiao(1994) any more. This implies that we need a priori information about the presence of the drift.

2.2 The likelihood ratio test

We modify the likelihood ratio test proposed by Ryu and Cho(1987) so that it can be used to detect the variance change in the time series with a unit root. Consider the log-likelihood function conditional on Y_1 given n observations generated by model (1).

$$\begin{aligned} \ln L(\Theta) = & -\frac{k-1}{2}\ln 2\pi\sigma_a^2 - \frac{n-k}{2}\ln 2\pi\sigma_b^2 - \frac{1}{2\sigma_a^2}\sum_{t=2}^{k}(Y_t-\phi Y_{t-1}-\delta)^2 \\ & -\frac{1}{2\sigma_b^2}\sum_{t=k+1}^{n}(Y_t-\phi Y_{t-1}-\delta)^2 \end{aligned} \qquad (6)$$

where $\Theta=(\delta,\sigma_a^2,\sigma_b^2,\phi=1)$. For fixed k, we obtain the maximum likelihood estimators of δ, σ_a^2 and σ_b^2 as follows

$$\hat{\delta}=\frac{Y_n}{n-1}, \quad \hat{\sigma}_a^{\,2}=\frac{1}{k-1}\sum_{t=2}^{k}\hat{\xi}_t \quad \text{and} \quad \hat{\sigma}_b^{\,2}=\frac{1}{n-k}\sum_{t=k+1}^{n}\hat{\xi}_t$$

where $\hat{\xi}_t=(Y_t-Y_{t-1}-\hat{\delta})^2$. Substituting $\Theta=(\hat{\delta}, \hat{\sigma}_a^{\,2}, \hat{\sigma}_b^{\,2}, \phi=1)$ into (6) we obtain $\ln L(\Theta^*)$ as a function of k only as follows

$$\ln L(\Theta^*)=-\frac{k-1}{2}\ln\ \hat{\sigma}_a^{\,2}-\frac{n-k}{2}\ln\ \hat{\sigma}_b^{\,2}-\frac{n-1}{2}\ln 2\pi-\frac{n-1}{2}.$$

We can obtain the estimator of k, $\hat{k}$, by maximizing $\ln L(\Theta^*)$ for $1<k<n$. Let $\Theta_0=(\hat{\delta}, \sigma_a^2=\sigma_b^2=\sigma^2, \phi=1)$ under H_0, then we obtain the LRT test statistic given by

$$\begin{aligned}\text{LRT} &= -2\ln\{\sup L(\Theta_0)/\sup L(\Theta)\}\\ &= (n-1)\ln\ \hat{\sigma}^2-(\hat{k}-1)\ln\ \hat{\sigma}_a^{\,2}-(n-\hat{k})\ln\ \hat{\sigma}_b^{\,2}\end{aligned}$$

where $\hat{\sigma}^2=\frac{1}{n-1}\sum_{t=2}^{n}\hat{\xi}_t$, $\quad \hat{\sigma}_a^{\,2}=\frac{1}{\hat{k}-1}\sum_{t=2}^{\hat{k}}\hat{\xi}_t$ and $\hat{\sigma}_b^{\,2}=\frac{1}{n-\hat{k}}\sum_{t=\hat{k}+1}^{n}\hat{\xi}_t$.

3. Power Comparisons and Results

Table 1 contains asymptotic percentiles of the test statistics LRT for some selected sample sizes. The result is obtained from simulation based on 10,000 replications where the ε_t are generated by the RNNOA subroutine of IMSL. Since the limiting distribution of the test statistic CUSUMD is the same as the test statistic of Incla'n and Tiao (1994), the critical values by Incla'n and Tiao(1994) will be used for the comparison. Hence we do not report those here.

Table 1. Critical Values for the LRT based on the likelihood ratio test

n	Probability of smaller value							
	0.05	0.10	0.50	0.90	0.95	0.975	0.99	0.995
100	2.072	2.536	4.997	9.168	10.913	12.538	14.702	16.649
200	2.373	2.849	5.394	9.875	11.462	13.074	15.098	16.972
300	2.558	3.053	5.668	10.083	11.724	13.335	15.402	16.942

To examine the power of CUSUMD and LRT we perform the simulation as follows. Samples of size $n=100$ are generated which are consisted of two separate regimes with different variances. We fix the the variance of the first regime $\sigma_1^2=1$ and vary the variance of the second regime σ_2^2 so that we consider the following values of the ratios of the variances, $\Delta=\sigma_2^2/\sigma_1^2$, 0.3, 0.5, 0.8, 1.0, 1.2, 1.5, and 2.0. Three locations of the change points, 25th, 50th and 75th observations, are used and three different values of drift δ=0, 3 ,7 are considered when ϕ=1. To see the performance of the proposed tests for the stationary case we also consider δ=0 when ϕ=0.9. The empirical powers of the two test statistics are obtained in Table 2 through Table 5 at the significance level 0.05 based on 10,000 replications.

Table 2. Empirical Powers of Size 0.05 Test ($\delta=0$ and $\phi=1$)

n	k/n	Test Type	$\Delta=\sigma_2^2/\sigma_1^2$ ($\sigma_1^2=1.0$)						
			0.3	0.5	0.8	1.0	1.2	1.5	2.0
100	0.25	LRT	1.0000	.9334	.1642	.0536	.0854	.3042	.8171
		CUSUMD	1.0000	.9550	.1996	.0458	.0976	.2896	.7330
	0.50	LRT	1.0000	.9741	.1913	.0532	.1097	.4996	.9660
		CUSUMD	1.0000	.9902	.2566	.0466	.2052	.7314	.9956
	0.75	LRT	1.0000	.8722	.1385	.0525	.1018	.4299	.9119
		CUSUMD	.9978	.6962	.0978	.0532	.1710	.6174	.9692

Table 3. Empirical Powers of Size 0.05 Test ($\delta=3$ and $\phi=1$)

n	k/n	Test Type	$\Delta=\sigma_2^2/\sigma_1^2$ ($\sigma_1^2=1.0$)						
			0.3	0.5	0.8	1.0	1.2	1.5	2.0
100	0.25	LRT	1.0000	.9303	.1680	.0554	.0830	.3012	.8219
		CUSUMD	1.0000	.9550	.1978	.0504	.0984	.3016	.7396
	0.50	LRT	1.0000	.9710	.1932	.0542	.1090	.4976	.9669
		CUSUMD	1.0000	.9900	.2560	.0490	.2068	.7324	.9932
	0.75	LRT	1.0000	.8687	.1409	.0552	.1030	.4239	.9104
		CUSUMD	.9976	.6900	.1112	.0474	.1702	.6176	.9674

Table 4. Empirical Powers of Size 0.05 Test ($\delta=7$ and $\phi=1$)

n	k/n	Test Type	$\Delta=\sigma_2^2/\sigma_1^2$ ($\sigma_1^2=1.0$)						
			0.3	0.5	0.8	1.0	1.2	1.5	2.0
100	0.25	LRT	1.0000	.9278	.1613	.0533	.0845	.2993	.8195
		CUSUMD	1.0000	.9568	.2078	.0500	.1100	.3048	.7272
	0.50	LRT	1.0000	.9712	.1852	.0562	.1160	.4939	.9653
		CUSUMD	1.0000	.9908	.2440	.0538	.2122	.7306	.9950
	0.75	LRT	1.0000	.8662	.1362	.0533	.0970	.4255	.9096
		CUSUMD	.9970	.6908	.1064	.0504	.1768	.6190	.9642

Table 5. Empirical Powers of Size 0.05 Test ($\delta=0$ and $\phi=.9$)

n	k/n	Test Type	$\Delta=\sigma_2^2/\sigma_1^2$ ($\sigma_1^2=1.0$)						
			0.3	0.5	0.8	1.0	1.2	1.5	2.0
100	0.25	LRT	.9998	.9270	.1677	.0513	.0903	.3113	.8399
		CUSUMD	1.0000	.9650	.1998	.0502	.1074	.3082	.7448
	0.50	LRT	1.0000	.9756	.1947	.0515	.1197	.5198	.9652
		CUSUMD	1.0000	.9908	.2730	.0508	.1984	.7314	.9930
	0.75	LRT	1.0000	.8747	.1435	.0522	.1012	.4311	.9152
		CUSUMD	.9972	.6988	.1098	.0540	.1712	.5920	.9612

From Tables 2 to 5 we observe that both tests keep the nominal levels relatively well except few cases. In general, CUSUMD performs better than LRT. Especially, when the change occurs in the middle CUSUMD outperforms LRT. When the change occurs earlier than the mid point, i.e., change point is at the 25th observation, LRT performs better than CUSUMD as the variance of the second regime gets larger. On the other hand when the change occurs later than the mid point CUSUMD performs better than LRT. Though it is not reported in the table when $\Delta \geq 3$ or $\Delta \leq 0.3$ both tests detect the variance change always. It is also observed that the performance of both tests are not much different as the values of the drift δ are changed from 0 to 7 when ϕ=1 and $\delta=0$ when $\phi=0.9$. This implies that we can use CUSUMD whether δ=0 or not in the time series with a unit root. Also CUSUMD can be used even in the stationary case as long as $\delta=0$. But we need a priori information about the stationarity of the time series if we want to use LRT.

4. Real Example and Conclusions

CUSUMD test is applied to the IBM stock prices from May 17, 1961, to November 2, 1962 in Box and Jenkins (1976). The detection procedure of Incla'n and Tiao(1994) are employed for the detection of change points. The series analyzed is the logarithm of the raw data, Figure 1. Many authors proposed test statistics for the variance change and used the IBM stock prices series to show the performance of the proposed tests, see Incla'n and Tiao(1994) among others. The performance of the detection procedures are summarized in Table 6. It should be noted that most authors applied the test to the differenced seires at lag=1. From the previous results we know that the series has a unit root with zero drift. Hence in order to employ the existing tests for the detection of variance change points, the series needs a first difference. While CUSUMD can be used without taking a difference.

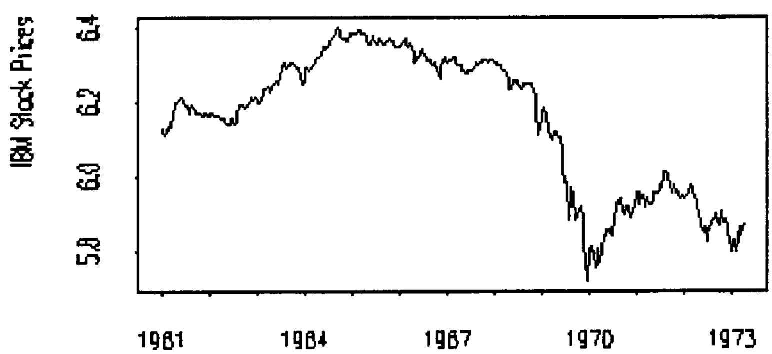

Figure 1. Time Series Plot of the IBM Stock Prices Series

Table 6. Comparison of several detetion procedures for the variance change

	Wichern et. al. (1976)	Baufays & Rasson (1985)	Ryu & Cho (1987)	Tsay (1988)	Incla'n & Tiao (1994)	Kim (1996)	Park & Cho
change points	180 235	235 280	181 235	237	235 279	235 279	235 278

The values are the location of the variance change points detected by each procedures.

It is observed that the detection performance using the cusum type tests of Incla'n and Tiao(1994), Kim(1996) and ours and using the maximum likelihood estimates of Baufays and Rasson(1985) are similar to each other while those of the other tests are different. This implies that CUSUMD is useful when the time series contains a unit root since it can be employed without taking a difference and can used even in the stationary case as long as it the series does not contain a drift. On the other hand LRT requires a priori information about the stationarity of the time series.

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