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Asymptotic Properties of Least Square Estimator of Disturbance Variance in the Linear Regression Model with MA(q)-Disturbances [1)]

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Abstract

The ordinary least squares estimator S^2 for the variance of the disturbances is considered in the linear regression model with autocorrelated disturbances. It is proved that the OLS-estimator of disturbance variance is asymptotically unbiased and weakly consistent, when the disturbances are generated by an MA(q) process. In particular, the asymptotic unbiasedness and consistency of S^2 is satisfied without any restriction on the regressor matrix.

1. Introduction

We consider the following linear regression model

$$y = X\beta + \varepsilon, \tag{1.1}$$

where y is the $n \times 1$ vector of observations on the dependent variables, X is the $n \times k$ non-stochastic regressor matrix and rank of X is $k < n$, β is the $k \times 1$ unknown parameter vector, and ε is the $n \times 1$ disturbance vector with expectation $E(\varepsilon) = 0$ and covariance matrix $E(\varepsilon\varepsilon') = \sigma_\varepsilon^2 V$, where V is assumed to be symmetric and positive definite.

In model (1.1), the generalized least squares (GLS) estimator of σ_ε^2 is

$$\widetilde{S^2} = \frac{1}{n-k}(\tilde{\varepsilon}'\tilde{\varepsilon}) = \frac{1}{n-k}(y - X\tilde{\beta})'(y - X\tilde{\beta}), \tag{1.2}$$

where $\tilde{\beta} = (X'V^{-1}X)^{-1}X'V^{-1}y$. Note that $\widetilde{S^2}$ is an unbiased and consistent estimator (Dhrymes (1978) and Fomby et al. (1984)). However, in practice the V is usually unknown,

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so that $\widehat{S^2}$ cannot be calculated. Taking the ordinary least squares (OLS) estimator of σ^2_ε instead, we get

$$S^2 = \frac{1}{n-k}(\hat{\varepsilon}'\hat{\varepsilon}) = \frac{1}{n-k}(y-X\hat{\beta})'(y-X\hat{\beta}), \tag{1.3}$$

where $\hat{\beta} = (X'X)^{-1}X'y$. Then it is well known that S^2 is in general a biased and inconsistent estimator of σ^2_ε, when $V \neq I$ (Dhrymes (1978) and Judge et al. (1985)).

The nature of this bias was investigated by many authors. For the case of the first-order autoregressive (AR(1)) disturbances, Sathe and Vinod (1974) and Neudecker (1977, 1978) tabulated the upper and lower bounds for the relative bias, $E(S^2/\sigma^2_\varepsilon)$. Kiviet and Kraemer (1992) showed that the relative bias tends to zero as autocorrelation increases whenever there is an intercept in the regression. Song (1995) evaluated the upper and lower bounds for the relative bias for the case of a first-order moving average MA(1) disturbances. However, Kraemer (1991) proved the asymptotic unbiasedness of S^2 for the AR(1) disturbance, and Song (1994) showed same results for the case of MA(1).

It has long been known (Klock (1972) or Drygas (1973)) that S^2 is a consistent estimator of the true disturbance variance under conditions which are much less restrictive than the ones needed for the consistency of $\hat{\beta}$. However, when the disturbances are correlated but still homoscedastic with general covariance $E(\varepsilon\varepsilon') = \sigma^2_\varepsilon V$, S^2 often remains consistent depending on the structure of V (Kraemer and Berghoff (1991)).

In this paper, we consider the linear regression model with an MA(q) disturbances. First, we will prove that the OLS esimator of the disturbance variance is asymptotically unbiased without any restriction on the regressor matrix X. Next, we provide the weak consistency of S^2.

2. Asymptotic Unbiasedness of S^2

Let the disturbances ε in model (1.1) be generated by an MA(q) process:

$$\begin{aligned}\varepsilon_t &= a_t - \theta_1 a_{t-1} - \cdots - \theta_q a_{t-q}, \qquad t = 1, 2, \cdots, n \\ &= \theta(B)a_t,\end{aligned} \tag{2.1}$$

where B is the backshift operator such that $Ba_t = a_{t-1}$, $\theta(B) = (1 - \theta_1 B - \theta_2 B^2 - \cdots - \theta_q B^q)$ is polynomial in B with order q and $\{a_t\}$ is a sequence of independent and identically distributed random variables with mean zero and constant variance σ^2_a. We also assume that all the zeros of $\theta(B)$ lie outside the unit circle. Thus, the $n \times n$ auto-covariance matrix is given by $E(\varepsilon\varepsilon') = \sigma^2_\varepsilon V$. Here the elements of V, v_{ij} , are given by Zinde-Walsh (1988). That is,

$$v_{ij} = \begin{cases} c_0 & \text{for } r = 0 \\ c_r & \text{for } 1 \le r \le q, \quad i, j = 1, 2, \cdots, n, \\ 0 & \text{for } r > q, \end{cases} \tag{2.2}$$

where $c_0 = \sigma^2_\varepsilon = (1 + \theta_1^2 + \cdots + \theta_q^2)\sigma^2_a$,

$$c_r = (-\theta_r + \theta_1\theta_{r+1} + \theta_2\theta_{r+2} + \cdots + \theta_{q-r}\theta_q)\sigma^2_a, \qquad r = |i-j|.$$

In what follows, we explore the asymptotic unbiasedness of S^2 in the context of the MA(q) disturbances. At first, we need the characteristic roots of V, which are obtained by the following result:

Lemma 2.1. (Horn and Johnson 1985, p. 346) Let V be $(n \times n)$-Hermitian matrix and let the characteristic roots $\lambda_t(V)$, $t = 1, 2, \cdots, n$ be arranged in decreasing order $\lambda_{\max} = \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n = \lambda_{\min}$. For each $t = 1, 2, \cdots, n$ we have

$$|\lambda_t(V)| \le \max_j \sum_i |v_{ij}|, \qquad i, j = 1, 2, \cdots, n. \tag{2.3}$$

Then, we have the following theorem.

Theorem 2.1. If the disturbances ε in model (1.1) follow an MA(q) process of (2.1), then S^2 is asymptotically unbiased for σ^2_ε .

Proof. From Watson (1955), Sathe and Vinod (1974) and Dufour (1986, 1988), we have the inequalities for the relative bias $E(S^2/\sigma^2_\varepsilon)$:

$$0 \le \begin{matrix}\text{mean of } n-k \text{ smallest} \\ \text{characteristic roots of } V\end{matrix} \le E\left(\frac{S^2}{\sigma^2_\varepsilon}\right)$$

$$\le \begin{matrix}\text{mean of } n-k \text{ greatest} \\ \text{characteristic roots of } V\end{matrix} \le \frac{n}{n-k}, \tag{2.4}$$

which implies that the upper bound for $E(S^2/\sigma^2_\varepsilon)$ tends to one as $n \to \infty$.

Now, it remains to show that the lower bound tends to one as well. By the Lemma 2.1 and mathmatical induction, the characteristic roots of V are bounded:

$$\lambda_t(V) \le \frac{(1+|\theta_1|+|\theta_2|+\cdots+|\theta_q|)^2}{1+\theta_1^2+\theta_2^2+\cdots+\theta_q^2}, \qquad t=1,2,\cdots,n, \tag{2.5}$$

where $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$ are the characteristic roots of V which depend only on the order q, not the sample size n.

Therefore, a lower bound for the mean of the $n-k$ smallest characteristic roots of V in (2.2) can be derived as follows:

$$\begin{aligned} \frac{1}{n-k}\sum_{t=1}^{n-k}\lambda_{t+k} &= \frac{1}{n-k}\left(\sum_{t=1}^{n}\lambda_t - \sum_{t=1}^{k}\lambda_t\right) = \frac{1}{n-k}\left(tr(V) - \sum_{t=1}^{k}\lambda_t\right) \\ &\ge \frac{n}{n-k} - \frac{k}{n-k}\left(\frac{(1+|\theta_1|+|\theta_2|+\cdots+|\theta_q|)^2}{1+\theta_1^2+\theta_2^2+\cdots+\theta_q^2}\right), \end{aligned} \tag{2.6}$$

from (2.5). Obviously, the first term on the right hand side in (2.6) tends to one and the second term to zero as $n \to \infty$. The proof of Theorem 2.1 is complete.

3. Consistency of S^2

When the disturbances are correlated but still homoscedastic, with general covariance $E(\varepsilon\varepsilon') = \sigma_\varepsilon^2 V$, Kraemer and Berghoff (1991) have given the following necessary condition for consistency of S^2.

Theorem 3.1. (Kraemer and Berghoff (1991)) Let the disturbances ε in model (1.1) be correlated but still homoscedastic, with general covariance $E(\varepsilon\varepsilon') = \sigma_\varepsilon^2 V$, and let $\lambda_{\max}$ be the largest characteristic root of V. Then S^2 is a weakly consistent estimator of σ_ε^2 irrespective of X if

$$\frac{1}{n}\varepsilon'\varepsilon \xrightarrow{p} \sigma_\varepsilon^2 \ \text{ and } \ \lambda_{\max} = o(n) \tag{3.1}$$

Then, we explore the consistency of S^2 in the context of the MA(q) disturbances in the following theorem.

Theorem 3.2. If the disturbances ε in model (1.1) follow an MA(q) process of (2.1) and

$E(a_t^4) = \eta < \infty$, then S^2 is a weakly consistent estimator of σ_ε^2.

Proof. It is enough to show that two conditions in Theorem 3.1 are hold.

Since $E(\frac{1}{n}\varepsilon'\varepsilon) = \frac{1}{n}E(\sum_{t=1}^{n}\varepsilon_t^2) = \frac{1}{n}E[\sum_{t=1}^{n}(a_t - \sum_{i=1}^{q}\theta_i a_{t-i})^2]$

$$= \sigma_a^2(1+\sum_{i=1}^{q}\theta_i^2) = \sigma_\varepsilon^2 \quad \text{for all } n, \quad \text{since } E(a_t, a_{t-m}) = 0 \quad \text{for } m > 0,$$

and

$$Var(\frac{1}{n}\varepsilon'\varepsilon) = E[(\frac{1}{n}\varepsilon'\varepsilon)^2] - [E(\frac{1}{n}\varepsilon'\varepsilon)]^2 = E[(\frac{1}{n}\varepsilon'\varepsilon)^2] - \sigma_\varepsilon^4$$

$$= \frac{1}{n^2}\sum_{t=1}^{n}E[(a_t - \sum_{i=1}^{q}\theta_i a_{t-i})^4]$$

$$+ \frac{2}{n^2}\sum_{t<s}\sum E[(a_t - \sum_{i=1}^{q}\theta_i a_{t-i})^2(a_s - \sum_{i=1}^{q}\theta_i a_{s-i})^2] - \sigma_\varepsilon^4 \quad . \qquad (3.2)$$

It remains to show that $E[(\frac{1}{n}\varepsilon'\varepsilon)^2] \xrightarrow{p} \sigma_\varepsilon^2$. After some calculations, the first term of the right-hand side of equation (3.2) can be reduced to the following:

$$\frac{1}{n^2}\sum_{t=1}^{n}E[(a_t - \sum_{i=1}^{q}\theta_i a_{t-i})^4] = \frac{1}{n^2}[\eta(1+\sum_{i=1}^{q}\theta_i^4) + 6\sigma_\varepsilon^4(\sum_{i=1}^{q}\theta_i^2)] \to 0 \quad , \text{for } n \to \infty,$$

since $E(a_t^4) = \eta < \infty$.

Also, the second term is

$$\frac{2}{n^2}\sum_{t<s}\sum E[(a_t - \sum_{i=1}^{q}\theta_i a_{t-i})^2(a_s - \sum_{i=1}^{q}\theta_i a_{s-i})^2]$$

$$= \frac{n(n-1)}{n^2}\sum_{t<s}\sum \sigma_a^4[(1+\sum_{i=1}^{q}\theta_i^4) + 2\sum_{i=1}^{q}\theta_i^2 + 2\sum_{t<s}\sum \theta_i^2\theta_j^2] \xrightarrow{p} \sigma_a^4(1+\sum_{i=1}^{q}\theta_i^2)^2 = \sigma_\varepsilon^4,$$

for $n \to \infty$.

Hence the first condition holds. Now, the maximum characteristic roots of V in (2.2) can be expressed as follows:

$$\lambda_{\max}(V) \le \frac{(1+|\theta_1|+|\theta_2|+\cdots+|\theta_q|)^2}{1+\theta_1^2+\theta_2^2+\cdots+\theta_q^2}, \tag{3.3}$$

which implies that the maximum characteristic root of V is $o(n)$. This completes the proof.

For the case of MA(1) process, the exact bounds for the bias of S^2 depend on the patterns of the regressor matrix X in finite observation (Song (1995)). However, as $n \to \infty$ we proved that S^2 is asymptotically unbiased and weakly consistent for σ^2_ε, regardless of the regressor matrix X, when the disturbances ε are generated by an MA(q) process.

Since the asymptotic properties of S^2 strongly depend on the structure of V, we may consider the asymptotic unbiasedness and weak consistency of S^2 in panel data when the disturbances follow an error component model with serially correlated time effects.

References

[1] Dhrymes, P.S. (1978). *Introductory Econometrics*, Springer Verlag, New York.

[2] Drygas, H. (1973). A note on a paper by T. Klock concerning the consistency of variance, *Econometrica*, Vol. 41, 175.

[3] Dufour, J.M. (1986). Bias of S^2 in linear regression with dependent errors, *The American Statistician*, Vol. 40, 284-285.

[4] Dufour, J.M. (1988). Estimator of the disturbance variance in econometric model. *Journal of Econometrics*, Vol. 37, 277-292.

[5] Fomby, T.B., Hill, R.C. and Johnson, S.R. (1984). *Advanced Econometric Methods*, Springer Verlag, New York.

[6] Horn, R.A. and Johnson, C.R. (1985). *Matrix Analysis*, Cambridge University Press, Cambridge.

[7] Judge, G.G., Griffiths, W.E., Hill, R.C., Luetkepohl, H. and Lee, T.C. (1985). *The Theory and Practice of Econometrics*, John Wiley and Sons, New York.

[8] Klock, T. (1972). Note on consistent estimation of the variance of the disturbances in the linear model. *Econometrica*, Vol. 40, 911-912.

[9] Kiviet, J.F. and Kraemer, W. (1992). Bias S^2 in the linear regression model with correlated error. *The Review of Economics and Statistics*, Vol. 74, 362-365.

[10] Kraemer, W. (1991). The asymptotic unbiasedness of S^2 in the linear regression model with AR(1)-disturbances. *Statistical Papers*, Vol. 32, 71-72.

[11] Kraemer, W. and Berghoff, S. (1991). Consistency of S^2 in the linear regression model with correlated errors. *Empirical Economics*, Vol. 16, 375-377.

[12] Neudecker, H. (1977). Bounds for the bias of the least squares estimator of σ^2 in case of a first-order autoregressive process (positive autocorrelation). *Econometrica*, Vol. 45, 1258-1262.

[13] Neudecker, H. (1978). Bounds for the bias of the LS estimator in case of a first-order (positive) autoregressive process where regression contains a constant term. *Econometrica*, Vol. 46, 1223-1226.

[14] Sathe, S.T. and Vinod, H.D. (1974). Bounds on the variance of regression coefficients due to heteroscedastic or autoregressive errors. *Econometica*, Vol 42, 333-340.

[15] Song, S. H. (1994). The asymptotic unbiasedness of S^2 in the linear regression model with moving average or particular s-th order autocorrelated disturbances. *Journal of the Korean Statistical Society*, Vol. 23, 33-38.

[16] Song, S. H. (1995). Consistency and bounds on the bias of S^2 in the linear regression model with moving average disturbances. *Journal of the Korean Statistical Society*, Vol. 24, 507-518.

[17] Watson, G.S. (1955). Serial correlation in regression analysis I. *Biometrika*, Vol. 42, 327-341.

[18] Zinde-Walsh, V. (1988). Some exact formulae for autoregressive moving average process. *Econometric Theory*, Vol. 4, 384-402.