The Korean Communications
in Statistics Vol. 4, No. 1, 1997
pp. 255-258

A Note on a Result of Yu. V. Prokhorov in General Banach Spaces

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Abstract

We prove a conjecture of Yu. V. Prokhorov in general Banach spaces ; let $\{X_n, n\geq 1\}$ be a sequence of independent identically and symmetrically distributed Banach valued random variables, then the relation $\|S_n\|/b_n \to 1$ a.s. cannot hold for any choice of constants b_n.

1. Introduction

Prokhorov [5] constructed an example of a sequence of independent identically and symmetrically distributed random vectors $X_1, X_2, \cdots$, with values in a separable Hilbert space H such that their sums $S_n = X_1+X_2+\cdots+X_n$ satisfy

$$\|S_n\|b_n^{-1} \to 1 \text{ in probability}$$

as $n\to\infty$ for some choice of constants b_n, where $\| \ \|$ is a norm on H. Prokhorov also assumed in [5] that the relation

$$\|S_n\|b_n^{-1} \to 1 \text{ a.s.} \tag{1}$$

cannot hold under the above assumptions as $n\to\infty$ for any choice of constants b_n.

Martikainen [4] proved that (1) cannot hold under the above assumptions. But it is still an open question whether relation (1) can hold in a Banach space (see [4,5]). In this paper we consider this open question in a Banach space and prove under the weaker condition than Martikainen's.

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This research is supported by Catholic University of Taegu-Hyosung.

2. Main results

Let $(B, \|\ \|)$ be a real separable Banach space. Throughout this section,let $\{X_n, n\geq 1\}$ be a sequence of pairwise independent identically distributed B-valued random variables.

We begin with following lemma whose proof, using Theorem 4.2.5 in [2], is similar to that of Lemma 1 and 2 in [1].

Lemma 1. If b_n/n is positive and non-decreasing, then

$$\sum_{n=1}^{\infty} P(\|X_1\|>b_n)=\infty \text{ implies } P(\limsup\|X_n\|/b_n=\infty)=P(\limsup\|S_n\|/b_n=\infty)=1.$$

Next lemma is crucial to the proof of the main theorem whose proof is quite elementary.

Lemma 2. If b_n is positive and non-decreasing and if, in addition, $E\|X_1\|=\infty$, then for some further subsequence $\{b_{n_k}\}$,

$$\sum_{n=1}^{\infty} P(\|X_1\|>b_n)<\infty \text{ implies } P(\lim\|S_{n_k}\|/b_{n_k}=0)=1.$$

Proof. Let $X_k'=X_k$ if $\|X_k\|\leq b_k$ and $X_k'=0$ if $\|X_k\|>b_k$. Then by Borel--Cantelli lemma, we conclude that

$$\|S_n\|=\|\sum_{k=1}^{n}X_k'\|+o(1) \text{ a.s.}$$

Hence, it suffices to prove that $\lim_{n\to\infty}\sum_{k=1}^{n}E\|X_k'\|/b_n=0$ (see [2] ex. 7. p75).

Now $b_n^{-1}\sum_{k=1}^{n}E\|X_k'\|=b_n^{-1}\sum_{k=1}^{n}\int_{|x|\leq b_k}|x|dF(x)$, where $F(x)=P(\|X_1\|\leq x)$. Clearly for any $N<n$, it is bounded by

$$nb_n^{-1}(b_N+\int_{b_N\leq|x|<b_n}|x|dF(x)). \tag{2}$$

Since $E\|X_1\|=\infty$ and $\sum_{n=1}^{\infty}P(\|X_1\|>b_n)<\infty$, $b_n n^{-1}$ cannot be bounded. Hence for fixed N the term $(n/b_n)b_N$ in (2) tends to 0 as $n\to\infty$ and the rest is bounded by

$$nb_n^{-1}\sum_{j=N+1}^{n}b_j\int_{b_{j-1}\leq|x|<b_j}dF(x)\leq\sum_{j=N+1}^{n}j\int_{b_{j-1}\leq|x|<b_j}dF(x)$$

because $nb_j/b_n\leq j$ for $j\leq n$. We may now replace the n in the righthand number above by ∞ ; as $N\to\infty$, it tends to 0 since

$$\sum_{k=2}^{\infty} k \int_{b_{k-1}\le|x|<b_k} dF(x) \le \sum_{n=1}^{\infty} P(\|X_1\|\ge b_n) < \infty.$$

Thus we prove Lemma 2.

Theorem. Suppose that $EX_1=0$ if $E\|X_1\|<\infty$, then for any sequence of constants b_n,either

$$P(\liminf \|\frac{S_n}{b_n}\|=0)=1 \text{ or } P(\limsup \|\frac{S_n}{b_n}\|=\infty)=1$$

and consequently, $P(\lim \|S_n/b_n\|=1)=0$.

Proof. Assume that the theorem fails. Then there is a sequence of constants b_n such that

$$P(\liminf \|\frac{S_n}{b_n}\|>0)>0 \text{ and } P(\limsup \|\frac{S_n}{b_n}\|<\infty)>0. \qquad (3)$$

Firstly, suppose that

$$\limsup \frac{|b_n|}{n} < \infty.$$

Then from (3), $P(\limsup \|S_n\|/n<\infty)>0$, hence by Lemma 1 for $b_n=n$, $\sum_{n=1}^{\infty} P(\|X_1\|>n) <\infty$ and hence $E\|X_1\|<\infty$ and $EX_1=0$. Thus it follows by Theorem 1 and Remark 2 of [3] that

$$P(\lim \|S_n/n\|=0)=1,$$

which contradicts the first half of (3).

Alternatively, suppose that

$$\limsup \frac{|b_n|}{n} = \infty.$$

Set

$$a_n = n \ \max\left[\frac{|b_1|}{1}, \frac{|b_2|}{2}, \cdots, \frac{|b_n|}{n}\right] \ge |b_n| \ge 0. \qquad (4)$$

Then the sequence a_n/n is positive and non-decreasing and there exists a sequence of integer $n_1<n_2<\cdots$ such that

$$a_{n_k} = |b_{n_k}|. \qquad (5)$$

From (3) and (4), it follows that

$$P(\limsup \frac{\|S_n\|}{a_n} < \infty) > 0,$$

and hence by Lemma 1 and 2,

$$P(\liminf \frac{\|S_{n_k}\|}{a_{n_k}} = 0) = 1 . \tag{6}$$

From (5) and (6), it follows that

$$P(\liminf \frac{\|S_n\|}{b_n} = 0) = 1$$

which contradicts the first half of (3) and the proof of the theorem is complete.

Hence we have the following result as the answer of a conjecture of Prokhorov.

Corollary. If $\{X_n, n \geq 1\}$ is a sequence of pairwise independent identically and symmetrically distributed B-valued random variables, then the relation (1) cannot hold for any choice of constants b_n.

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