※연구논문

Burr 고장모형에서 신뢰도와 고장률의 베이지안 추정

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Bayesian Estimation of the Reliability and Failure Rate Functions for the Burr Type-XII Failure Model

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Abstract

In this paper, we consider a hierarchical Bayes estimation of the parameter, the reliability and failure rate functions based on type-II censored samples from a Burr type-XII failure time model. The Gibbs sampler approach brings considerable conceptual and computational simplicity to the calculation of the posterior marginals and reliability. A numerical study is provided.

1. Introduction

The two parameter Burr type-XII distribution which simply written Burr(c, k) is one of the general parametric families that covers a large portion of the curve shape characteristics of types I, IV, VI in the Pearson family and many other transitional ones such as type III. Burr(1942) classified and explored twelve types of distributions. Dubey(1972, 1973) discussed the usefulness and the properties of the Burr(c, k) distribution as a lifetime model. Papadopoulos(1978) developed a Bayes estimator of k and reliability for the Burr(c, k) by using the gamma conjugate prior based on a complete sample. Lewis(1981) noted that the Weibull and exponential distributions are special limiting cases of the parameter values of the Burr(c, k) distribution. She proposed the use of the Burr(c, k) distribution as a model in accelerated life test data representing times to breakdown of an insulating fluid. Evans and Ragab(1983) obtained Bayes estimators of the two parameters and reliability of the Burr(c, k) by assuming discrete values on a finite set of points of the prior. AL-Hussaini and Jaheen(1992) developed approximate Bayes estimators of the two parameters, reliability and failure rate functions of the Burr(c, k) failure model by using the method of Lindley based on type-II censored samples. AL-Hussaini and Jaheen(1994) developed Bayes estimators of the two parameters, reliability and failure rate functions of the Burr(c, k) failure model by using the approximation method of Tierney and Kadane based on type-II censored samples.

Using Gibbs sampler approach, Dey and Lee(1992) considered Bayesian computation for the parameters and the reliability function of 2-parameter exponential distribution and considered Bayesian computation in constrained parameter and truncated data problem in multivariate life distributions. Tiwari, Yang and Zalkikar(1996) considered Bayesian estimation of the parameters and the reliability function based on type-II censored samples from a Pareto failure model.

In this paper, hierachical Bayes approach is considered for estimating the parameter, reliability and failure rate functions of Burr(c, k) distribution under type-II censoring. In Section 2, we give the model and describe the computation methods for Bayes estimation. In Section 3, we implement the Burr(c, k) failure model with an illustration from the simulated data.

2. Model Development and Gibbs Sampler

In problems such life-testing, when n items are to be tested, the ordered observations are a common occurrence. In that case, time and cost can be saved by stopping the experiment after the $r(\leq n)$ ordered observations have occurred, rather than waiting for all n failures.

We assume that the Burr(c, k) model represent the lifetimes of all item. The Burr(c, k) distribution has a probability density function(pdf) of the form

$$f(x \mid c,k) = ckx^{c-1}(1+x^{c})^{-(k+1)}, \ x>0, \ c>0, \ k>0. \tag{2.1}$$

A random sample of n items is drawn from the Burr(c, k) failure model and is put on life test. The observed sample consists of, for a preassigned r, the ordered failure times, $x_1 < x_2 < \cdots < x_r$ and $(n-r)$ survivors. We assume that the parameter c is known. Then the likelihood function and reliability of the censored sample are given, respectively, by

$$f(\underline{x} \mid c, k) = \frac{n!}{(n-r)!} c^r k^r \prod_{i=1}^{r} \left(\frac{x_i^{c-1}}{1+x_i^c} \right) \exp(-kT), \tag{2.2}$$

$$R(t) = (1+t^c)^{-k}, \tag{2.3}$$

where $\underline{x} = (x_1, \cdots, x_r)$ and $T = \sum_{i=1}^{r} \ln(1+x_i^c) + (n-r)\ln(1+x_r^c)$. The hazard rate is given by

$$h(t) = \frac{kct^{c-1}}{(1+t^c)}. \tag{2.4}$$

In our hierarchical Bayesian model, at the first stage, the prior distribution on k is gamma distribution with parameters $\alpha_1 + 1$ and β_1, where α_1 is known. The gamma density is given by

$$f(k \mid \alpha_1, \beta_1) = \frac{1}{\Gamma(\alpha_1+1)\beta_1^{\alpha_1+1}} k^{\alpha_1} \exp\left(-\frac{k}{\beta_1}\right), \quad \alpha_1 + 1 > 0, \quad \beta_1 > 0. \tag{2.5}$$

We denote the above distribution of $(k \mid \alpha_1, \beta_1)$ as Gamma$(\alpha_1 + 1, \beta_1)$. At the second stage, the distribution on hyperparameter β_1 is inverted gamma with parameter α_2 and β_2 and both parameters are known. The inverted gamma density is given by

$$f(\beta_1 \mid \alpha_2, \beta_2) = \frac{\beta_2^{\alpha_2}}{\Gamma(\alpha_2)\beta_1^{\alpha_2+1}} \exp\left(-\frac{\beta_2}{\beta_1}\right), \quad \alpha_2 > 0, \quad \beta_2 > 0. \tag{2.6}$$

We denote the above distribution as $(\beta_1 \mid \alpha_2, \beta_2) \sim IG(\alpha_2, \beta_2)$. Then the joint posterior density of k and β_1 is

$$f(k,\beta_1 \mid \underline{x}) \propto f(\underline{x} \mid k) f(k \mid \beta_1) f(\beta_1)$$
$$\propto \exp[-k(T+\frac{1}{\beta_1}) - \frac{\beta_2}{\beta_1}] \frac{1}{\beta_1^{(\alpha_1+\alpha_2+2)}} k^{r+\alpha_1}. \qquad (2.7)$$

From (2.7), the full conditional posterior pdfs are

$$f(k \mid \beta_1, \underline{x}) \propto \exp[-k(T+\frac{1}{\beta_1})] k^{r+\alpha_1},$$
$$\text{that is, } (k \mid \beta_1, \underline{x}) \sim \text{Gamma}[r+\alpha_1+1, (T+\frac{1}{\beta_1})^{-1}], \qquad (2.8)$$

and

$$f(\beta_1 \mid k, \underline{x}) \propto \exp[-\frac{1}{\beta_1}(k+\beta_2)] \frac{1}{\beta_1^{\alpha_1+\alpha_2+2}},$$
$$\text{that is, } (\beta_1 \mid k, \underline{x}) \sim IG[\alpha_1+\alpha_2+1, k+\beta_2]. \qquad (2.9)$$

Thus the conditional posterior pdf of k and β_1 are gamma pdf and inverted gamma pdf, respectively.

The Gibbs sampler is an iterative Monte Carlo integration method, developed formally by Geman and Geman(1984) in the context of image restoration. In statistical framework Tanner and Wong(1987) used essentially this algorithm in their substitution sampling approach. Gelfand and Smith(1990) developed the Gibbs sampler for fairly general parametric settings.

Now, we introduce the Gibbs sampler approach briefly.

1. We have a collection of p r.v.'s $U_1, \cdots, U_p$ whose full conditional distributions, denoted generically by $f(U_s \mid U_r, r \neq s), s=1,\cdots,p$ are available for sampling.

2. Under mild conditions(Gelfand and Smith(1990)), these full conditional distributions uniquely determine the full joint distribution $f(U_1,\cdots,U_p)$ and hence all the marginal distributions $f(U_s), s=1,\cdots,p$.

The Gibbs sampler generates from the conditional distributions as follows.

1. Given an arbitrary starting set of values $U_1^{(0)}, \cdots, U_p^{(0)}$, draw

$$U_1^{(1)} \text{ from } f(U_1 \mid U_2^{(0)}, \cdots, U_p^{(0)}),$$
$$U_2^{(1)} \text{ from } f(U_2 \mid U_1^{(1)}, U_3^{(0)}, \cdots, U_p^{(0)}),$$
$$\cdots$$
$$U_p^{(1)} \text{ from } f(U_p \mid U_1^{(1)}, \cdots, U_{p-1}^{(1)})$$

to complete one iteration of the scheme.

2. After t such iterations we arrive at a joint sample $(U_1^{(t)}, \cdots, U_p^{(t)})$ from $f(U_1, \cdots, U_p)$. Geman and Geman(1984) showed under mild conditions that

$$(U_1^{(t)}, \cdots, U_p^{(t)}) \xrightarrow{d} (U_1, \cdots, U_p) \sim f(U_1, \cdots, U_p) \text{ as } t \to \infty.$$

Hence for sufficiently large t, $U_s^{(t)}$ can be regarded as a sample from $f(U_s)$.

3. Parallel replications m times yields m i.i.d p-tuples:

$$(U_{1j}^{(t)}, \cdots, U_{pj}^{(t)}),\ j=1, \cdots, m.$$

For any function T of $U_1, \cdots, U_p$ whose expectation exists,

$$\frac{1}{m} \sum_{i=1}^{m} T(U_1^{(i)}, \cdots, U_p^{(i)}) \to E[T(U_1, \cdots, U_p)] \text{ as } m \to \infty,$$

almost surely. The distribution of $(U_1, \cdots, U_p)$ can be approximated by the empirical distribution of $(U_{1j}^{(t)}, \cdots, U_{pj}^{(t)})$, $j=1, \cdots, m$. Similarly the marginal of U_s can be approximated by the empirical distribution of $U_{sj}^{(t)}$, $j=1, \cdots, m$. When a lower dimensional marginal is required, for example $f(U_s)$, and if $f(U_s \mid U_r, r \neq s)$ can be computed, the estimate of pdf is

$$\hat{f}(U_s) = \frac{1}{m} \sum_{j=1}^{m} f(U_s \mid U_{rj}^{(t)}, r \neq s).$$

For any $T(U_1, \cdots, U_p)$, let

$$T_j^{(t)} \equiv T(U_{1j}^{(t)}, \cdots, U_{pj}^{(t)}),\ j=1, \cdots, m,$$

the empirical distribution of $T_1^{(t)}, \cdots, T_m^{(t)}$ provides an estimate of $f(T(U_1, \cdots, U_p))$.

3. Numerical Example

In this section, an illustrative example is presented by simulated data. In our simulation data, we take $n=30$, $r=15$, $c=5.0$, $k=4.2542$, $\alpha_1=6.0$, $\beta_1=$

0.5846, $\alpha_2=3.0$ and $\beta_2=0.25$ and generate the observations x_j from the Burr(c, k) failure model with c and k. For hierarchical Bayesian analysis, using Gibbs sampler, we need the marginal posterior densities which are as follows:

$$(k \mid \beta_1, \underline{x}) \sim \text{Gamma}[r+\alpha_1+1, (T+\frac{1}{\beta_1})^{-1}], \tag{3.1}$$

$$(\beta_1 \mid k, \underline{x}) \sim IG[\alpha_1+\alpha_2+1, k+\beta_2]. \tag{3.2}$$

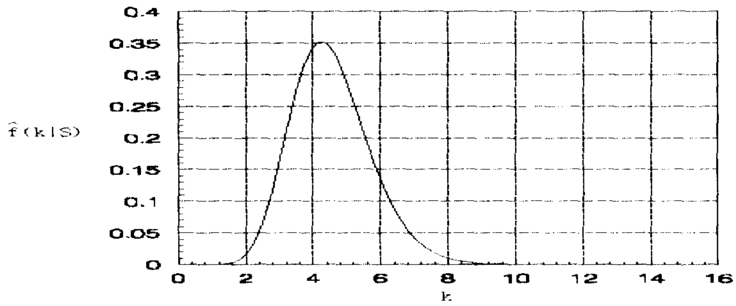
We place vague second-stage prior on β_1, letting $\alpha_2=1.0\times10^{-5}$ and $\beta_2=1.0\times10^{-5}$. In Gibbs sampler, use 100 iterations and then $m=1000$ replications for 50% censoring observations. Based on $m=1000$ replications, $\hat{k}=4.5283$. The Figures 1, 2 and 3 are graphs of $\hat{f}(k \mid S)$, $\hat{f}(h(t) \mid S)$, $\hat{f}(R(t) \mid S)$ at mission time $t=0.6$. From the Gibbs sampler, $\{R_1^{(t)}, \cdots, R_m^{(t)}\}$ is a sample from $f(R(t) \mid S)$ and $\{h_1^{(t)}, \cdots, h_m^{(t)}\}$ is a sample from $f(h(t) \mid S)$. The 90% confidence intervals are $(R_{(0.05m)}^{(t)}, R_{(0.95m)}^{(t)})$ and $(h_{(0.05m)}^{(t)}, h_{(0.95m)}^{(t)})$. $(0.05m)$ and $(0.95m)$ are $0.05m^{th}$ and $0.95m^{th}$ the order statistics, respectively. For various time t, Tables 1 and 2 give the true values, the posterior means, the residuals and 90% confidence intervals for $R(t)$ and $h(t)$ on the same censored data.

< Table 1 > Posterior Mean and 90% Confidence Interval of $R(t)$

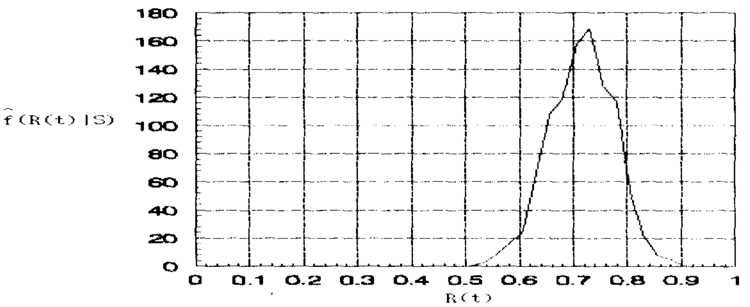
t	$R(t)$	$\hat{R}(t)$	$R(t)-\hat{R}(t)$	90% Confidence Interval
0.5	0.8773	0.8705	0.0068	(0.8208, 0.9140)
0.6	0.7272	0.7149	0.0122	(0.6185, 0.8035)
0.7	0.5164	0.5024	0.0140	(0.3691, 0.6352)
0.8	0.2995	0.2910	0.0084	(0.1623, 0.4370)
0.9	0.1389	0.1390	0.0001	(0.0509, 0.2578)
1.0	0.0524	0.0571	0.0047	(0.0117, 0.1320)

< Table 2 > Posterior Mean and 90% Confidence Interval of $h(t)$

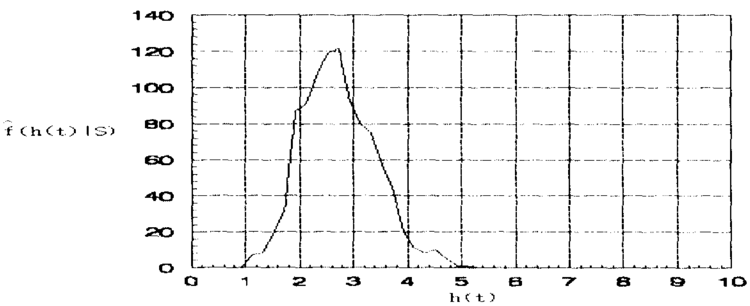
t	$h(t)$	$\hat{h}(t)$	$h(t)-\hat{h}(t)$	90% Confidence Interval
0.5	1.2891	1.3722	0.0831	(0.8852, 1.9443)
0.6	2.5578	2.7226	0.1648	(1.7563, 3.8577)
0.7	4.3723	4.6540	0.2817	(3.0022, 6.5943)
0.8	6.5622	6.9850	0.4228	(4.5059, 9.8971)
0.9	8.7745	9.3399	0.5654	(6.0250, 13.2337)
1.0	10.6354	11.3206	0.6853	(7.3027, 16.0403)



< Figure 1> Estimated pdf : $\hat{f}(k \mid S)$



< Figure 2 > Estimated pdf : $\hat{f}(R(t) \mid S)$



< Figure 3 > Estimated pdf : $\hat{f}(h(t) \mid S)$

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