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Markov Chain Method for Monitoring Several Correlated Quality Characteristics with Variable Sampling Intervals

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Abstract

Markov chain method to evaluate the properties of control charts with variable sampling intervals(VSI) for simultaneously monitoring several correlated quality characteristics under multivariate normal process are investigated. For comparing the efficiencies and properties of multivariate control charts, we consider multivariate Shewhart, CUSUM and EWMA charts in terms of average time to signal(ATS) and average number of samples to signal(ANSS). We obtained stabilized numerical results with Markov chain method when the number of transient state is greater than 100.

1. Introduction

Markov chain method and integral equation method have traditionally been used to obtain aymptotic numerical properties of the univariate CUSUM and EWMA control charts which have continuous chart statistic. Compared to Markov chain method, integral equation method usually provides great accuracy for the same computational effort. But, Markov chain method is relatively easy to use and offers more flexibility for computing some quantities that can not be easily obtained with integral equation method. In recent years, Markov chain approach has become increasingly popular.

In industrial quality control, the quality of a product is often characterized by joint levels of several correlated variables. In this case, one could obtain better sensitivity by using multivariate control chart than separate control charts for each of quality parameters. Therefore, a multivariate quality control procedure for simultaneously monitoring several correlated variables is reasonable. For example, strength and elasticity are correlated quality characteristics in plastics products. The multivariate approach to quality control was first introduced by Hotelling (1947). Alt(1982) described various types of multivariate Shewhart type T^2 charts based on Hotelling's T^2 statistic and provided recommendations for implemenation. Woodall and Ncube(1985) considered a single multivariate CUSUM procedure for controlling the means of multivarite normal process. They described how a p dimensional multivariate normal process can be monitored by using p two-sided univariate CUSUM charts.

2. Evaluating Multivariate Control Statistic

Assume that the process of interest has $p(p\geq 2)$ related quality characteristics represented by the random vector $\underline{X}=(X_1, X_2, \cdots, X_p)'$ and we obtain a sequence of random vectors $\underline{X}_1, \underline{X}_2, \cdots,$ where $\underline{X}_i=(\underline{X}'_{i1}, \cdots, \underline{X}'_{in})'$ is a sample of observations at the sampling time i and $\underline{X}_{ij}=(X_{ij1}, \cdots, X_{ijp})$. It will be assumed that the sequential observation vectors between and within samples are independent and identically distributed. Since the process quality variables have a multivariate normal distribution, the distribution of $\underline{X}$ is indexed by a set of parameters $\underline{\theta}=(\underline{\mu}, \Sigma)$ where $\underline{\mu}$ is the mean vector and Σ is the covariance matrix of $\underline{X}$. Let $\underline{\theta}_o=(\underline{\mu}_0, \Sigma_0)$ be the known target values for $\underline{\theta}$. For simplicity, we assume that $\underline{\mu}_0 = \underline{0}'$, all diagonal and off-diagonal elements of Σ_0 are 1 and 0.3, respectively.

We can obtain multivariate control statistic for monitoring $\underline{\mu}$ by using the likelihood ratio statistic for testing $H_0: \underline{\mu} = \underline{\mu}_0$ vs $H_1: \underline{\mu} \neq \underline{\mu}_0$ where Σ_0 is known. Likelihood ratio λ at the ith sample ($i=1,2,\cdots$) can be expressed as

$$\lambda = \exp\left[-\frac{n}{2}(\overline{\underline{X}_i}-\underline{\mu})'\,\Sigma_0^{-1}(\overline{\underline{X}_i}-\underline{\mu})\right],$$

where $\overline{\underline{X}_i}$ is a sample mean vector of the n observations at sampling time i.

Let Z_i^2 be $-2\ln\lambda$. Then

$$Z_i^2 = n(\overline{X}_i - \mu)' \Sigma_0^{-1} (\overline{X}_i - \mu). \quad (1)$$

Thus, the statistic Z_i^2 can be used as a control statistic for monitoring μ. Since the statistic Z_i^2 has a chi-square distribution with p degrees of freedom, the percentage point of Z_i^2 can be obtaind from a chi-square distribution. When the process has shifted to μ from the target μ_0, Z_i^2 has a non-central chi-square distribution with p degrees of freedom and noncentrality parameter $\tau^2 = n(\mu - \mu_0)' \Sigma_0^{-1}(\mu - \mu_0)$.

The null hypothesis will be rejected if $Z_i^2 > \chi^2_{1-\alpha}(p)$. Thus, the likelihood ratio test statistic Z_i^2 can be the Shewhart control statistic for μ. Hence a multivariate Shewhart chart based on Z_i^2 signals when $Z_i^2 \geq \chi^2_{1-\alpha}(p)$.

3. Variable Sampling Interval Charts

Traditional fixed sampling interval(FSI) control charts take samples from a process at fixed time interval. Recently, the application of variable sampling interval(VSI) control charts has become quite frequent. Reynolds et al.(1988) considered the properties and details of the application of VSI feature to the Shewhart $\overline{X}$-chart and showed that VSI feature substantially reduces ATS. Cui and Reynolds(1988) considered VSI Shewhart $\overline{X}$-chart with runs rules using Markov chain. Reynolds(1995) showed that the optimal VSI uses only two sampling intervals spaced as apart as possible. Saccucci et al.(1990) investigated properties of VSI EWMA charts and showed that VSI EWMA charts have similar performance to those of VSI CUSUM charts.

To evaluate the performance of a VSI control chart, it is necessary to obtain time and number of samples separately. Therefore, we use ATS and ANSS for evaluating and comparing the properties of the FSI and VSI charts. In fact, ANSS is the same definition as the average run length(ARL) in FSI chart. In FSI control chart, $t_{i+1} - t_i$, the length of sampling interval between sampling times, is constant for all $i\,(i=0,1,\cdots)$. But for a VSI chart, the sampling times are random variables and sampling interval $t_{i+1} - t_i$ is a function of chart statistic and depends on $X_1, X_2, \cdots, X_i$.

For VSI control charts, if we use a finite number of interval lengths

$d_1, d_2, \cdots, d_\eta$ where $d_1 < d_2 < \cdots < d_\eta$, these possible interval lengths must be chosen to satisfy $l_1 < d_i < l_2$. The minimum possible interval length $l_1 > 0$ might be determined by the physical considerations such as the shortest time required to take a sample. The maximum interval length l_2 might be determined by the maximum amount of time that the process engineers are willing to allow the process to run without sampling. The interval of chart statistic can be divided into in-control region C and out-of-control region C'. Then the region C can be partitioned into η regions $C_1, C_2, \cdots, C_\eta$ to apply η sampling interval VSI chart where C_i is the region in which the sampling interval d_i is used.

In multivariate Shewhart chart, the ATS and the variance of the time to signal is given as

$$E(T) = d_0 + \sum_{j=1}^{\eta} \frac{d_j p_j}{q}$$

and

$$Var(T) = \sum_{j=1}^{\eta} \frac{d_j^2 p_j}{q} + \frac{(\sum_{j=1}^{\eta} d_j p_j)^2}{q^2},$$

where d_0 is the sampling interval used before the first sample, $p_j = P(d(\underline{X}_i) = d_j)$ and $q = 1 - \sum_{j=1}^{\eta} p_j > 0$.

One disadvantage of VSI scheme is that frequent switching between different sampling intervals requires more cost and effort to evaluate properties of the process than corresponding FSI scheme. Amin and Letsinger(1991) described general procedures for combining VSI feature and examined switching behavior and runs rules for switching between different sampling intervals. They also presented that the average number of switches to signal (ANSW) of the CUSUM and EWMA procedures is far fewer than that of the Shewhart procedure.

4. Markov Chain Method

To design and evaluate performances of the control chart, it is very convenient when the distribution of control statistic is known. When the multivariate control statistic is continuous, the continuous state space of the chart statistic is partitioned into a finite number of discrete intervals and the probability distribution of the chart statistic is discretized to apply Markov chain approach. Suppose that in-control region C for the chart statistic Y_j which depends on $X_1, X_2, \cdots, X_j$ is partitioned into r states $E_1, E2, \cdots, E_r$ where each interval corresponds to a state of Markov chain and absorbing state $C' = \{\underset{\sim}{x} \mid Y_j > h\}$ is a signal region.

Let C be the continuous region and is partitioned into η regions $C_1, C_2, \cdots, C_\eta$ to apply Markov chain method with VSI and the sampling interval d_i is used when $Y_j \in C_i\ (i = 1, 2, \cdots, \eta)$. Since Y_j is continuous, let a discretized version $\tilde{Y}_j$ of $Y_j \in E_i$ be the midpoint of E_i. The probability of moving from any state i to any other state j can be denoted as $p_{ij}(k) = P(Y_{k+1} \in E_j \mid Y_k \in E_i)$ for $i, j = 1, 2, \cdots, r+1$ and $k = 0, 1, 2, \cdots$. In this paper, $p_{ij}(k)$ will be written briefly as p_{ij}. The transition probability matrix $P = [\,p_{ij}\,]$ can be partitioned as

$$P = \begin{bmatrix} Q & (I-Q)\cdot \underset{\sim}{1} \\ \underset{\sim}{0}' & 1 \end{bmatrix},$$

where Q is the $r \times r$ transition matrix corresponding to the transient state, I is the identity matrix, $\underset{\sim}{0}$ is an $r \times 1$ vector of 0's and $\underset{\sim}{1}$ is the $r \times 1$ vector of 1's. From the transition matrix P, we can obtain the fundamental matrix M as

$$M = (I - Q)^{-1} = [\,m_{ij}\,],$$

where m_{ij} is the expected number of visits to the transient state j before absorption, given that the Markov chain starts in transient state i.

Let $\underset{\sim}{b} = (b_1, b_2, \cdots, b_r)$, $\underset{\sim}{N} = (N_1, N_2, \cdots, N_r)$ and $\underset{\sim}{T} = (T_1, T_2, \cdots, T_r)$ be the vectors of sampling interval, number of samples to signal and time to signal, respectively. Then, $E(N_i) = \sum_{j=1}^{r} m_{ij}$ is ANSS when the process starts in state i. The ANSS vector is

$$E(\underset{\sim}{N}) = M\underset{\sim}{1}$$

and

$$Var(\underline{N}) = (2M-I)\cdot E(\underline{N}) - (E(\underline{N}))^{(2)},$$

where $(E(\underline{N}))^{(2)}$ is a vector whose ith component is the square of the ith component of $E(\underline{N})$. Following Reynolds(1988), the ATS vector is

$$E(\underline{T}) = M\underline{b}$$

and

$$Var(\underline{T}) = MB[2M-I]\underline{b} - (M\underline{b})^{(2)},$$

where B is a diagonal matrix with elements of corresponding sampling interval and $(M\underline{b})^{(2)}$ is a vector whose ith component is the square of the ith component of $M\underline{b}$.

Let ATS(r) be ATS calculated using r states. Lucas and Crosier(1982) showed that an approximation of the continuous state ATS with a second degree polynomial in $1/r^2$ is a good approximation. This polynomial is of the form

$$ATS(r) = asymptotic\ ATS + A/r^2 + B/r^4,$$

where A and B are the coefficients. For large r, we also obtain the ATS by taking the asymptotic ATS. The ATS(r) tends to stabilize as the number of transient states r increases. For univariate FSI CUSUM charts, several authors have concluded that a relatively small value of r is adequate for calculating the ARL for most practical purposes. For example Yashin(1985) reported that $r=10$ is sufficient for rough estimates of ARL and that $r=30$ should be satisfactory for most situations.

In the following sections, we present p_{ij}'s of transition probability matrix P when $\eta=2$ and we denote $F(\cdot)$ as the distribution function of control statistic. In this case the sampling interval d_1 is used when $Y_j \in C_1$, the sampling interval d_2 is used when $Y_j \in C_2$ and the chart signals when $Y_j \in C'$. Then, ATS when the Markov chain starts in state i is

$$ATS_i = d_2 \sum_{j=1}^{m} m_{ij} + d_1 \sum_{j=m+1}^{r} m_{ij}.$$

4.1 Multivariate CUSUM Charts

For the purpose of using Markov chain method and adding VSI feature to the CUSUM chart, Reynolds et al.(1990) proposed a modified CUSUM statistic

$$Y_{Z_i^2, i} = \max(Y_{Z_i^2, i-1}, 0) + (Z_i^2 - k), \tag{2}$$

where $Y_{Z_i^2, 0} = \omega\ (\omega \geq 0)$ and $k \geq 0$. This chart signals whenever $Y_{Z_i^2, i} \geq h$. Let the interval $(-\infty, \infty)$ be divided into in-control regions $C_1 = (-\infty, g]$, $C_2 = (g, h]$ and out-of-control region $C' = (h, \infty)$. Thus, g is the boundary between the regions specifying d_1 and d_2.

Suppose that states $1, 2, \cdots, m$ used d_2 and states $m+1, m+2, \cdots, r$ used d_1. Consider first the case $g > 0$. Then the state 1 corresponds to $Y_i \leq 0$ and $\widetilde{Y}_i = 0$. Let $w = g/(m-1)$. Thenfor $j = 2, 3, \cdots, m$ state j corresponds to $(j-2)w < Y_i \leq (j-1)w$ and $\widetilde{Y}_i = (j-3/2)w$. Let $v = (h-g)/(r-m)$. Then for $j = m+1, \cdots, r$, state j corresponds to $g + (j-m-1)v < Y_i \leq g + (j-m)v$ and $\widetilde{Y}_i = g + (j - m - \frac{1}{2})v$.

The transition probability p_{ij} for Q is as follows : For $i = 1$,

$$p_{1j} = \begin{cases} F(k) & j = 1 \\ F[(j-1)w + k] - F[(j-2)w + k] & j = 2, \cdots, m \\ F[g + (j-m)v + k] - F[g + (j-m-1)v + k] & j = m+1, \cdots, r \end{cases}$$

For $i = 2, 3, \cdots, m$,

$$p_{ij} = \begin{cases} F[-(i - \frac{3}{2})w + k] & j = 1 \\ F[(j - i + \frac{1}{2})w + k] - F[(j - i - \frac{1}{2})w + k] & j = 2, 3, \cdots m \\ F[(m - i + \frac{1}{2})w + (j-m)v + k] & \\ \quad - F[(m - i + \frac{1}{2})w + (j - m - 1)v + k] & j = m+1, \cdots, r. \end{cases}$$

< Table 1 > ATS and ANSS values for multivariate charts (p=2)

shift	Shewhart FSI	Shewhart VSI	CUSUM FSI	CUSUM VSI	EWMA FSI	EWMA VSI
no shift	200.0	200.0	200.0	200.0	200.0	200.0
τ = 0.5	115.5	107.2	85.4	70.6	93.5	77.2
τ = 1.0	41.9	31.5	22.1	13.1	26.7	17.0
τ = 1.5	15.8	8.8	8.9	5.1	10.7	7.6
τ = 2.0	6.9	3.1	4.9	3.2	5.9	5.0
τ = 2.5	3.5	1.6	3.2	2.5	3.8	3.7
τ = 3.0	2.2	1.2	2.3	2.2	2.8	2.9
τ = 3.5	1.5	1.1	1.8	2.0	2.2	2.5
			k = 2.5		λ = 0.2	

For $i=m+1, m+2, \cdots, r$,

$$p_{ij}=\begin{cases} F[-g-(i-m-\frac{1}{2})v+k] & j=1 \\ F[-(m-j)w-(i-m-\frac{1}{2})v+k] & \\ \qquad -F[-(m-j+1)w-(i-m-\frac{1}{2})v+k] & j=2,3,\cdots,m \\ F[(j-i+\frac{1}{2})v+k]-F[(j-i-\frac{1}{2})v+k] & j=m+1,\cdots,r. \end{cases}$$

For the case $g<0$, two states are needed for nonpositive values Y_i. State 1 corresponds to $Y_i \le g$ where d_2 is used, and state 2 corresponds to $g < Y_i \le 0$ where d_1 is used. Thus $m=2$. Let $w=h/(r-2)$. Then for state 1

$$p_{1j}=\begin{cases} F(g+k) & j=1 \\ F(k)-F(g+k) & j=2 \\ F[(j-2)w+k]-F[(j-3)w+k] & j=3,\cdots,r \end{cases}$$

For state 2, $p_{2j}=p_{1j}$ for $j=1,2,\cdots,r$.

For $i=3,\cdots,r$,

$$p_{ij}=\begin{cases} F[g-(i-\frac{5}{2})w+k] & j=1 \\ F[-(i-\frac{5}{2})w+k]-F[g-(i-\frac{5}{2})w+k] & j=2 \\ F[(j-i+\frac{1}{2})w+k]-F[(j-i-\frac{1}{2})w+k] & j=3,4,\cdots,r. \end{cases}$$

For the case $g=0$, only one state corresponding to $Y_i \le 0$ is needed to use d_2 and thus $m=1$. Let w be $h/(r-1)$. Then for $i=1$,

$$p_{1j}=\begin{cases} F[-(i-\frac{3}{2})w+k] & j=1 \\ F[(j-i+\frac{1}{2})w+k]-F[(j-i-\frac{1}{2})w+k] & j=2,3,\cdots,r. \end{cases}$$

For $i=2,3,\cdots,r$,

$$p_{ij}=\begin{cases} F[-(i-\frac{3}{2})w+k] & j=1 \\ F[(j-i+\frac{1}{2})w+k]-F[(j-i-\frac{1}{2})w+k] & j=2,3,\cdots,r. \end{cases}$$

4.2 Multivariate EWMA Charts

As in the multivariate CUSUM chart, designing and evaluating the performance of the EWMA chart with Markov chain method is very convenient. A multivariate EWMA chart based on the statistic Z_i^2 $(i=1,2,\cdots)$ is

$$Y_{Z_i^2,i}=(1-\lambda)Y_{Z_i^2,i-1}+\lambda Z_i^2, \tag{3}$$

where $Y_{Z_i^2,0}=\omega\ (\omega\ge 0)$ and $0<\lambda\le 1$. This chart signals whenever $Y_{Z_i^2,i}\ge h$.

Let the interval $(0,\infty)$ be divided into in-control region $C=(0,h]$ and out-of-control region $C'=(h,\infty)$. Suppose that this chart signals when $Y_j\in C'$, the sampling interval d_1 is used when $Y_j\in(g,h)$ and d_2 is used when $Y_j=(0,g]$. Assume that the interval $(0,g]$ is divided into m states and $(0,g]$ is divided into $(r-m)$ states then ω and v are g/m and $(h-g)/(r-m)$, respectively.

< Table 2 > ATS and ANSS values for multivariate charts (p=4)

	Shewhart		CUSUM		EWMA	
shift	FSI	VSI	FSI	VSI	FSI	VSI
no shift	200.0	200.0	200.0	200.0	200.0	200.0
τ = 0.5	138.1	130.5	111.2	98.6	110.9	98.4
τ = 1.0	61.0	48.7	33.8	22.1	40.6	34.9
τ = 1.5	24.6	15.2	12.3	6.8	19.6	19.8
τ = 2.0	10.6	5.0	6.3	3.7	11.8	13.5
τ = 2.5	5.2	2.2	3.9	2.7	8.1	9.8
τ = 3.0	2.9	1.4	2.8	2.3	5.9	7.6
τ = 3.5	1.9	1.1	2.1	2.1	4.6	6.0
			k = 5.0		λ = 0.1	

Then the transition probability p_{ij} is as follows : For $i = 1, 2, \cdots, m$,

$$p_{ij} = \begin{cases} F\left[(i-\frac{1}{2})w + (j-i+\frac{1}{2})w/\lambda\right] - \\ \qquad F\left[(i-\frac{1}{2})w + (j-i-\frac{1}{2})w/\lambda\right] & j=1,\cdots,m \\ F\left[(i-\frac{1}{2}) + \left\{g+(j-m)v-(i-\frac{1}{2})w\right\}/\lambda\right] \\ \qquad -F\left[(i-\frac{1}{2})w + \left\{g+(j-m-1)v-(i-\frac{1}{2})w\right\}/\lambda\right] & j=m+1,\cdots r \\ 1-F\left[(i-\frac{1}{2})w + h-(i-\frac{1}{2})w/\lambda\right] & j=r+1 \end{cases}$$

For $i = m+1, m+2, \cdots, r,$

$$p_{ij} = \begin{cases} F\left[g+(i-m-\frac{1}{2})v + \left\{jw-g-(i-m-\frac{1}{2})v\right\}/\lambda\right] \\ \qquad -F\left[g+(i-m-\frac{1}{2})v + \left\{(j-1)w-g-(i-m-\frac{1}{2})v\right\}/\lambda\right] & j=1,\cdots,m \\ F\left[g+(i-m-\frac{1}{2})v + \left\{(j-i+\frac{1}{2})v\right\}/\lambda\right] \\ \qquad -F\left[g+(i-m-\frac{1}{2})v + \left\{(j-m-\frac{1}{2})v\right\}/\lambda\right] & j=m+1,\cdots,r \\ 1-F\left[g+(i-m-\frac{1}{2})v + \left\{h-g-(i-m-\frac{1}{2})v\right\}/\lambda\right] & j=r+1 \end{cases}$$

5. Numerical Results and Concluding Remarks

In order to evaluate the performances and compare the matched FSI and VSI multivariate charts with proposed control schemes, we let that the sampling interval of unit time $d=1$ in FSI charts and two sampling intervals $d_1=0.1$ and $d_2=1.9$ for the VSI charts. In our computation, the ATS and ANSS of the chart when the process is in-control were fixed to be 200 and the sample size for each characteristic was 5 for p = 2, 4.

The parameters h and g values of Shewhart chart are obtained from the percentage points of chi-square distributions to guarantee an in-control ATS and ANSS. After the reference value and smoothing constant of the proposed multivariate charts in (2) and (3) have been determined, the parameters h and g were caluculated by Markov chain method.

The ATS values for matched FSI and VSI charts are given in <Table 1> and <Table 2>. As in the univariate charts, multivariate CUSUM and EWMA charts are more efficient than multivariate Shewhart charts in small or moderate shifts and VSI schemes are more efficient than FSI schemes.

Our various computation shows that large reference value k is efficient for large shifts and smaller reference value k is efficient for small shifts of the parameters in terms of ANSS and ATS. This result coincides with univariate CUSUM charts. And in multivariate EWMA chart we also found that smaller values of smoothing constant λ is efficient for small shift.

The asymptotic ATS and ANSS values with different number of transient states r are given in <Table 3>. When r is small, for example r=10 or r=20, theevaluated numerical values are not stabilized. In our computation, ATS(r) tends to stabilize when the number r is greater than 100.

< Table 3 > Asymptotic ATS and ANSS with different number of transient state r ($\tau^2=0.25$, $p=4$)

	CUSUM				EWMA			
	FSI	VSI	FSI	VSI	FSI	VSI	FSI	VSI
r = 10	98.1	80.5	107.8	84.8	89.4	117.2	110.4	104.3
r = 20	102.3	85.3	110.5	95.9	106.9	106.8	113.7	101.8
r = 30	102.9	86.6	111.0	97.3	109.5	102.8	114.4	101.1
r = 50	103.2	87.9	111.2	98.3	110.4	99.7	114.8	100.6
r = 100	103.3	88.5	111.2	98.6	110.9	98.4	115.0	100.3
r = 130	103.3	88.6	111.3	98.7	110.9	98.3	115.0	100.2
r = 160	103.3	88.6	111.3	98.7	111.0	98.1	115.0	100.2
	k = 4.5		k = 5.0		λ = 0.1		λ = 0.2	

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