⊠연구논문

Variable Sampling Interval Control Charts for Number of Defectives

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Abstract

Previous VSI control chart works have been done on quality variable whose distribution is normal. But there are many situations in which the assumption of normality is not appropriate. Also, in many industrial processes, the interest is to monitor the number of defectives.

In this paper, we will take the existing properties of VSI control chart developed for the normal distribution and apply them to the np-chart based on the discrete binomial distribution. We will consider the CUSUM chart for the number of defectives. Here, the interesting object is to compute the VSI ATS for CUSUM control chart using Markov chain approach and to compare FSI ATS and VSI ATS.

1. Introduction

Control charts are used to monitor a quality variable to detect shifts in the parameter of the distribution of this variable. A control chart is maintained by taking samples from a process and plotting a quality characteristic that has been measured or computed from a sample versus the sample number or time. Usually, a control chart has two other horizontal lines, called the upper control limit(UCL) and the lower control limit(LCL). But CUSUM chart has one control limit since CUSUM chart is restarted at zero if Sn is plotted below zero.

Page(1954) introduced Cumulative Sum(CUSUM) control chart. This chart directly incorporates all the information in the sequence of sample values by plotting the cumulative sums of the statistics computed from each sample. Brook and

Evans(1972) computed average run length(ARL) of a discrete one-sided CUSUM chart by using a Markov chain approach.

The usual practice in using a control chart is to take samples from the process with fixed sampling interval(FSI), say per hour. Arnold(1970) introduced a sampling procedure using variable sampling interval(VSI). Proposed VSI control chart is the logical extension of FSI control chart. The basic idea is that the time interval until the next sample should be short if some indication of a shift in the process appears in the current sample and long if there is no indication of a shift. He used a Markovian structure and developed an expected sample size needed for various sampling plans. Reynolds, Amin, Arnold, and Nachlas(1988) developed evaluation for the average time to signal(ATS) and the average number of samples to signal (ANSS) in a VSI $\overline{X}$-chart. They considered details of the application of the VSI property to the Shewhart $\overline{X}$-chart and showed that it was more efficient than the FSI $\overline{X}$-chart. Reynolds et al.(1990) considered VSI CUSUM charts. Vaughan(1993) presented a new form of policy for the design of np charts.

2. Introduction to np-chart

A np-chart plots the number of defectives in the sample to detect a shift in the target value. Thus the monitored variable is discrete. In this case, we obtain the ATS values for a FSI control chart and a VSI control chart and find which one is more efficient in detecting a shift. A control chart based on the binomial distribution is designed as the following:

$$P(x \text{ defectives in a sample of } n \text{ items}) = P(x)\ ,$$

$$P(x) = \binom{n}{x} p^x (1-p)^{n-x}\ , \quad x = 0, 1, \cdots, n\ ,$$

where x = number of defectives in the current sample,

p = fraction defectives,

p_0 = target value of p,

p_1 = value of p that should be detected quickly,

np_0 = target number of defectives per sample,

np_1 = number of defectives per sample to be detected quickly.

The np-chart consists of the target value line and the control limit(h) to signaling a shift and the signal region and the in control region. The procedure for the FSI np-chart is to take samples from the process at fixed length sampling intervals, every d time units and if the number of defectives in the current sample is greater than control limit, we determine that the process is out of control. In the variable sampling interval(VSI) np-chart, let the in control region be partitioned into $I_1, I_2, \cdots, I_\eta$ regions and use $d_1, d_2, \cdots, d_\eta$ sampling intervals depending on which region the current sample is plotted.

3. VSI CUSUM Chart for np-Chart

Cumulative Sum(CUSUM) control charts are widely used in industry for process and measurement control. This chart detailed by Lucas(1985) detects a process shift by plotting the cumulative sum of the statistic $x_i - k$. Here x_i is the observed data in the i^{th} sample and a reference value k is a function of the target value and the shift in defectives to be detected. Therefore the CUSUM control statistic is $S_n = \sum_{i=1}^{n} (x_i - k)$.

Whenever S_n is greater than the control limit h the chart signals, and whenever S_n is plotted below zero the chart restarts at zero. By resetting at zero whenever S_n goes below zero, the CUSUM chart can be considered a sequence of probability ratio tests(SPRT's). In the binomial distribution, using the critical region $0 \le \sum z_i \le a$ where $z_i = \ln \dfrac{p(x_i | p_1)}{p(x_i | p_0)}$, the SPRT reduces to

$$0 \le \sum_{i=1}^{n} \left(x_i + n \frac{\ln[(1-p_1)/(1-p_0)]}{\ln[(p_1(1-p_0))/(p_0(1-p_1))]} \right) \le \frac{a}{\ln \dfrac{p_1(1-p_0)}{p_0(1-p_1)}},$$

$$\text{where } -n \frac{\ln[(1-p_0)/(1-p_1)]}{\ln[(p_1(1-p_0))/(p_0(1-p_1))]} = k \text{ and } \frac{a}{\ln \dfrac{p_1(1-p_0)}{p_0(1-p_1)}} = h.$$

4. Average Time to Signal(ATS)

How long a VSI control chart takes to signal is based on the number of samples to signal (N) and the time interval used at each sample. The time inteval d_i depends on the region I_i in which the control statistic S_n plotted. The time to signal is defined to be the length of time required for the chart to signal and average time to signal(ATS) is defined to be the the expected value of the time to signal. Brook and Evans(1972) first used a Markov chain approach to determining the ANSS for a FSI CUSUM chart. Reynolds, Amin, and Arnold(1990) extended a Markov Chain approach to the VSI CUSUM chart. This approach will yield an exact ANSS for the discrete case such as binomial distribution and a good approximation of the ANSS for continuous case. In order to use Markov chain approach, it is designed to the upper one-sided CUSUM chart as follows :
Let

$(-\infty, 0]$ = state zero, $[h, +\infty)$ = absorbing state.

Assume that k is a rational number, that is $\frac{a}{b}$, where a and b are integers. The control region $(0, h)$ can be partitoned into $bh-1$ states such that an inteval of length is $\frac{1}{b}$. Let $H=bh$ and

statistic value	$(-\infty, 0]$	$\frac{1}{b}$	$\frac{2}{b}$	$\cdots$	$\frac{u}{b}$	$\cdots$	$\frac{v}{b}$	$\cdots$	$\frac{bh-1}{b}$	$[h, +\infty)$
"matching"										
state	zero	1	2	$\cdots$	U	$\cdots$	V	$\cdots$	$H-1$	H

Here, there are a total of $H+1$ states in the Markov Chain. The next step is to evaluate the transition matrix $P_{H+1 \times H+1}$. p_{UV} is the probability of the process change from state U to state V or the statistic x_i-k changing from $\frac{u}{b}$ to $\frac{v}{b}$. It is obtained by

$$p_{UV} = \begin{cases} P\left(x_i \le \frac{v-u+a}{b} \mid np \right) & \text{for } 0 \le u \le H, \; v=0 \\ P\left(x_i = \frac{v-u+a}{b} \mid np \right) & \text{for } 0 \le u \le H, \; 0 < v \le H-1 \\ P\left(x_i \ge \frac{v-u+a}{b} \mid np \right) & \text{for } 0 \le u \le H, \; v=H \end{cases}$$

The transition matrix P is written as

state	:	0	1	2	$\cdots$	H
statistic value	:	0	$1/b$	$2/b$	$\cdots$	h

$$\begin{matrix} 0=0 \\ 1=1/b \\ 2=2/b \\ \cdot \\ \cdot \\ \cdot \\ H=h \end{matrix} \qquad \begin{pmatrix} p_{00} & p_{01} & p_{02} & \cdots & p_{0H} \\ p_{10} & p_{11} & p_{12} & \cdots & p_{1H} \\ p_{20} & p_{21} & p_{22} & \cdots & p_{2H} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ p_{H0} & p_{H1} & p_{H2} & \cdots & p_{HH} \end{pmatrix} .$$

The fundamental matrix(FM) is obtained by

$$FM = (I_H - Q)^{-1},$$

where I_H = $H \times H$ identity matrix, Q = matrix whose column and row corresponding to state H is deleted.
If the process being monitored starts in a given state j , the ANSS is

$$\text{ANSS } (j) = \sum_{v=0}^{H-1} m_{jv} \ ,$$

where m_{jv} = elements of FM. Here, we assume that the starting state is zero. Therefore,

$$\text{ANSS}(0) = \text{ANSS} = \sum_{v=0}^{H-1} m_{0v} \ .$$

In a FSI CUSUM chart, the ATS is

$$\text{FSI ATS} = d \times (\text{ANSS}) ,$$

where d = fixed time interval.

In a VSI CUSUM chart, the ATS is

$$\text{VSI ATS} = \sum_{v=0}^{H-1} m_{0v} b_v ,$$

where b_v is the time interval used when the Markov chain is in state V. The vector $\underline{b}$ is composed of the time intervals $d_1, d_2, \cdots, d_\eta$ where each d_i can be used for more than one state.

As in comparing the standard np-chart to the VSI np-chart, this is done by matching the in control FSI ATS and the VSI ATS and comparing each ATS for given shifts. The two charts will have the same ATS at $np = np_0$ if

$$d \sum_{v=0}^{H-1} m_{uv} = \sum_{v=0}^{H-1} m_{uv} b_{uv} , \tag{1}$$

where u is the starting state and m_{uv} is an element of matrix FM and b_{uv} is time interval. Reynolds, Amin, Arnold, and Nachlas(1988) showed that the properties and results are optimal when d_1 and d_2 are used approximately the same number of times at $\mu = \mu_0$ for the VSI $\overline{X}$-chart. This extends to the VSI CUSUM chart by partitioning the in control ANSS into two regions, that is, $(-\infty, G]$ and $(G, H-1]$, where the expected number of samples in $(-\infty, G]$ is approximately equal to that of in $(G, H-1]$. This can be applied to the VSI CUSUM chart for number of defectives. If a statistic S_n is plotted in state G or less, use the time interval d_2, and if S_n is plotted in a state greater than G, use the time interval d_1. The VSI ATS can be computed as

$$\text{VSI ATS} = d_2 \sum_{v=0}^{G} m_{uv} + d_1 \sum_{v=G+1}^{H-1} m_{uv} .$$

Reynolds and Arnold(1989) and Reynolds(1989) showed that the optimal VSI chart uses only the shortest possible interval and longest possible interval from a range of possible sample intervals.

If $d_1 = d \times 0.1$ then d_2 is derived by the equation (1). d_2 is generally expressed as

$$d_2 = \frac{d \times \text{ANSS}(u) - d_1 \sum_{v=G+1}^{H-1} m_{uv}}{\sum_{v=0}^{G} m_{uv}} .$$

In this paper, since starting state is assumed state zero it becomes

$$d_2 = \frac{d \times \text{ANSS} - d_1 \sum_{v=G+1}^{H-1} m_{0v}}{\sum_{v=0}^{G} m_{0v}} .$$

Example <Compute the VSI ATS for CUSUM chart>

- $n = 20$, $p_0 = 0.05$: in control .
- $k = \frac{1}{2} \left(= \frac{a}{b}\right)$.
- $h = 2$.

Find the FSI ATS and VSI ATS for CUSUM chart.

Generating the transition matrix P

$$P = \begin{pmatrix} P(x_i \le \frac{1}{2}) & P(x_i = \frac{2}{2}) & P(x_i = \frac{3}{2}) & P(x_i = \frac{4}{2}) & P(x_i \ge \frac{5}{2}) \\ P(x_i \le 0) & P(x_i = \frac{1}{2}) & P(x_i = \frac{2}{2}) & P(x_i = \frac{3}{2}) & P(x_i \ge \frac{4}{2}) \\ P(x_i \le \frac{-1}{2}) & P(x_i = 0) & P(x_i = \frac{1}{2}) & P(x_i = \frac{2}{2}) & P(x_i \ge \frac{3}{2}) \\ P(x_i \le \frac{-2}{2}) & P(x_i = \frac{-1}{2}) & P(x_i = 0) & P(x_i = \frac{1}{2}) & P(x_i \ge \frac{2}{2}) \\ P(x_i \le \frac{-3}{2}) & P(x_i = \frac{-2}{2}) & P(x_i = \frac{-1}{2}) & P(x_i = 0) & P(x_i \ge \frac{1}{2}) \end{pmatrix} .$$

Using the binomial distribution,

$$P = \begin{pmatrix} 0.3585 & 0.3773 & 0 & 0.1887 & 0.0755 \\ 0.3585 & 0 & 0.3773 & 0 & 0.2642 \\ 0 & 0.3585 & 0 & 0.3773 & 0.2642 \\ 0 & 0 & 0.3585 & 0 & 0.6415 \\ 0 & 0 & 0 & 0.3585 & 0.6415 \end{pmatrix},$$

$$Q = \begin{pmatrix} 0.3585 & 0.3773 & 0 & 0.1887 \\ 0.3585 & 0 & 0.3773 & 0 \\ 0 & 0.3585 & 0 & 0.3773 \\ 0 & 0 & 0.3585 & 0 \end{pmatrix},$$

and

$$FM = (I_H - Q)^{-1} = \begin{pmatrix} 2.1312 & 1.0240 & 0.6135 & 0.6337 \\ 0.9056 & 1.6205 & 0.7780 & 0.4645 \\ 0.3754 & 0.6719 & 1.4788 & 0.6290 \\ 0.1347 & 0.2407 & 0.5302 & 1.2255 \end{pmatrix}.$$

The in control ANSS is

$$\text{ANSS}(0) = 2.1312 + 1.0240 + 0.6135 + 0.6337 = 4.4024 \text{ samples.}$$

Thus,

$$\text{FSI ATS} = 4.4024 \times d\ .$$

$$\text{VSI ATS} = [\,2.1312 \quad 1.0240 \quad 0.6135 \quad 0.6337\,] \begin{bmatrix} d_2 \\ d_1 \\ d_1 \\ d_1 \end{bmatrix}, \text{ given } G = \text{state zero.}$$

5. Numerical Results and Conclusions

In this paper, we applied the variable sampling interval(VSI) scheme to CUSUM charts for the number of defectives based on the discrete binomial distribution. The FSI ATS and VSI ATS for CUSUM chart were evaluated by using a Markov chain approach.

Here, we made in control ATS to be about 235 to compare results. As the reference value k is changed, we evaluated the FSI and VSI ATS. The values of k are usually taken as the larger values than np_0. The ATS, h and G values of the CUSUM chart with several k values are in the Tables 1-3 and <Figure 1>.

The Tables 1-3 and <Figure 1> show that VSI ATS values are smaller than those of FSI ATS values. So, VSI CUSUM control charts are more efficient to detecting shifts of a process than corresponding FSI CUSUM control charts in terms of ATS values.

< Table 1 > ATS values of FSI and VSI CUSUM charts for number of defectives ($n=20$, $p_0=0.02$) when $k=0.45$

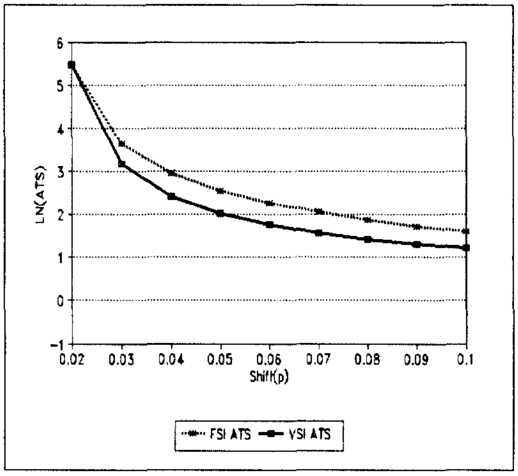
* $h=6.5$, $d=1.0$, $d_1=0.1$

shift (p)	$G=51$ $d_2=1.285$	
	FSI ATS	VSI ATS
0.02	236.29	236.29
0.03	37.95	27.99
0.04	18.92	12.93
0.05	12.65	8.49
0.06	9.57	6.41
0.07	7.74	5.21
0.08	6.51	4.44
0.09	5.63	3.90
0.10	4.96	3.50
0.11	4.45	3.20
0.12	4.04	2.96
0.13	3.71	2.76
0.14	3.43	2.59
0.15	3.20	2.45
0.16	3.00	2.32
0.17	2.83	2.21
0.18	2.68	2.12
0.19	2.55	2.03
0.20	2.43	1.95

< Table 2 > ATS values of FSI and VSI CUSUM charts for number of defectives ($n=20$, $p_0=0.02$) when $k=0.6$

$* h=4.0$, $d=1.0$, $d_1=0.1$

shift (p)	$G=15$ $d_2=1.139$	
	FSI ATS	VSI ATS
0.02	232.71	232.71
0.03	39.16	32.10
0.04	16.61	11.93
0.05	10.18	6.92
0.06	7.36	4.91
0.07	5.80	3.87
0.08	4.82	3.24
0.09	4.15	2.82
0.10	3.66	2.52
0.11	3.29	2.30
0.12	2.99	2.12
0.13	2.75	1.98
0.14	2.55	1.86
0.15	2.38	1.76
0.16	2.24	1.68
0.17	2.11	1.61
0.18	1.99	1.54
0.19	1.89	1.49
0.20	1.80	1.44



< Figure 1 > FSI and VSI ATS's for CUSUM Chart ($n=20$, $p_0=0.02$, $k=0.45$)

< Table 3 > ATS values of FSI and VSI CUSUM charts for number of defectives ($n=20$, $p_0=0.02$) when $k=0.8$

$* h=2.7$, $d=1.0$, $d_1=0.1$

shift (p)	$G=9$ $d_2=1.104$	
	FSI ATS	VSI ATS
0.02	234.81	234.81
0.03	46.45	40.96
0.04	18.46	14.41
0.05	10.43	7.47
0.06	7.11	4.84
0.07	5.40	3.57
0.08	4.37	2.86
0.09	3.69	2.41
0.10	3.21	2.10
0.11	2.85	1.88
0.12	2.57	1.72
0.13	2.34	1.60
0.14	2.15	1.51
0.15	2.00	1.43
0.16	1.86	1.37
0.17	1.75	1.33
0.18	1.65	1.29
0.19	1.57	1.26
0.20	1.49	1.23

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