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A Note on Derivations in prime rings*

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Abstract

Derivation은 Lie group, Lie ring 그리고 Lie Algebra에서 정의되어 사용되며 발전하였으며 ring에서 일반화 되었다. 역시 prime ring에서 연구되어지는 derivation의 성질들은 prime near-ring에서 일반화 시키려고 하고 있다. 1957년 E. Posner는 prime ring에서 두개의 derivation의 곱(함수합성)이 derivation이면 이들중 하나의 derivation이 0임을 밝혔다. 본 논문에서는 prime ring에서 derivation이 연구된 역사적인 배경을 소개하고 몇가지 성질을 찾는다. 즉, D. F를 prime ring R의 derivation들이라 할 때 정수 $n\geq 1$에 대하여 $DF^n=0$이면 D=0이거나 또는 $F^{3n-1}=0$이고, $D^nF=0$이면 $D^{9n-7}=0$ 이거나 또는 $F^2=0$ 이다.

0. Introduction and Historical Background of Derivations

Let R be a ring. An additive map D : R -> R is said to be derivation of R if D(xy) = D(x)y + xD(y) for all x, y ∈ R. For example, if R =ℝ[x] where ℝ is the set of real numbers, then the ordinary derivative d/dx of elementary calculus is a derivation. Our next example of a derivation is interesting only when R is noncommutative. Fix an element a∈R and define the map $D=D_a : R\rightarrow R$ by $D(x)=ax-xa$ for x∈R. Then D is a derivation and it is called the inner derivation. The set Der(R) of all derivations of R is closed under addition. In general, however Der(R) is not closed under multiplication(function composition). If D,F∈Der(R), then the map [D,F] = DF-FD is also a derivation. Thus Der(R) becomes a Lie ring.

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Suppose F is a field and R is a ring that happens to be an F-algebra. A derivation D $\in$Der(R) is said to be an F-derivation if D is an F-linear operator on R. For example, the calculus derivative d/dx is an $\mathbb{R}$-derivation on $\mathbb{R}$[x]. Let R=F[x]. Then there exists an unique F-derivation D of R with D(x) = 1 and thus we have $D(\sum_{i=0}^{n} a_i X^i) = \sum_{i=0}^{n} i a_i X^{i-1}$ where $a_i \in F$ for i= 0, 1, ...n. In this case, D is called formal differentiation.

Let R be a prime ring(for some a, b$\in$R, aR_b = 0 implies either a=0 or b=0). In 1979, I.N. Herstein proved that if D is an inner derivation of R satisfying $D(x)^n = 0$ for all x $\in$R and n a fixed integer, then D=0[12]. This result was extended to arbitrary derivations in semiprime ring[8]. In 1983 B. Felzenszwalb and C. Lanski considered derivations satisfying $D(x)^{n(x)} = 0$ for all x$\in$ I, an ideal of R and proved that D(I)=0 when R has no nipotent right ideal[7]. Using the general approach in [7], in 1985, L. Carin and A. Giambruno study the situation when $D(x)^{n(x)} = 0$ for all x$\in$V, a Lie ideal of R[2]. They prove first that D(v) = 0 when $charR \neq 2$ and R contains no nilpotent right ideal, and then obtain the same conclusion when n(x) = n is fixed and R is a 2 -tortion free semiprime ring. In 1990, C. Lanski proved that if R contains no nilpotent right ideal, V is a non commutative Lie ideal of R and D is nilpotent derivation on V, then D=0[17]. In 1992, he considered derivations which are nilpotent on certain subsets of prime rings, Lie ideals, one-sided ideasls and when a given ring has an involution on the sets of symmetric or skew symmetric elements. Combining the results and ideals of V.K. Kharchenko(1978)[15] and W.S. Martindale and C.R. Miers(1983) in [20] he proved that (under some necessary assumption an charateristic typical for derivations) derivations become inner when extended to the Martindale quotient ring and are adjoint to nilpotent elements. Also, in 1984, L.o. Chung and J. Luh proved that if I is its two-sided ideal and D is a derivations of R such that $D^n(I) = 0$, then $D^n(R) = 0$[5]. In 1994, C. Lanski proved that D is a nonzero derivation of R, L a nonzero left ideal of R, and $n \geq 1$is a fixed integer so that D(x)=0 for all x$\in$L, then LD(L)=0[16].

On the other hand, in 1957, E. Posner proved that if the product of two derivations of R with characteristics not 2, is a derivation, then one of these two derivation must be zero[21]. In particular, if D is a derivation of R and $D^2 = 0$ then $D = 0$. In 1983, L.O. Chung and J. Luh proved that if D is a derivation of 2-tortion free semiprime ring and $D^{2n} = 0$ where n is a positive integer, then $D^{2n-1} = 0$([4]). Recently Chung and Luh proved that if D is a derivation of R with charateristics not 2 and $D^{2n} = 0$ then

$D^{2n-1}=0$, i.e., the index of a nilpotent derivation of a prime ring is necessarily odd. Also, in 1996, Y. Ye and J. Luh proved that if D^{2n} is a derivation of R then $D^{2n-1}=0$[22].

In this paper, we proove that if for a prime ring with no charateristic restriction, D and F are derivations of R and $DF^n=0$ for an integer n>1, then either D=0 and $F^{3n-1}=0$ and if $D^mF=0$ for an integer m>1, then either $D^{9m-7}=0$ or $F^2=0$.

2. Properties of derivations

Lemma 1. Let D be a dervation of a prime ring R and let α be an element of R. If D(x)a=0 for all $x \in$ R then either D=0 or a=0.

proof : For all $x,y \in$ R, we have

$$0 = D(xy)\alpha = D(x)y\alpha + xD(y)\alpha = D(x)y\alpha$$

that is, D(x)Rα=0. Hence since R is a prime ring, either D=0 or α=0.

Lemma 2. Let D and F be derivations of a prime ring R. If DF=0, then either D=0 or F^2=0.

proof: For all x, $y \in$ R we have

$$0 = DF(xy) = D(x)F(y) + F(x)D(y)$$

Replace y by F(y). we get

$$D(x)F^2(y) = 0$$

Hence by Lemmer 1, either D=0 or F^2=0

Theorem 3. Let D and F be derivations of a prime ring R. If for an integer n>1, DF^n=0 then either D=0 or $F^{3n-1} = 0$.

proof. Since for all $x,y \in$ R $DF^n(xy) = 0$ Using the Leibniz' rule we have

$$\sum_{i=0}^{n}\binom{n}{i}\{DF^{n-i}(x)F^i(y) + F^{n-i}(x)DF^i(y)\} = 0 \tag{1}$$

Replace x by $F^{n-1}(x)$ and y by $F^n(y)$ in (1).

We get

$$D^{n-1}(x)\ F^{2n}(y) = 0 \tag{2}$$

Replace x by $F^{n-2}(x)$ and y by $F^{n+1}(y)$ in (1) and using (2). We obtain

$$DF^{n-2}(x)\ F^{2n+1}(y) = 0 \tag{3}$$

Continuing this process we have

$$D(x)\ F^{3n-1}(y) = 0$$

That is, for all $x,y \in R$ $D(x)F^{3n-1}(y) = 0$

Hence by Lemma 1, either D=0 or $F^{3n-1}=0$

Theorem 4. Let D and F be derivations of a prime ring R. If for an integer m>1 $D^mF = 0$ then $D^{9m-7} = 0$ or $F^2 = 0$

proof. Since [D, F] = DF - FD is a derivation, [D, [D,F]] = $D^2F - 2DFD + FD^2$ is a derivation and also $D^3F - 3D^2FD + 3DFD^2 - FD^3$ is a derivation.

Continuing,

$$\sum_{i=0}^{2m-1}\binom{2m-1}{i}(-1)^i D^{2m-1-i}FD^i \quad \text{is a derivation.}$$

Using the fact that $D^mF = 0$, $\left\{\sum_{i=0}^{2m-1}\binom{2m-1}{i}(-1)^i D^{2m-1-i}FD^i\right\}F = 0$

Applying Lemma 2, we have that either $F^2 = 0$ or

$$\sum_{i=m}^{2m-1}\binom{2m-1}{i}(-1)^i D^{2m-1-i}FD^i = 0 \qquad (4)$$

If $F^2 \neq 0$ then premultipling (4) by D^{m-1}, we get

$$D^{m-1}\ FD^{2m-1} = 0 \qquad (5)$$

and premultipling (4) by D^{m-2}, we obtain

$(2m-1)D^{m-1}FD^{2m-2} - D^{m-2}FD^{2m-1} = 0$

and

$((2m-1)D^{m-1}FD^{2m-2} - D^{m-2}FD^{2m-1})D = 0.$

Using (5) we get

$$D^{n-2}\ FD^{2n} = 0 \qquad (6)$$

Premultipling (4) by D^{m-3}, we have

$$-\binom{2m-1}{2m-3}D^{m-3}FD^{2m-1} + (2m-1)D^{m-2}FD^{2m-2} - D^{m-1}FD^{2m-3} = 0,$$

and

$$\left\{-\binom{2m-1}{2m-3}D^{m-3}FD^{2m-1} + (2m-1)D^{m-2}FD^{2m-2} - D^{m-1}FD^{2m-3}\right\}D^2 = 0.$$

Using (5) and (6), $D^{m-3}FD^{2m+1} = 0$

Continuing, we have $FD^{3m-2} = 0$

Hence, since $F \neq 0$, by Theorem 3, $D^{9m-7} = 0$.

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