A REMARK ON HALF-FACTORIAL DOMAINS

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Abstract. An atomic integral domain \( R \) is a half-factorial domain (HFD) if whenever \( x_1 \cdots x_m = y_1 \cdots y_n \) with each \( x_i, y_j \in R \) irreducible, then \( m = n \). In this paper, we show that if \( R[X] \) is an HFD, then \( Cl_t(R) \cong Cl_t(R[X]) \), and if \( G_1 \) and \( G_2 \) are torsion abelian groups, then there exists a Dedekind HFD \( R \) such that \( Cl(R) = G_1 \oplus G_2 \).

1. Introduction

Let \( R \) be an integral domain and \( R^* = R - \{0\} \). A nonunit \( r \in R^* \) is said to be irreducible if whenever \( r = ab \), \( a, b \in R \), either \( a \) or \( b \) is a unit of \( R \). An integral domain \( R \) is atomic if every nonzero nonunit of \( R \) can be factored as a product of irreducible elements of \( R \). Following Zaks [13], we say that an atomic integral domain \( R \) is a half-factorial domain (HFD) if whenever \( x_1 \cdots x_m = y_1 \cdots y_n \) with each \( x_i, y_j \in R \) irreducible, then \( m = n \). It is well known that any Krull domain \( R \) with divisor class group \( Cl(R) = \mathbb{Z}_2 \) is a HFD, but not a UFD. It is classical that a ring of integers \( R \) of a number field is a UFD if and only if \( Cl(R) = \{0\} \). The first arithmetic description of rings of integers with nontrivial divisor class groups was given in [8]. He proved that \( |Cl(R)| \leq 2 \) if and only if any two factorizations of an element of \( R \) into irreducible elements have the same number of factors. Thus a Dedekind domain \( R \) with the property that each nonzero ideal class contains a prime ideal is a HFD if and only if \( |Cl(R)| \leq 2 \). The ring of integers in a finite algebraic number field over the rationals is an example of a Dedekind domain which satisfies the condition of having a prime ideal in each ideal class. In order to measure how far an atomic integral domain \( R \) is from being a HFD, we define the elasticity of \( R \)

\[ \rho(R) = \sup \{ m/n | x_1 \cdots x_m = y_1 \cdots y_n, \text{ for } x_i, y_j \in R \text{ irreducible} \} \]

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Thus $1 \leq \rho(R) \leq \infty$ and $\rho(R) = 1$ if and only if $R$ is a HFD. This concept was introduced by Valenza [12], who studied $\rho(R)$ for $R$ the ring of integers in an algebraic number field. In this paper, we show that if $R[X]$ is an HFD, then $Cl_t(R) \cong Cl_t(R[X])$, and if $G_1$ and $G_2$ are torsion abelian groups, then there exists a Dedekind HFD $R$ such that $Cl(R) = G_1 \oplus G_2$.

For general references on factorization in integral domains, see [1], [3] or [5].

2. Half-factorial and locally half-factorial domains

Let $R$ be an integral domain. We say that $R$ is a $GCD$-domain if any two elements of $R$ have a GCD in $R$. In [4], an integral domain $R$ is said to be a locally half-factorial domain (LHFD) if each localization $R_S$ of $R$ is a HFD.

Given a Dedekind domain $R$, let $Cl(R)$ denote its divisor class group, and $[I]$ the ideal class of $I$ in $Cl(R)$. If for a given abelian group $G$ and subset $C \subseteq G - \{0\}$ there exists a Dedekind domain $R$ such that $Cl(R) \cong G$ and $C = \{c|c \in G \text{ and } c \text{ contains a nonprincipal prime ideal of } R\}$, then the pair $\{G, C\}$ is called realizable [11].

Let $G$ be an abelian group and $C \subseteq G$. We say that $C$ is an independent set in $G$ if $n_1c_1 + \cdots + n_kc_k = 0, n_i \in \mathbb{Z}$, distinct $c_i \in C$, implies that each $n_ic_i = 0$. In fact, if $R$ is a Dedekind domain with torsion divisor class group $\{Cl(R), A\}$ with $A$ independent, then $R$ is an LHFD [4, Theorem 2.5].

We start with following example:

**Example 2.1.** (1) Let $R$ be a Dedekind domain with realizable pair $\{Z, A\}$, where $A = \{-2, 1, 2, 3, \cdots \}$ [11, Theorem 2.4]. Then $R$ is a HFD [7, Corollary 3.3]. Thus there exists a Dedekind HFD $R'$ such that $\{Z \bigoplus Z, A \bigoplus A\}$ is a realizable pair, where $A \bigoplus A = \{(-2, 0), (1, 0), (2, 0), \cdots (0, -2), (0, 1), (0, 2), \cdots \}$ [6, Theorem 3.1]. Now, for each nonzero nonunit $f \in R'$, we have that $Cl(R'_f)$ is one of the followings:

$Z \bigoplus Z, Z_2 \bigoplus Z, \{0\} \bigoplus Z, Z \bigoplus \{0\}, Z \bigoplus Z_2, Z_2 \bigoplus Z_2, \{0\}$.

Suppose now that $Cl(R'_f) = Z_2 \bigoplus Z$. Then the prime ideals of $R'_f$ are distributed in the classes $\{(1, 0), (0, -2), (0, 1), (0, 2), \cdots \}$. If $Cl(R'_f) = Z_2 \bigoplus Z_2$, then the prime
ideals of $R'_f$ are distributed in the classes $\{(1,0),(0,1)\}$. Since it is independent, $R'_f$ is a HFD, and hence $R'$ is a LHFD.

**Example 2.2.** Let $R$ be a Dedekind domain with realizable pair $\left(\bigoplus_{i=1}^{\infty} \mathbb{Z}_n, A\right)$, where $A = \{(1,0,0,\cdots),(0,1,0,\cdots),\cdots\}$. Then $A$ is independent, and hence $R$ is a HFD and LHFD. On the other hand, if $R$ is a Dedekind domain associated to the realizable pair $\{\mathbb{Z}_9,\{1,3\}\}$, then $R$ is a HFD and LHFD. But $\{1,3\}$ is not an independent set.

A saturated multiplicative set $S$ of $R$ is called a splitting multiplicative set [2] if for each nonzero $d \in R$, $d = sa$ for some $s \in S$ and $a \in R$ with $s' R \cap aR = s'aR$ for all $s' \in R$. The set $T = \{0 \neq t | sR \cap tR = stR$ for all $s \in S\}$ is also a splitting set and we call $T$ the complementary multiplicative set for $S$. A splitting multiplicative set $S$ of $R$ is said to be an lcm-splitting set if for each $s \in S$ and $d \in R$, $sR \cap dR$ is principal. $S$ is a splitting set of $R$ if and only if $R_T$ is a GCD-domain, where $T$ is the complementary multiplicative set [2, Proposition 2.4].

For an integral domain $R$, let $Cl_t(R)$ denote the $t$-class group of $R$, i.e., the group of $t$-invertible $t$-ideals of $R$ modulo its subgroup of principal fractional ideals. For example, if $R$ is an integral domain such that each nonunit element of $R$ is a product of primary element, then $Cl_t(R) = \{0\}$. In particular, if $R$ is atomic, then $\rho(R) = \sup \{\rho(R_P)|htP = 1\}$ [6, Corollary 2.5].

**Theorem 2.3.** Let $R$ be an atomic domain and let $S$ be an lcm-splitting multiplicative set. Then

(1) $\rho(R) = \rho(R_S)$ and $Cl_t(R) \cong Cl_t(R_S)$.

(2) If $R[X]$ is a HFD, then $Cl_t(R) \cong Cl_t(R[X])$.

**Proof.** (1) Let $T$ be the complementary multiplicative set of $S$. Then $\rho(R) = \max \{\rho(R_S), \rho(R_T)\}$ [6, Theorem 2.3]. By [2, Proposition 2.4], $R_T$ is a GCD-domain; so $R_T$ is a UFD. Thus $\rho(R_T) = 1$ and hence $\rho(R) = \rho(R_S)$. By [2, Theorem 4.1], $Cl_t(R) \cong Cl_t(R_S)$. (2) Suppose now that $R[X]$ is a HFD. Then $R$ is integrally closed [9, Theorem 2.2], and hence $Cl_t(R) \cong Cl_t(R[X])$ [10, Theorem 3.6].

With the notation in Theorem 2.3, $S$ is an lcm-splitting multiplicative set if and only if $S$ is generated by principal primes [2, Corollary 2.7], [6, Theorem 1.6]. In particular, $R$ is a HFD if and only if $R_S$ is a HFD.
Theorem 2.4. Let $G_1$ and $G_2$ be torsion abelian groups. Then there exists a Dedekind HFD $R$ such that $Cl(R) = G_1 \oplus G_2$.

Proof. Let $R_1, R_2$ be a Dedekind HFDs with $Cl(R_i) = G_i$ [1, Theorem 3.2]. Suppose that $R_i$ is associated to $\{G_i, A_i\}$ with $i = 1, 2$. Define $A = \{(a, 0), (0, b) | a \in A_1, b \in A_2\}$. Then $A$ is realizable [11]. Let $R$ be a Dedekind domain associated to $\{G, A\}$. Then $\rho(R) = \max\{\rho(R_1), \rho(R_2)\} = 1$ [6, Theorem 3.1]. Thus $R$ is a HFD. \qed

In view of the above theorem, we have:

Corollary 2.5. Let $\{R_i\}$ be a finite family of Dedekind HFDs with torsion divisor class groups $\{Cl(R_i) = G_i\}$. Then there exists a Dedekind HFD $R$ such that $Cl(R) \cong \bigoplus G_i$. \qed

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