MINIMAL CLOZ-COVERS OF NON-COMPACT SPACES

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ABSTRACT. Observing that for any dense weakly Lindelöf subspace of a space Y, X is \( Z^\# \)-embedded in Y, we show that for any weakly Lindelöf space X, the minimal \( \text{Cloz} \)-cover \( (E_{cc}(X), z_X) \) of X is given by \( E_{cc}(X) = \{(\alpha, x) : \alpha \text{ is a } G(X)\text{-ultrafilter on } X \text{ with } x \in \bigcap \alpha\} \), \( z_X((\alpha, x)) = x \), \( z_X \) is \( Z^\# \)-irreducible and \( E_{cc}(X) \) is a dense subspace of \( E_{cc}(\beta X) \).

1. Introduction

All spaces in this paper are Tychonoff and \( \beta X \) denotes the Stone-Čmech compactification of a space X.

In [5], it is shown that the minimal cloz-cover \( (E_{cc}(X), z_X) \) of a compact space X is characterized as follows:

\[ E_{cc}(X) = \{(\alpha, x) : \alpha \text{ is a } G(X)\text{-ultrafilter on } X \text{ with } x \in \bigcap \alpha\} \]

which is a subspace of \( \mathcal{L}(G(X)) \times X \) and \( z_X((\alpha, x)) = x \),

where \( \mathcal{L}(G(X)) \) is the ultrafilter-space of \( G(X) \). In [9] ([4], resp.), a theory of the minimal basically disconnected cover \( (\Lambda X, \Lambda_X) \) (quasi-F cover \( (QF(X), \Phi_X) \), resp.) of a Tychonoff space X is developed and the relation between \( \Lambda X \) and \( \Lambda \beta X \) \( (QF(X) \text{ and } QF(\beta X), \text{ resp.}) \) is explored. In [6], the minimal basically disconnected (quasi-F, resp.) cover of a locally weakly Lindelöf space X is characterized by the filter space \( \Lambda X = \{\alpha : \alpha \text{ is a fixed } \sigma Z(X)^\# \text{-ultrafilter on } X\} \) \( (QF(X) = \{\alpha : \alpha \text{ is a fixed } Z(X)^\# \text{-ultrafilter on } X\}, \text{ resp.}) \).

In this paper, we show that every (non-compact) weakly Lindelöf space X has the minimal cloz-cover \( (E_{cc}(X), z_X) \) and that \( E_{cc}(X) \) is characterized by the space \( \{(\alpha, x) : \alpha \text{ is a } G(X)\text{-ultrafilter on } X \text{ with } x \in \bigcap \alpha\} \) which is a dense subspace of

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E_{cc}(\beta X). Moreover, we find suitable conditions for a space for which the minimal cloz-cover is basically disconnected. For the terminology, we refer to [1] and [7]

2. Covering maps and \(Z^\#\)-irreducible maps

Definition 2.1. Let \(f : X \to Y\) be a continuous map. Then \(f\) is said to be

(a) perfect if \(f\) is closed and for any \(y \in Y\), \(f^{-1}(y)\) is a compact subset in \(X\),

(b) irreducible if \(f\) is onto and for any closed set \(A\) in \(X\) with \(A \neq X\),
\(f(A) \neq Y\), and

(c) a covering map if \(f\) is a perfect irreducible map.

Proposition 2.2. Consider the following commutative diagram:

\[
\begin{array}{ccc}
Z & \xrightarrow{f} & X \\
\downarrow{j_1} & & \downarrow{j_2} \\
W & \xrightarrow{g} & Y,
\end{array}
\]

where \(f, g\) are continuous maps and \(j_1, j_2\) are dense embeddings. Then we have the following:

(a) if \(f\) and \(g\) are perfect onto maps, then \(g(W - Z) = Y - X\),

(b) if \(g\) is a covering map and \(f\) is a perfect onto map, then \(f\) is a covering map, and

(c) if \(W, Y\) are compact spaces and \(f\) is a covering map, then \(g\) is also a covering map.

proof. (a) It is trivial ([7]).

(b) Take any closed set \(A\) in \(W\) with \(f(A \cap Z) = X\). By (a), \(g^{-1}(X) = Z\) and hence \(X = f(A \cap Z) = g(A \cap g^{-1}(X)) = g(A) \cap X\). Thus \(X \subseteq g(A)\). Since \(X\) is dense in \(Y\) and \(g(A)\) is closed, \(g(A) = Y\). Since \(g\) is irreducible, \(A = W\) and so \(A \cap Z = Z\). Thus \(f\) is irreducible.

(c) Clearly \(g\) is a perfect continuous map. Since \(g(W) = g(\text{cl}_W(Z)) = \text{cl}_Y(g(Z)) = \text{cl}_Y(f(Z)) = \text{cl}_Y(X) = Y\), \(g\) is onto. Take any closed set \(A\) in \(W\) with \(A \neq W\). Then \(A \cap Z \neq Z\) and hence \(f(A \cap Z) \neq X\). Since \(f(A \cap Z) = g(A) \cap X\) by (a), \(g(A) \cap X \neq X\); hence \(g(A) \neq Y\). Thus \(g\) is irreducible.
Notation 2.3. For any space $X$, let

(a) $C(X) = \{ f : f : X \to R \text{ is a continuous map} \}$ and $C^*(X) = \{ f : f : X \to R \text{ is a bounded continuous map} \}$, where $R$ is the space of real numbers endowed with the usual topology,

(b) for any $f \in C(X)$, $f^{-1}(0) = Z(f)$ which will be called a zero-set in $X$, and complements of zero-sets in $X$ will be called cozero-sets in $X$,

(c) $Z(X) = \{ Z : Z \text{ is a zero-set in } X \}$,

(d) $Z(X)^\# = \{ \text{cl}_X(\text{int}_X(A)) : A \in Z(X) \}$,

(e) $B(X) = \{ B : B \text{ is a clopen set in } X \}$, and

(f) $R(X) = \{ A : A \text{ is a regular closed set in } X \}$.

It is well-known that $R(X)$ is a complete Boolean algebra under the inclusion relation and $B(X)$, $Z(X)^\#$ are sublattices of $R(X)$ and that for any covering map $f : X \to Y$, the map $\phi : R(X) \to R(Y)$, defined by $\phi(A) = f(A)$, is a Boolean isomorphism. Moreover for any dense subspace $Y$ of a space $X$, the $\psi : R(X) \to R(Y)$, defined by $\psi(A) = A \cap Y$, is a Boolean algebra isomorphism ([8]).

In a lattice, meets and joins will be denoted by $\wedge$ and $\vee$, respectively and for any map $f : X \to Y$ and $B \subseteq 2^X$, let $f(B) = \{ f(B) : B \in B \}$.

Definition 2.4. A covering map $f : X \to Y$ is said to be $Z^\#$-irreducible if $f(Z(X)^\#) = Z(Y)^\#$.

We note that for any covering map $f : X \to Y$, $Z(Y)^\# \subseteq f(Z(X)^\#)$ and hence $f$ is $Z^\#$-irreducible if and only if $Z(Y)^\# \supseteq f(Z(X)^\#)$.

Proposition 2.5. Let $g : Y \to W$, $h : W \to X$ be covering maps. Then $h \circ g$ is $Z^\#$-irreducible if and only if $h$ and $g$ are $Z^\#$-irreducible.

proof. Assume that $h \circ g$ is $Z^\#$-irreducible and take any $A \in Z(Y)^\#$, then there is $B \in Z(X)^\#$ with $h \circ g(A) = B$; $h(g(A)) = B = h(\text{cl}_W(h^{-1}(\text{int}_X(B))))$. Since $h$ is a covering map and $g(A)$, $\text{cl}_W(h^{-1}(\text{int}_X(B)))$ are regular closed in $W$, $\text{cl}_W(h^{-1}(\text{int}_X(B))) = g(A)$. Since $\text{cl}_W(h^{-1}(\text{int}_X(B))) = \text{cl}_W(\text{int}_W(h^{-1}(B)))$, $g(A) \in Z(W)^\#$. Thus $g$ is $Z^\#$-irreducible.

Take any $A \in Z(W)^\#$. Since $g$ is a covering map, $\text{cl}_Y(g^{-1}(\text{int}_W(A))) \in Z(Y)^\#$. Since $h \circ g$ is $Z^\#$-irreducible, $h \circ g(\text{cl}_Y(g^{-1}(\text{int}_W(A)))) \in Z(X)^\#$. But $h \circ g(\text{cl}_Y(g^{-1}(\text{int}_W(A)))) = h(g(\text{cl}_Y(g^{-1}(\text{int}_W(A))))) = h(A)$. Thus $h$ is $Z^\#$-irreducible. The converse is immediate from the definition.
Definition 2.6. Let $Y$ be a space and $X$ a subspace of $Y$. Then $X$ or $j : X \hookrightarrow Y$ is said to be $Z$-embedded in $Y$ if for any $A \in Z(X)\#$, there is a $B \in Z(Y)\#$ such that $A = B \cap X$.

**Proposition 2.7.** Consider the following commutative diagram:

$$
\begin{array}{ccc}
P & \xrightarrow{f} & X \\
j_1 \downarrow & & \downarrow j_2 \\
Y & \xrightarrow{g} & W,
\end{array}
$$

where $j_1$, $j_2$ are dense embeddings and $f$, $g$ are covering maps. Then $g$ is $Z$-irreducible and $j_1$ is $Z$-embedded if and only if $f$ is $Z$-irreducible and $j_2$ is $Z$-embedded.

**proof.** $(\implies)$ Take any $A \in Z(P)\#$. Since $j_1$ is $Z$-embedded, there is a $B \in Z(Y)\#$ such that $A = B \cap P$. Note that $f(A) = f(B \cap P) = g(B) \cap X$. Since $g$ is $Z$-irreducible, $f(A) \in Z(X)\#$. Thus $f$ is $Z$-irreducible. Let $C \in Z(X)\#$. Then $\text{cl}_P(f^{-1}(\text{int}_X(C))) \in Z(P)\#$. Since $j_1$ is $Z$-embedded, there is a $D \in Z(Y)\#$ such that $D \cap P = \text{cl}_P(f^{-1}(\text{int}_X(C)))$. Then $C = f(D \cap P) = g(D) \cap X$. Since $g$ is $Z$-irreducible, $g(D) \in Z(X)\#$; therefore $j_2$ is $Z$-embedded.

$(\impliedby)$ Take any $A \in Z(Y)\#$. Then $A \cap P \in Z(P)\#$ for $P$ is dense in $Y$ and $f(A \cap P) = g(A \cap P) = g(A) \cap X$. Since $f$ is $Z$-irreducible, $g(A) \cap X \in Z(X)\#$. Since $j_2$ is $Z$-embedded, there is a $B \in Z(W)\#$ with $g(A) \cap X = B \cap X$. Since $j_2$ is a dense embedding and $g(A)$, $B$ are regular closed, $g(A) = B$. Thus $g$ is $Z$-irreducible.

Take any $C \in Z(P)\#$. Since $f$ is $Z$-irreducible, $f(C) \in Z(X)\#$. Since $j_2$ is $Z$-embedded, there is a $D \in Z(W)\#$ with $f(C) = D \cap X$. Since $g$ is a covering map, $\text{cl}_Y(g^{-1}(\text{int}_W(D))) \in Z(Y)\#$. Then $f(\text{cl}_Y(g^{-1}(\text{int}_W(D))) \cap P) = g(\text{cl}_Y(g^{-1}(\text{int}_W(D)))) \cap X = D \cap X = f(C)$. Hence $\text{cl}_Y(g^{-1}(\text{int}_W(D))) \cap P = C$. Thus $j_1$ is $Z$-embedded.

**Definition 2.8.** A pair $(Y, f)$ is said to be a cover of a space $X$ if $f : Y \longrightarrow X$ is a covering map.

Let $X$, $Y$ be spaces and $f : X \longrightarrow Y$ a continuous map. For any $U \subseteq Y$, let $f_U : f^{-1}(U) \longrightarrow U$ be the restriction and corestriction of $f$ with respect to $f^{-1}(U)$ and $U$, respectively.
Lemma 2.9. Let $X$ be a space and $(E, f)$ a cover of $\beta X$. Then $(f^{-1}(X), f_X)$ is also a cover of $X$.

proof. Clearly, we have a pullback diagram:

$$
\begin{array}{ccc}
  f^{-1}(X) & \xrightarrow{f_X} & X \\
  j \downarrow & & \beta_X \downarrow \\
  E & \xrightarrow{f} & \beta X.
\end{array}
$$

Since $f$ is perfect, $f_X$ is also perfect and clearly, $f_X$ is onto. Since $X$ is dense in $\beta X$ and $f$ is a covering map, $f^{-1}(X)$ is dense in $E$. Thus $j$ is a dense embedding and hence $f_X$ is a covering map by Proposition 2.2.

3. Minimal Cloz-covers of non-compact spaces

Definition 3.1. Let $X$ be a space.

(a) A cozero-set $C$ in $X$ is said to be a complemented cozero-set if there is a cozero-set $D$ in $X$ such that $C \cap D = \emptyset$ and $C \cup D$ is dense in $X$. In case, $\{C, D\}$ is called a complementary pair of cozero-sets in $X$.

(b) $G(X) = \{\text{cl}_X(C) : C$ is a complemented cozero-set in $X\}$.

For any space $X$, $G(X) = \{A' \in Z(X)^\# : A' \in Z(X)^\#\}$, where $A'$ denotes the complement of $A$ in $R(X)$, that is, $A' = \text{cl}_X(X - A)$ and $G(X)$ is a subalgebra of $R(X)$ ([5]).

Recall that a subspace $Y$ of a space $X$ is called $C^*$-embedded in $X$ if for any $f \in C^*(Y)$, there is a $g \in C^*(X)$ with $g|_Y = f$.

Definition 3.2. A space $X$ is said to be

(a) a cloz-space if $G(X) = B(X)$, and

(b) a quasi-$F$ space if every dense cozero-set in $X$ is $C^*$-embedded, equivalently, for any zero-sets $Z_1, Z_2$ in $X$ such that $\text{int}_X(Z_1) \cap \text{int}_X(Z_2) = \emptyset$, $\text{cl}_X(\text{int}_X(Z_1)) \cap \text{cl}_X(\text{int}_X(Z_2)) = \emptyset$.

Proposition 3.3. (a) A space $X$ is a cloz-space if and only if every element of $X$ has a cloz open neighborhood.
(b) Every dense $Z^\#$-embedded subspace of a cloz-space is again a cloz-space.

proof. (a) ($\implies$) It is trivial.

($\impliedby$) Let $\{C, D\}$ be a complemented pair of cozero-sets in $X$ and $x \in cl_X(C)$. Let $V$ be a cloz open neighborhood of $x$ in $X$. Since $cl_V((C \cup D) \cap V) = cl_X(C \cup D) \cap V = X \cap V = V$, $\{C \cap V, D \cap V\}$ is a complemented pair of cozero-sets in $V$. Since $V$ is a cloz-space, $cl_V(C \cap V)$ is clopen in $V$. Moreover, $cl_X(C) \cap V = cl_V(C \cap V) = int_V(cl_V(C \cap V)) = int_X(cl_X(C) \cap V)$. Hence $x \in int_X(cl_X((C)))$. Thus $cl_X(C)$ is again clopen in $X$ and therefore $X$ is a cloz-space.

(b) Let $Y$ be a cloz-space and $X$ a dense $Z^\#$-embedded subspace of $Y$. Let $\{C, D\}$ be a complemented pair of cozero-sets in $X$. Since $G(X) \subseteq Z(X)^\#$ and $X$ is $Z^\#$-embedded in $Y$, there are $A, B \in Z(Y)^\#$ with $cl_X(C) = A \cap X$ and $cl_X(D) = B \cap X$. Note that

$$\emptyset = cl_X(C) \cap cl_X(D) = (A \cap X) \cap (B \cap X)$$

$$= cl_X(int_X((A \cap X) \cap (B \cap X))) = cl_X(int_X((A \cap B) \cap X))$$

$$= cl_X(int_Y(A \cap B) \cap X) = cl_Y(int_Y(A \cap B)) \cap X = (A \cap B) \cap X.$$

Since $X$ is dense in $Y$, $A \cap B = \emptyset$. Since $X = X \cap Y = (A \cup B) \cap X$, $Y = A \cup B$. Hence $A' = B$. Thus $A \in G(Y)$. Since $Y$ is a cloz-space, $A$ is clopen in $Y$ and hence $cl_X(C)$ is clopen in $X$.

Definition 3.4. Let $\mathcal{C}$ be a full subcategory of the category $Tych$ of Tychonoff spaces and continuous maps and $X \in Tych$. Then

(a) a pair $(Y, f)$ is called a $\mathcal{C}$-cover of $X$ if $(Y, f)$ is a cover of $X$ and $Y \in \mathcal{C}$,

(b) a $\mathcal{C}$-cover $(Y, f)$ is called a minimal $\mathcal{C}$-cover of $X$ if for any $\mathcal{C}$-cover $(Z, g)$ of $X$, there is a covering map $h : Z \to Y$ with $foh = g$.

Lemma 3.5. ([6]) Let $\mathcal{C}$ be a full subcategory of $Tych$ such that $Y \in \mathcal{C}$ if and only if $\beta Y \in \mathcal{C}$. Suppose that $X \in Tych$ and $(E, f)$ is a minimal $\mathcal{C}$-cover of $\beta X$. If $f^{-1}(X) \in \mathcal{C}$, then $(f^{-1}(X), f_X)$ is a minimal $\mathcal{C}$-cover of $X$.

Let Cloz (QF, resp.) denote the full subcategory of $Tych$ determined by cloz-spaces (quasi-F spaces, resp).

It is known that every compact space $X$ has the minimal Cloz-cover $(E_{cc}(X), z_X)$ and moreover $E_{cc}(X) = \{(\alpha, x) : \alpha$ is a $G(X)$-ultrafilter on $X$ with $x \in \cap \alpha \}$ and $z_X((\alpha, x)) = x$ ([5]). It is a natural question whether every space has a Cloz-cover. We will give some partial answers for this problem in this section.
Definition 3.6. A space $X$ is said to be \emph{weakly Lindelöf} if for any open cover $\mathcal{U}$ of $X$, there is a countable subfamily $\mathcal{V}$ of $\mathcal{U}$ such that $\bigcup \mathcal{V}$ is dense in $X$ and a space $X$ is called \emph{locally weakly Lindelöf} if every element of $X$ has a weakly Lindelöf neighborhood.

In [3], the minimal $QF$-cover $(QF(X), \Phi_X)$ of a compact space $X$ is constructed as an inverse limit space and in [8], Vermeer constructed the minimal $QF$-cover $(QF(X), \Phi_X)$ of arbitrary space $X$. In [4] ([6], resp), the minimal $QF$-cover $(QF(X), \Phi_X)$ of a compact (locally weakly Lindelöf, resp.) space $X$ is characterized by $QF(X) = \{ \alpha : \alpha$ is a fixed $Z(X)^\#$-ultrafilter on $X \}$ and $\Phi_X(\alpha) = \cap \alpha$. Moreover, $\Phi_X$ is $Z(X)^\#$-irreducible if $X$ is compact ([4]).

For any space $X$, let $(QF(\beta X), \Phi_\beta)$ ($(E_{cc}(\beta X), z_\beta)$, resp.) denote the minimal $QF(\mathsf{Cloz})$ (resp.)-cover of $\beta X$.

Theorem 3.7. Let $X$ be a space such that $\Phi_\beta^{-1}(X)$ is $Z^\#$-embedded in $QF(\beta X)$. Then $(\beta^{-1}(X), z_{\beta_X})$ is the minimal $\mathsf{Cloz}$-cover of $X$, $\beta^{-1}(X)$ is dense in $E_{cc}(\beta X)$ and $z_{\beta_X}$ is $Z^\#$-irreducible.

proof. Clearly, $\beta^{-1}(X)$ is dense in $E_{cc}(X)$. Since every quasi-$F$ space is a cloz-space, there is a covering map $g : QF(\beta X) \rightarrow E_{cc}(\beta X)$ with $z_\beta \circ g = \Phi_\beta$. Since the following diagram

$$
\begin{array}{ccc}
z_{\beta^{-1}}(X) & \xrightarrow{z_{\beta_X}} & X \\
\downarrow j_2 & & \downarrow \beta \\
E_{cc}(\beta X) & \xrightarrow{z_\beta} & \beta X
\end{array}
$$

is a pullback, there is a unique continuous map $g^0 : \Phi_\beta^{-1}(X) \rightarrow \beta^{-1}(X)$ such that $z_{\beta_X} \circ g^0 = \Phi_X$ and $g \circ j_1 = j_2 \circ g^0$, where $j_1 : \Phi_\beta^{-1}(X) \hookrightarrow QF(\beta X)$ is the inclusion map. Since $j_1, \beta$ are $Z^\#$-embedded and $\Phi_\beta$ is $Z^\#$-irreducible, by Proposition 2.7, $\Phi_X$ is $Z^\#$-irreducible. Let $a \in \beta^{-1}(X)$. Then there is $b \in QF(\beta X)$ with $g(b) = a$. Hence $\Phi_\beta(b) = z_\beta(g(b)) = z_\beta(a) \in X$ and so $b \in \Phi^{-1}(X)$. Hence $g(b) = g^0(b) = a$. Thus $g^0$ is onto. Since $z_{\beta_X} \circ g^0 = \Phi_X$ is a covering map, by Proposition 2.2, $g^0$ is a covering map. Since $z_{\beta_X} \circ g^0 = \Phi_X$ is $Z^\#$-irreducible, by Proposition 2.5, $z_{\beta_X}$ and $g^0$ are $Z^\#$-irreducible. Consider the following commutative diagram:
\[
\Phi^{-1}_\beta(X) \xrightarrow{g^0} z^{-1}_\beta(X)
\]
\[
\downarrow j_1 \quad \downarrow j_2
\]
\[
QF(\beta X) \xrightarrow{g} E_{cc}(\beta X).
\]

By Proposition 2.7, \(j_2\) is \(Z^\#\)-embedded. So, by Proposition 3.3, \(z^{-1}_\beta(X)\) is a cloz-space. Thus, by Lemma 3.5, \((z^{-1}_\beta(X), z_{\beta X})\) is the minimal \(\text{Cloz}\)-cover of \(X\).

Recall that a dense weakly Lindelöf subspace of a space is \(Z^\#\)-embedded and that for any covering map \(f : X \rightarrow Y\) such that \(Y\) is weakly Lindelöf, \(X\) is weakly Lindelöf ([4]). Using this, we have the following corollary:

**Corollary 3.8.** For a weakly Lindelöf space \(X\), \((z^{-1}_\beta(X), z_{\beta X})\) is the minimal \(\text{Cloz}\)-cover of \(X\) and \(z_{\beta X}\) is \(Z^\#\)-irreducible.

For any weakly Lindelöf space \(X\), \((E_{cc}(X), z_X)\) denotes the minimal \(\text{Cloz}\)-cover of \(X\).

For any space \(X\), the isomorphism \(\psi : R(\beta X) \rightarrow R(X) (\psi(A) = A \cap X, A \in R(\beta X))\) induces a lattice isomorphism \(G(\beta X) \rightarrow G(X)\). Thus \((\alpha, x) \in z^{-1}_\beta(X)\) if and only if \(\alpha_X = \{A \cap X : A \in \alpha\}\) is a \(G(X)\)-ultrafilter and \(x \in \cap \alpha_X\). Therefore we have the following corollary:

**Corollary 3.9.** For any weakly Lindelöf space \(X\), \(E_{cc}(X)\) is the space \(\{(\alpha, x) : \alpha\text{ is a }G(X)\text{-ultrafilter on }X\text{ with }x \in \alpha\}\) which is a subspace of \(L(G(X)) \times X\), where \(L(G(X))\) is the ultrafilter space of \(G(X)\).

**Definition 3.10.** A space \(X\) is said to be a **basically disconnected space** if for any zero-set \(Z\) in \(X\), \(\text{int}_X(Z)\) is closed in \(X\).

Recall that a sublattice \(A\) of \(R(X)\) is called **\(\sigma\)-complete** if it is closed under countable joins and meets.

**Proposition 3.11.** Let \(X\) be a weakly Lindelöf space. Then the following are equivalent:

1. \(G(X) = Z(X)^\#\),
2. \(G(X) = \{\text{cl}_X(C) : C\text{ is a cozzero-set in }X\}\),
3. \(G(E_{cc}(X)) = Z(E_{cc}(X))^\#\),
4. \(E_{cc}(X)\) is basically disconnected,
(5) \( Z(X)^\# \) is a \( \sigma \)-complete Boolean subalgebra of \( R(X) \).

Proof. (1) \( \Rightarrow \) (2) Clearly, \( G(X) \subseteq \{ \text{cl}_X(C) : C \text{ is a cozero-set in } X \} \). Let \( C \) be a cozero-set in \( X \), then \( \text{cl}_X(\text{int}_X(X - C)) \subseteq Z(X)^\# = G(X) \) and hence \( \text{cl}_X(X - \text{cl}_X(X - C)) = \text{cl}_X(C) \subseteq G(X) \).

(2) \( \Rightarrow \) (3) Let \( A \in Z(E_{cc}(X))^\# \). Since \( z_X \) is \( Z^\# \)-irreducible, \( z_X(A) \in Z(X)^\# \) and hence \( z_X(A') = z_X(A') \subseteq G(X) \subseteq Z(X)^\# \). Hence \( A' \in Z(E_{cc}(X))^\# \) and so \( A \in G(E_{cc}(X)) \).

(3) \( \Rightarrow \) (4) Let \( Z \) be a zero-set in \( E_{cc}(X) \), then by (3), \( \text{cl}_{E_{cc}(X)}(\text{int}_{E_{cc}(X)}(Z)) \subseteq G(E_{cc}(X)) \). Since \( E_{cc}(X) \) is a cloz-space, \( \text{cl}_{E_{cc}(X)}(\text{int}_{E_{cc}(X)}(Z)) \) is clopen in \( E_{cc}(X) \) and hence \( \text{int}_{E_{cc}(X)}(Z) \) is closed. Thus \( E_{cc}(X) \) is basically disconnected.

(4) \( \Rightarrow \) (5) Since \( z_X \) is \( Z^\# \)-irreducible, \( z_X(Z(E_{cc}(X))^\#) = Z(X)^\# \) and since \( E_{cc}(X) \) is basically disconnected, \( Z(E_{cc}(X))^\# \) is a \( \sigma \)-complete Boolean subalgebra of \( R(E_{cc}(X)) \) and hence \( Z(X)^\# \) is a \( \sigma \)-complete Boolean subalgebra of \( R(X) \).

(5) \( \Rightarrow \) (1) Since \( Z(X)^\# \) is a subalgebra of \( R(X) \), \( G(X) = Z(X)^\# \).

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