THE COMPLETIONS OF FUZZY NORMED LINEAR SPACES

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1. Introduction

We know that the rational line $\mathbb{Q}$ is not complete but can be "enlarged" to the real line $\mathbb{R}$ which is complete. And this "completion" $\mathbb{R}$ of $\mathbb{Q}$ includes the meaning that $\mathbb{Q}$ is dense in $\mathbb{R}$. It is quite important that an arbitrary incomplete space can be "completed" in a similar fashion. For example, every metric space, normed space and pre-Hilbert space has complete metric space, Banach space and Hilbert space, respectively.

The theory of fuzzy sets introduced by L.A. Zadeh [4] in 1965 has developed in the last thirty years and has been applied in many fields. Many authors have introduced the concepts of fuzzy normed linear spaces in different ways. In this paper, to discuss the completions of fuzzy normed linear spaces, we use the concept of fuzzy normed linear spaces introduced by Felbin [1], which satisfies the concept of fuzzy metric spaces [2].

2. Definitions and preliminaries

We briefly review some definitions and preliminaries to needed discuss this paper.

Definition 2.1. A fuzzy real number is a fuzzy set on $\mathbb{R}$, i.e., a mapping $\eta : \mathbb{R} \longrightarrow [0,1]$ associating each real number t with its grade of membership $\eta(t)$.

Definition 2.2. A fuzzy real number η is convex if

$$\eta(t) \geq \eta(s) \wedge \eta(r) = \min(\eta(s), \eta(r)),$$

where $s \leq t \leq r$.

Definition 2.3. Let η be a fuzzy real number. If there exists an $t_0 \in \mathbb{R}$ such that $\eta(t_0) = 1$, then η is called a normal fuzzy real number.

Definition 2.4. The α-level set of a fuzzy real number η, $\alpha \in (0,1]$, denoted by $[\eta]_\alpha$, is defined as $[\eta]_\alpha = \{t \mid \eta(t) \geq \alpha\}$.

The following proposition is seen in [4].

Proposition 2.1. A fuzzy real number η is convex if and only if each of its α-level sets $[\eta]_\alpha, \alpha \in (0,1]$, is a convex set in $\mathbb{R}$.

Definition 2.5. A fuzzy real number η is called upper semi-continuous if for all $t \in \mathbb{R}$ and $\epsilon > 0$ with $\eta(t) = a$, there is $c > 0$ such that

$$|s - t| < c = c(t) \Rightarrow \eta(s) < a + \epsilon,$$

i.e., $\eta^{-1}([0, a+\epsilon))$ for all $a \in [0,1]$ and $\epsilon > 0$ is open in $\mathbb{R}$ with the usual topology.

It can be seen in [2] that the α-level set of an upper semi-continuous convex normal fuzzy real number for each $\alpha \in (0,1]$ is a closed interval $[a^\alpha, b^\alpha]$, where $a^\alpha = -\infty$ and $b^\alpha = +\infty$ are also admissible. Let us denote the set of all upper semi continuous normal convex fuzzy real numbers by E. Since each $r \in \mathbb{R}$ can be considered as a fuzzy real number $\bar{r}$ defined by

$$\bar{r}(t) = \begin{cases} 1 & \text{if} \quad t = r, \\ 0 & \text{if} \quad t \neq r, \end{cases} \tag{2.1}$$

$\mathbb{R}$ can be embedded in E.

Definition 2.6. A fuzzy real number η is called non-negative if $\eta(t) = 0$ for all $t < 0$. The set of all non-negative fuzzy real numbers of E is denoted by G.

Arithmetic operations $\oplus, \ominus, \odot$ and $\oslash$ on E $\times$ E can be defined as in [3];

$$(\eta \oplus \delta)(t) = \sup_{s \in \mathbb{R}} \{\eta(s) \wedge \delta(t - s)\}, \quad t \in \mathbb{R}, \tag{2.2}$$

$$(\eta \ominus \delta)(t) = \sup_{s \in \mathbb{R}} \{\eta(s) \wedge \delta(s - t)\}, \quad t \in \mathbb{R}, \tag{2.3}$$

$$(\eta \odot \delta)(t) = \sup_{\substack{s \in \mathbb{R} \\ s \neq 0}} \{\eta(s) \wedge \delta(t/s)\}, \quad t \in \mathbb{R},$$

$$(\eta \oslash \delta)(t) = \sup_{s \in \mathbb{R}} \{\eta(st) \wedge \delta(s)\}, \quad t \in \mathbb{R}.$$

Remark 2.1. The additive and multiplicative identities in E are $\bar{0}$ and $\bar{1}$, respectively. Let $\ominus\eta$ be defined to be $\bar{0} \ominus \eta$. From (2.2) and (2.3) it follows that $(\ominus\eta)(t)=\eta(-t)$, $t \in \mathbb{R}$, and $\eta \ominus \delta = \eta \oplus (\ominus\delta)$.

Definition 2.7.. Let $\mathbb{R}^+$ be the set of all nonnegative real numbers. For $k \in \mathbb{R}^+\backslash\{0\}$, $k\eta$ is defined to be $(k\eta)(t) = \eta(t/k)$ and 0η is defined to be $\bar{0}$.

We recall the following two propositions in [2].

Proposition 2.2. Let $\eta, \delta \in$ E and $[\eta]_\alpha = [a_1^\alpha, b_1^\alpha]$ and $[\delta]_\alpha = [a_2^\alpha, b_2^\alpha]$ for $\alpha \in (0, 1]$. Then

$$[\eta \oplus \delta]_\alpha = [a_1^\alpha + a_2^\alpha, b_1^\alpha + b_2^\alpha],$$
$$[\eta \ominus \delta]_\alpha = [a_1^\alpha - b_2^\alpha, b_1^\alpha - 2_2^\alpha],$$
$$[\eta \odot \delta]_\alpha = [a_1^\alpha a_2^\alpha, b_1^\alpha b_2^\alpha],$$
$$[\bar{1} \oslash \delta]_\alpha = [\frac{1}{b_2^\alpha}, \frac{1}{a_2^\alpha}], \quad a_2^\alpha > 0.$$

Proposition 2.3. Let $\{[a^\alpha, b^\alpha] \mid \alpha \in (0, 1]\}$ be a given family of non-empty intervals with

(a) $[a^{\alpha_1}, b^{\alpha_1}] \supset [a^{\alpha_2}, b^{\alpha_2}]$ for all α_1 in $(0, \alpha_2)$ and

(b) $[\lim_{k\to\infty} a^{\alpha_k}, \lim_{k\to\infty} b^{\alpha_k}] = [a^\alpha, b^\alpha]$ such that

whenever (α_k) is an increasing sequence in (0,1] converging to α, then the family $[a^\alpha, b^\alpha]$ represents the α-level sets of some fuzzy real number η in E.

Conversely, if $\{[a^\alpha, b^\alpha] \mid \alpha \in (0, 1]\}$ is a family of α-level sets of some fuzzy real number $\eta \in$ E, then the conditions (a) and (b) are satisfied.

Definition 2.8. Define a partial ordering $\preccurlyeq$ in E by $\eta \preccurlyeq \delta$ if and only if $a_1^\alpha \le a_2^\alpha$ and $b_1^\alpha \le b_2^\alpha$ for all $\alpha \in (0, 1]$, where $[\eta]_\alpha = [a_1^\alpha, b_1^\alpha]$ and $[\delta]_\alpha = [a_2^\alpha, b_2^\alpha]$. We write $\eta \preccurlyeq \delta$ as $\delta \succcurlyeq \eta$ when desired. The strict inequality in E is defined by $\eta \prec \delta$ if and only if $a_1^\alpha < a_2^\alpha$ and $b_1^\alpha < b_2^\alpha$ for each $\alpha \in (0, 1]$.

Definition 2.9. The sequence (η_n) in E converges to η in E, denoted by $\eta = \lim_{n\to\infty} \eta_n$, if $\lim_{n\to\infty} a_n^\alpha = a^\alpha$ and $\lim_{n\to\infty} b_n^\alpha = b^\alpha$ for all $\alpha \in (0, 1]$, where $[\eta_n]_\alpha = [a_n^\alpha, b_n^\alpha]$ and $[\eta]_\alpha = [a^\alpha, b^\alpha]$.

This convergence is called an α-level convergence.

Definition 2.10. The sequence (η_n) in E is called a Cauchy sequence if $\lim\limits_{\substack{m\to\infty\\ n\to\infty}} [\eta_m \ominus \eta_n] = \bar{0}$, i.e., $\lim\limits_{\substack{m\to\infty\\ n\to\infty}} [\eta_m \ominus \eta_n]_\alpha = \{0\}$.

Definition 2.11. E is complete if every Cauchy sequence in E converges in E.

Proposition 2.4 [1]. Let $\eta, \delta, \mu \in$ E, $[\eta]_\alpha = [a^\alpha, b^\alpha]$, $[\delta]_\alpha = [c^\alpha, d^\alpha]$ and $[\mu]_\alpha = [e^\alpha, f^\alpha]$ for $\alpha \in (0,1]$. Then:

(1) $[\eta \oplus \delta] \ominus \mu = \eta \oplus [\delta \ominus \mu]$;

(2) if $\mu \preccurlyeq \eta$, then $\mu \ominus \delta \preccurlyeq \eta \ominus \delta$;

(3) if $\mu \preccurlyeq \eta$ and $\delta \succ \bar{0}$, then

 (a) $\mu \odot \delta \preccurlyeq \eta \odot \delta$;

 (b) $\mu \oslash \delta \preccurlyeq \eta \oslash \delta$;

(4) $(\eta \odot \mu) \oslash \delta = \eta \odot (\mu \oslash \delta)$ for $c^\alpha > 0$.

3. The completions of fuzzy normed linear spaces

The concept of fuzzy normed linear spaces was introduced by Felbin [1]. For the sake of convenience, we recall some definitions and results in [1], and construct the completions of fuzzy normed linear spaces.

Definition 3.1. Let X be a vector space over $\mathbb{R}$ and let $\|\cdot\| : X \longrightarrow G$ be a fuzzy number valued function. Let mappings $L, R : [0,1] \times [0,1] \longrightarrow [0,1]$ be symmetric, non-decreasing in both arguments and satisfy $L(0,0) = 0$ and $R(1,1) = 1$. Write

$$[\|x\|]_\alpha = [\|\|x\|\|_1^\alpha, \|\|x\|\|_2^\alpha] \quad \text{for} \quad x \in X, \quad \alpha \in (0,1].$$

Suppose that for all $x(\neq 0) \in$ X, there exists $\alpha_0 \in (0,1]$ independent of x such that for all $\alpha \leq \alpha_0$,

(a) $\|\|x\|\|_2^\alpha < \infty$,

(b) $\inf \|\|x\|\|_1^\alpha > 0$.

The quadruple (X, $\|\cdot\|$, L, R) is called a fuzzy normed linear space and $\|\cdot\|$ a fuzzy norm, if

(FN-1) $\|x\| = \bar{0}$ if and only if x = 0;

(FN-2) $\|rx\| = |r|\|x\|, x \in \mathrm{X}, r \in \mathbb{R}$;

(FN-3) for all $x, y \in \mathrm{X}$,

(a) whenever $s \leq |||x|||_1^1$, $t \leq |||y|||_1^1$ and $s + t \leq |||x + y|||_1^1$, $\|x + y\|(s + t) \geq L(\|x\|(s), \|y\|(t))$,

(b) whenever $s \geq |||x|||_1^1$, $t \geq |||y|||_1^1$ and $s + t \geq |||x + y|||_1^1$, $\|x + y\|(s + t) \leq R(\|x\|(s), \|y\|(t))$.

Example 3.1. If $\mathrm{X} = \mathbb{R}^n$ and $\|\cdot\|$ is defined by

$$\|(x_1, x_2, \ldots, x_n)\|(t) = \begin{cases} 1, & \text{if } t = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} \\ 0, & \text{otherwise} \end{cases}$$

for $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, then $\|\cdot\|$ is a fuzzy norm according to Definition 3.1 with the choice of $L = \text{Min}$ and $R = \text{Max}$.

Lemma 3.1. *When L = Min in Definition 3.1 the condition (FN-3)(a) is equivalent to the triangle inequality*

$$|||x + y|||_1^\alpha \leq |||x|||_1^\alpha + |||y|||_1^\alpha$$

for all $\alpha \in (0, 1]$ and $x, y \in X$.

Lemma 3.2. *When R = Max in Definition 3.1 the condition (FN-3)(b) is equivalent to the triangle inequality*

$$|||x + y|||_2^\alpha \leq |||x|||_2^\alpha + |||y|||_2^\alpha$$

for all $\alpha \in (0, 1]$ and $x, y \in X$.

By Lemma 3.1 and Lemma 3.2 we have :

Theorem 3.1. *In a fuzzy normed linear space (X, $\|\cdot\|$, Min, Max), the condition (FN-3) in Definition 3.1 is equivalent to*

$$\|x + y\| \preccurlyeq \|x\| \oplus \|y\|.$$

Remark 3.1. (I) With the choice of L = Min and R = Max, we can show that $|||\cdot|||_2^\alpha : \mathrm{X} \longrightarrow \mathbb{R}^+$ is a usual norm in X. In fact,

(i) $x = 0$ if and only if $\|x\| = \bar{0}$ if and only if $|||x|||_2^\alpha = 0$ for $\alpha \in (0, 1]$.

(ii) if $r = 0$, then $|||rx|||_2^\alpha = |||0|||_2^\alpha = 0$ and $|r||||x|||_2^\alpha = 0$ for $\alpha \in (0, 1]$. Thus $|||rx|||_2^\alpha = |r||||x|||_2^\alpha$ for $\alpha \in (0, 1]$.

if $r \neq 0$, then

$$\begin{aligned}[\|rx\|]_\alpha &= \{t \in \mathbb{R} \mid \|rx\|(t) \geq \alpha\} \\ &= \{t \in \mathbb{R} \mid (|r|\|x\|)(t) \geq \alpha\} \\ &= \{t \in \mathbb{R} \mid \|x\|(t/|r|) \geq \alpha\}.\end{aligned}$$

Put $k = t/|r|$, then

$$\begin{aligned}[\|rx\|]_\alpha &= \{k|r| \in \mathbb{R} \mid \|x\|(k) \geq \alpha\} \\ &= [|r||||x|||_1^\alpha, |r||||x|||_2^\alpha].\end{aligned}$$

Since

$$[\|rx\|]_\alpha = [|||rx|||_1^\alpha, |||rx|||_2^\alpha]$$

for $\alpha \in (0, 1]$, $|||rx|||_1^\alpha = |r||||x|||_1^\alpha$ and $|||rx|||_2^\alpha = |r||||x|||_2^\alpha$.

(iii) Since

$$\|x + y\| \preccurlyeq \|x\| \oplus \|y\|,$$

for any $x, y \in \mathrm{X}$, by definition of addition $\oplus$,

$$|||x + y|||_2^\alpha \leq |||x|||_2^\alpha + |||y|||_2^\alpha.$$

(II) Since $x = 0$ implies $|||x|||_1^\alpha = 0$, similarly we can show that $|||\cdot|||_1^\alpha : \mathrm{X} \longrightarrow \mathbb{R}^+$ is a usual semi-norm in X with the choice of L = Min, R = Max.

(III) The Remark 3.1 is silightly different Remark of Felbin[1].

Definition 3.2. Let (X, $\|\cdot\|, L, R$) be a fuzzy normed linear space. A sequence (x_n) in X is said to converge to x in X, denoted by $\lim_{n\to\infty} x_n = x$, if

$$\lim_{n\to\infty} \|x_n - x\| = \bar{0},$$

i.e., $\lim_{n\to\infty} |||x_n - x|||_2^\alpha = 0$ for $\alpha \in (0, 1]$.

Definition 3.3. A sequence (x_n) in a fuzzy normed linear space $(X, \|\cdot\|, L, R)$ is called a Cauchy sequence if

$$\lim_{\substack{m\to\infty \\ n\to\infty}} \|x_m - x_n\| = \bar{0},$$

i.e., $\lim_{\substack{m\to\infty \\ n\to\infty}} |\|x_m - x_n\||_2^\alpha = 0$ for $\alpha \in (0, 1]$.

Definition 3.4. A fuzzy normed linear space is complete if each Cauchy sequence in X converges in X.

Definition 3.5. Let $(X_1, \|\cdot\|_{X_1}, L_1, R_1)$ and $(X_2, \|\cdot\|_{X_2}, L_2, R_2)$ be fuzzy normed linear spaces. Then we have the following definition :

(a) A mapping T of X_1 into X_2 is said to be fuzzy-isometry if for all $x, y \in X_1$,

$$\|Tx - Ty\|_{X_2} = \|x - y\|_{X_1}.$$

(b) The spaces X_1 is said to be fuzzy-isometric with the spaces X_2 if there exists a bijective fuzzy-isometry of X_1 into X_2. The spaces X_1 and X_2 are called fuzzy-isometric sapces.

Now we establish the completions of fuzzy normed linear spaces.

Theorem 3.2. *Let G be complete. For a fuzzy normed linear sapces* $X = (X, \|\cdot\|_X,$ *Min, Max), there exists a complete fuzzy normed linear spaces* $\hat{X} = (\hat{X}, \|\cdot\|_{\hat{X}},$ *Min, Max) which has a subspace W being fuzzy-isometric with X and dense in* $\hat{X}$. *This space* $\hat{X}$ *is unique except for fuzzy-isonetries.*

Proof. We subdivide the proof into six steps (a) to (f).

(a) Construction of $\hat{X} = (\hat{X}, \|\cdot\|_{\hat{X}}$, Min, Max).

Let (x_n) and (y_n) be Cauchy sequences in X. Define (x_n) to be equivalent to (y_n), written $(x_n) \sim (y_n)$ if for all $\alpha \in (0, 1]$,

$$\lim_{n\to\infty} \|x_n - y_n\|_X = \bar{0} \quad \text{if and only if} \quad \lim_{n\to\infty} |\|x_n - y_n\|_X|_2^\alpha = 0. \tag{3.1}$$

This obviously defines an equivalent relation. In fact,

$i)$ Since $\lim_{n\to\infty} \|x_n - y_n\|_X = \bar{0}$, $\quad (x_n) \sim (x_n)$,

$ii)$ $(x_n) \sim (y_n)$ if and only if $\lim_{n\to\infty} \|x_n - y_n\|_X = \bar{0}$
$= \lim_{n\to\infty} \|y_n - x_n\|_X$ if and only if $(y_n) \sim (x_n)$,

$iii)$ Since $\lim_{n\to\infty} \|x_n - y_n\|_X \preccurlyeq \lim_{n\to\infty} \|x_n - z_n\|_X \oplus \lim_{n\to\infty} \|z_n - y_n\|_X$,
$(x_n) \sim (z_n)$ and $(z_n) \sim (y_n)$ implies $(x_n) \sim (y_n)$.

Let $\hat{X}$ be the set of all equivalence classes $\hat{x}, \hat{y}, \cdots$ of Cauchy sequences thus obtained. We write $(x_n) \in \hat{x}$ to mean that (x_n) is a member of $\hat{x}$. We now set

$$\|\hat{x} - \hat{y}\|_{\hat{X}} = \lim_{n\to\infty} \|x_n - y_n\|_X, \tag{3.2}$$

where $(x_n) \in \hat{x}$, $(y_n) \in \hat{y}$. We show that this limit exists. We have

$$\|x_n - y_n\|_X \preccurlyeq \|x_n - x_m\|_X \oplus \|x_m - y_m\|_X \oplus \|y_m - y_n\|_X,$$

hence by Proposition 1.4 (2) we obtain

$$\|x_n - y_n\|_X \ominus \|x_m - y_m\|_X \preccurlyeq \|x_n - x_m\|_X \oplus \|y_m - y_n\|_X.$$

Similarly an inequality with m and n interchanged can be obtained. Thus we obtain the following inequality

$$|\|x_n - y_n\|_X \ominus \|x_m - y_m\|_X| \preccurlyeq \|x_n - x_m\|_X \oplus \|y_m - x_m\|_X \longrightarrow \bar{0}$$

as $m, n \longrightarrow \infty$.

Thus $(\|x_n - y_n\|_X)_n$ is a Cauchy sequence in G, since G is complete, hence there exists the limit. Now we show that the limit in (3.2) is independent of the particular choice of representatives. In fact, if $(x_n) \sim (x'_n)$ and $(y_n) \sim (y'_n)$, then by (3.1),

$$|\|x_n - y_n\|_X \ominus \|x'_n - y'_n\|_X| \preccurlyeq \|x_n - x'_n\|_X \oplus \|y_n - y'_n\|_X \longrightarrow \bar{0}$$

as $n \longrightarrow \infty$, which implies the assertion

$$\lim_{n\to\infty} \|x_n - y_n\|_X = \lim_{n\to\infty} \|x'_n - y'_n\|_X.$$

(b) Construction of fuzzy normed linear space $\hat{\mathrm{X}}$.

To define two algebraic operations on $\hat{\mathrm{X}}$, we consider $\hat{x}, \hat{y} \in \hat{\mathrm{X}}$ and any representatives $(x_n) \in \hat{x}, (y_n) \in \hat{y}$. We set $z_n = x_n + y_n$ for each n. Then (z_n) is a Cauchy sequence in X since

$$\begin{aligned}\lim_{\substack{m\to\infty\\ n\to\infty}} |||z_n - z_m||_{\mathrm{X}}|_2^\alpha &= \lim_{\substack{m\to\infty\\ n\to\infty}} |||(x_n + y_n) - (x_m + y_m)||_{\mathrm{X}}|_2^\alpha \\ &\leq \lim_{\substack{m\to\infty\\ n\to\infty}} |||x_n - x_m||_{\mathrm{X}}|_2^\alpha + \lim_{\substack{m\to\infty\\ n\to\infty}} |||y_n - y_m||_{\mathrm{X}}|_2^\alpha\end{aligned}$$

for $\alpha \in (0, 1]$.
We define the sum $\hat{z} = \hat{x} + \hat{y}$ of $\hat{x}$ and $\hat{y}$ to be the equivalence class of (z_n). Then $+$ is a binary operation in $\hat{\mathrm{X}}$. This definition is independent of the particular choice of Cauchy sequences, belong to $\hat{x}$ and $\hat{y}$, respectively.
In fact, if $(x_n) \sim (x'_n)$ and $(y_n) \sim (y'_n)$ then $(x_n + y_n) \sim (x'_n + y'_n)$ because

$$\begin{aligned}&\lim_{n\to\infty} |||x_n + y_n - (x'_n + y'_n)||_{\mathrm{X}}|_2^\alpha \\ &\leq \lim_{n\to\infty} |||x_n - x'_n||_{\mathrm{X}}|_2^\alpha + \lim_{n\to\infty} |||y_n - y'_n||_{\mathrm{X}}|_2^\alpha\end{aligned}$$

for all $\alpha \in (0, 1]$.
Similarly we define the product $\alpha\hat{x} \in \hat{\mathrm{X}}$ of a scalar α and $\hat{x}$ to be the equivalence class of (αx_n), where (x_n) is a Cauchy sequence belonging to $\hat{x}$. Again, this definition is independent of the particular choice of a representative of $\hat{x}$.
In fact, if $(x_n) \sim (x'_n)$ then $(\alpha x_n) \sim (\alpha x'_n)$ because

$$\lim_{n\to\infty} |||\alpha x_n - \alpha x'_n||_{\mathrm{X}}|_2^\alpha = |\alpha| \lim_{n\to\infty} |||x_n - x'_n||_{\mathrm{X}}|_2^\alpha$$

for all $\alpha \in (0, 1]$.
The zero element of $\hat{\mathrm{X}}$ is the equivalence class containing all Cauchy sequence which converge to zero. It is not difficult to see that those two algebraic operations have all the properties required by the definition of a vector space, so that $\hat{\mathrm{X}}$ is a vector space.
Furthermore, We setting $||\hat{x}||_{\hat{\mathrm{X}}} = ||\hat{0} - \hat{x}||_{\hat{\mathrm{X}}}$ for every $\hat{x} \in \hat{\mathrm{X}}$. Then we prove that $\hat{\mathrm{X}}$

is a fuzzy normed linear space. In fact,

$$i)\quad \|\hat{x}\|_{\hat{X}} = \bar{0} \quad \text{if and only if} \quad \hat{x} = \hat{0},$$

$$\begin{aligned} ii)\quad \|\alpha\hat{x}\|_{\hat{X}} &= \|\hat{0} - \alpha\hat{x}\|_{\hat{X}} \\ &= \lim_{n\to\infty} \|0 - \alpha x_n\|_X \\ &= |\alpha| \lim_{n\to\infty} \|0 - x_n\|_X \\ &= |\alpha| \|\hat{0} - \alpha\hat{x}\|_{\hat{X}} \\ &= |\alpha| \|\hat{x}\|_{\hat{X}}, \end{aligned}$$

$$\begin{aligned} iii)\quad \|x + y\|_{\hat{X}} &= \|\hat{0} - (\hat{x} + \hat{y})\|_{\hat{X}} \\ &= \lim_{n\to\infty} \|0 - (x_n + y_n)\|_X \\ &\preccurlyeq \lim_{n\to\infty} \|0 - x_n\|_X \oplus \lim_{n\to\infty} \|0 - y_n\|_X \\ &= \|\hat{0} - \hat{x}\|_{\hat{X}} \oplus \|\hat{0} - \hat{y}\|_{\hat{X}} \\ &= \|\hat{x}\|_{\hat{X}} \oplus \|\hat{y}\|_{\hat{X}}. \end{aligned}$$

(c) Construction of a fuzzy-isometry $T : X \longrightarrow W \subset \hat{X}$.

With each $b \in X$ we associate the class $\hat{b} \in \hat{X}$ which contains the constant Cauchy sequence. This defines a mapping $T : X \longrightarrow W$ onto the subspace $W = T(X) \subset \hat{X}$. The mapping T is given by $b \longmapsto \hat{b} = Tb$, where $(b, b, \cdots) \in \hat{b}$. We easily see that T is a fuzzy-isometry since

$$\|\hat{b} - \hat{c}\|_{\hat{X}} = \|Tb - Tc\|_{\hat{X}} = \|b - c\|_X,$$

here $\hat{c}$ is the class of (y_n) with $y_n = c$ for all n. Any fuzzy-isometry is injective and $T : X \longrightarrow W$ is surjective since $T(X) = W$. Hence W and X are fuzzy-isometric.

(d) Denseness of W in $\hat{X}$.

We consider any $\hat{x} \in \hat{X}$. Let $(x_n) \in \hat{x}$. Since (x_n) is a Cauchy sequence in X, $\lim_{\substack{m\to\infty \\ n\to\infty}} \|x_m - x_n\|_X = \bar{0}$, that is, for every $\epsilon > 0$ there exists a positive intege N such that $\|x_m - x_n\|_X \preccurlyeq \overline{(\frac{\epsilon}{2})}$ for $m, n \geq N$. Then $|\|x_m - x_n\|_X|_2^\alpha \leq \frac{\epsilon}{2}$ for each $\alpha \in (0, 1]$ and $m, n \geq N$. Consider the constant Cauchy sequence $(x_N, x_N, \cdots)$ and let $\hat{x}_N$ be its equivalence class. Since

$$\lim_{n\to\infty} |\|x_n - x_N\|_X|_2^\alpha \leq \frac{\epsilon}{2} < \epsilon \quad \text{for all} \quad \alpha \in (0, 1],$$

$\|\hat{x} - \hat{x}_N\|_{\hat{X}} \preccurlyeq \bar{\epsilon}$ and hence $\hat{x}_N \in$W. Thus W is dense in $\hat{X}$.

(e) Completeness of $\hat{X}$.

Let $(\hat{x}_n)$ be any Cauchy sequence in $\hat{X}$. Since W is dense in $\hat{X}$ and $\hat{x}_n \in \hat{X}$ there is a $\hat{z}_n \in$ W such that

$$|\|x_n - z_n\|_X|_2^\alpha < \frac{1}{n} \quad \text{for any } n \in \mathbb{N}. \tag{3.3}$$

Since $|\|z_m - z_n\|_X|_2^\alpha \leq |\|z_m - x_m\|_X|_2^\alpha + |\|x_m - x_n\|_X|_2^\alpha + |\|x_n - z_n\|_X|_2^\alpha$,

$$\begin{aligned}
&\lim_{\substack{m\to\infty\\ n\to\infty}} |\|z_m - z_n\|_X|_2^\alpha \\
&\leq \lim_{m\to\infty} |\|z_m - x_m\|_X|_2^\alpha + \lim_{\substack{m\to\infty\\ n\to\infty}} |\|x_m - x_n\|_X|_2^\alpha + \lim_{n\to\infty} |\|x_n - z_n\|_X|_2^\alpha \\
&\leq \lim_{m\to\infty} \frac{1}{m} + \lim_{\substack{m\to\infty\\ n\to\infty}} |\|x_m - x_n\|_X|_2^\alpha + \lim_{n\to\infty} \frac{1}{n} = 0, \quad \text{for all} \quad \alpha \in (0,1].
\end{aligned}$$

Thus $\|\hat{z}_m - \hat{z}_n\|_{\hat{X}} \longrightarrow \bar{0}$, as $m, n \longrightarrow \infty$, that is, $(\hat{z}_m)$ is a Cauchy sequence in W. Since T : X $\longrightarrow$ W is a fuzzy-isometry and $(\hat{z}_m) \in$ W, if we define $z_m = T^{-1}\hat{z}_m$ for each m then the sequence (z_m) is a Cauchy sequence in X. Let $\hat{x} \in \hat{X}$ be the equivalent class to which (z_m) belongs. We show that $\hat{x}$ is the limit of the Cauchy sequence $(\hat{x}_n)$. By (3.3)

$$\begin{aligned}
\lim_{n\to\infty} |\|x_n - x\|_X|_2^\alpha &\leq \lim_{n\to\infty} |\|x_n - z_n\|_X|_2^\alpha + \lim_{n\to\infty} |\|z_n - x\|_X|_2^\alpha \\
&< \lim_{n\to\infty} \frac{1}{n} + \lim_{n\to\infty} |\|z_n - x\|_X|_2^\alpha = 0, \quad \text{for all} \quad \alpha \in (0,1].
\end{aligned}$$

Thus $\|\hat{x}_n - \hat{x}\|_{\hat{X}} \longrightarrow \bar{0}$ as $n \longrightarrow \infty$, that is, $\lim_{n\to\infty} \hat{x}_n = \hat{x}$.
This shows that $\hat{X}$ is complete.

(f) Uniqueness of $\hat{X}$ except for fuzzy-isometries.

Let $(\tilde{X}, \ \|\cdot\|_{\tilde{X}}$, Min, Max) be another complete fuzzy metric space with a subspace $\tilde{W}$ dense in $\tilde{X}$ and fuzzy-isometric with X. Then for any $\tilde{x}, \tilde{y} \in \tilde{X}$ we have sequences $(\tilde{x}_n), (\tilde{y}_n)$ in $\tilde{W}$ such that $\tilde{x}_n \longrightarrow \tilde{x}$ and $\tilde{y}_n \longrightarrow \tilde{y}$. Hence the following equality

$$\|\tilde{x} - \tilde{y}\|_{\tilde{X}} = \lim_{n\to\infty} \|\tilde{x}_n - \tilde{y}_n\|_{\tilde{X}}$$

follows from

$$|\|\tilde{x} - \tilde{y}\|_{\tilde{X}} \ominus \|\tilde{x}_n - \tilde{y}_n\|_{\tilde{X}}| \preccurlyeq \|\tilde{x} - \tilde{x}_n\|_{\tilde{X}} \oplus \|\tilde{y} - \tilde{y}_n\|_{\tilde{X}} \longrightarrow \bar{0} \quad \text{as} \quad n \longrightarrow \infty.$$

Since $\tilde{W}$ is fuzzy-isometric with W $\subset \hat{X}$ and cl(W) $= \hat{X}$, the closure of W the fuzzy metries $\tilde{d}$ on $\tilde{X}$ and $\hat{d}$ on $\hat{X}$ must be same. Hence $\tilde{X}$ and $\hat{X}$ are fuzzy-isometric.

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