A NOTE ON THE KY FAN'S RESULT

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1. Introduction

In a normed space, the distance δ from an element $y \in E$ to a nonempty subset X of E is defined to be $\delta = \inf_{x \in X} \|x - y\|$. If there exists an $x \in X$ such that

$$d(X, y) = \|x - y\|,$$

then x is said to be an element of best approximation to y.

In recent years, the study of fixed point theory and the best approximation theory from self-mappings to nonself- mappings has become a very active topic in nonlinear functional analysis [1,5,6,7,8]. A continuous nonself-mapping $f : X \to E$, which is defined on a compact convex subset X of a normed vector space E does not necessarily have a fixed point (for example, if $f(X) \cap X = \emptyset$, then f does not have a fixed point in X). But in 1969, Ky Fan [4] proved the following interesting result

Let E be a normed vector space, X a compact convex subset of E and $f : X \to E$ a continuous mapping, then the following variational inequality

$$\|y - f(y)\| \leqq \|f(y) - x\|, \text{ for } x \in X$$

has a solution in X, i.e., there exists a $y \in X$ such that

$$\|y - f(y)\| = \min_{x \in X} \|x - f(y)\|.$$

This kind of scalar variational inequalities is called Ky Fan's variational inequality and y is called an element of best approximation to f on X.

In this note, we discuss scalar variational inequality problems for set-valued mappings, which are generalization of Ky Fan's result for single-valued mappings.

2. Definitions and preliminaries

Definition 2.1. Let X, Y be two topological spaces and $F : X \to 2^Y$ a set-valued mapping. F is said to have a closed-values, compact-values and convex-values (when Y is a vector space) if for any $x \in X$, $F(x)$ is a closed set, a compact set and a convex set, respectively.

Definition 2.2. Let X, Y be two topological spaces and $F : X \to 2^Y$ a set-valued mapping. An $x_0 \in X$ is called a fixed point of F, if $x_0 \in F(x_0)$.

Example 2.1. The mapping $F : \mathbb{R} \to 2^{\mathbb{R}}$, defined by $F(x) = [x^2, x^2 + 1]$ for each $x \in \mathbb{R}$. Since $x^2 \leqq x \leqq x^2 + 1$ for all $x \in [0, 1]$, $x \in F(x)$ for each $x \in [0, 1]$. Thus for each $x \in [0, 1]$, x is a fixed point of F.

Defintion 2.3. Let X, Y be topological spaces. $F : X \to 2^Y$ is said to be upper semi-continuous at $x_0 \in X$ if for any neighborhood $V(\subset Y)$ of $F(x_0)$ there exists a neighborhood U of x_0 such that $F(U) \subset V$. F is said to be upper semi-continuous if F is upper semi-continuous at all points of X.

Example 2.2. The mapping $F : \mathbb{R} \to 2^{\mathbb{R}}$, defined by

$$F(x) = \begin{cases} [-1, +1] & \text{if } x = 0, \\ \{0\} & \text{if } x \neq 0 \end{cases}$$

is upper semi-continuous.

Lemma 2.1. A map $f : X \to Y$ from a topological space X to a topological space Y is continuous if and only if the inverse image $f^{-1}(V)$ of each open subset V of Y is open in X.

Lemma 2.2 [2]. Let X, Y be Hausdorff topological spaces and $F : X \to 2^Y$ be a set valued mapping. Then the necessary and sufficient condition for the upper semi-continuity of F is the inverse image of an open set is open. i.e., for any open set $V \subset Y$,

$$F^{-1}(V) = \{x \in X : F(x) \subset V\} \text{ is open in } X.$$

Remark 2.1. Lemma 2.1 is a single-valued case of F in Lemma 2.2.

3. Main theorems

In this section we discuss our main result, which can be called a best approximation theorem or variational inequality for set-valued mappings.

Lemma 3.1 [9]. Let X, Y be two topological spaces and $F : X \to 2^Y$ an upper semi-continuous set-valued mapping with compact-values. If $\{x_\alpha\}_{\alpha \in \Gamma}$ be a net in X converging to x and $y_\alpha \in F(x_\alpha)$, $\alpha \in \Gamma$, then there exist a $y \in F(x)$ and a subnet $\{y_\beta\} \subset \{y_\alpha\}_{\alpha \in \Gamma}$ such that $y_\beta \to y$.

Lemma 3.2 [4]. Let X be a nonempty compact convex subset of Hausdorff topological vector space. Let A be a closed subset of $X \times X$ satisfying the following properties:
(1) $(x, x) \in A$ for every $x \in X$.
(2) For any fixed point $y \in X$, the set $\{x \in X \mid (x, y) \notin A\}$ is convex (or empty).
Then there exists a point $y_0 \in X$ such that $X \times \{y_0\} \subset A$.

The following theorem is our main result which is a generalization of Ky Fan's result in [4].

Theorem 3.1. Let X be a nonempty compact convex subset of a normed vector space E. For any nonempty compact and convex-valued upper semi-continuous mapping $F : X \to 2^E$, there exists a point $y_0 \in X$ such that

$$\min_{z \in F(y_0)} \|y_0 - z\| = \min_{x \in X, z \in F(y_0)} \|x - z\|.$$

Proof. Let

$$A := \{(x, y) \in X \times X : \min_{z \in F(y)} \|x - z\| \geqq \min_{z \in F(y)} \|y - z\|\}.$$

Firstly, we show that A is a closed subset of $X \times X$. In fact, let $\{(x_n, y_n)\}$ be a sequence in A converging to (x, y). Then for each n, $\min_{z \in F(y_n)} \|x_n - z\| \geqq \min_{z \in F(y_n)} \|y_n - z\|$, i.e., for each n and for $z_n \in F(y_n)$,

$$\|x_n - z_n\| \geqq \min_{\{z_n : z_n \in F(y_n)\}} \|x_n - z_n\| \geqq \min_{\{z_n : z_n \in F(y_n)\}} \|y_n - z_n\|.$$

Hence for each n and for all $z_n \in F(y_n)$, there exists $z'_n \in F(y_n)$ such that $\|x_n - z_n\| \geqq \|y_n - z'_n\|$. By Lemma 3.1, there exists $z, z' \in F(y)$ and subsequences $\{z_{n_i}\}, \{z'_{n_i}\}$ of $\{z_n\}$ and $\{z'_n\}$ such that $z_{n_i} \to z$ and $z'_{n_i} \to z'$, respectively. Thus $\lim_{i \to \infty} \|x_{n_i} - z_{n_i}\| \geqq \lim_{i \to \infty} \|y_{n_i} - z'_{n_i}\|$. Since $\|\cdot\|$ is continuous,

$$\|\lim_{i \to \infty} x_{n_i} - \lim_{i \to \infty} z_{n_i}\| \geqq \|\lim_{i \to \infty} y_{n_i} - \lim_{i \to \infty} z'_{n_i}\|.$$

Hence $\|x - z\| \geqq \|y - z'\|$. Consequently for all $z \in F(y)$, there exists $z' \in F(y)$ such that $\|x - z\| \geqq \|y - z'\|$. This implies $\min_{z \in F(y)} \|x - z\| \geqq \|y - z'\|$. Since $\|y - z'\| \geqq \min_{z \in F(y)} \|y - z\|$,

$$\min_{z \in F(y)} \|x - z\| \geqq \min_{z \in F(y)} \|y - z\|.$$

Thus we have $(x, y) \in A$. This implies that A is a closed subset of $X \times X$.

Secondly, it is obvious that $(x, x) \in A$ for all $x \in X$.

Thirdly, we show that for each fixed $y \in X$,
$B(y) = \{x \in X \mid \min_{z \in F(y)} \|x - z\| < \min_{z \in F(y)} \|y - z\|\}$ is a convex subset of X. In fact, let $x_1, x_2 \in B(y)$ and $\lambda \in (0, 1)$, then

$$\min_{z \in F(y)} \|x_1 - z\| < \min_{z \in F(y)} \|y - z\| \text{ and } \min_{z \in F(y)} \|x_2 - z\| < \min_{z \in F(y)} \|y - z\|.$$

Since $F(y)$ is compact set, there exist $z' \in F(y)$ such that $\min_{z \in F(y)} \|y - z\| = \|y - z'\|$. And there exist $z_1, z_2 \in F(y)$ such that

$$\min_{z \in F(y)} \|x_1 - z\| \leqq \|x_1 - z_1\| < \|y - z'\|,$$
$$\min_{z \in F(y)} \|x_2 - z\| \leqq \|x_2 - z_2\| < \|y - z'\|.$$

We have

$$\begin{aligned} \lambda\|x_1 - z_1\| &< \lambda\|y - z'\|, \\ (1-\lambda)\|x_2 - z_2\| &< (1-\lambda)\|y - z'\|. \end{aligned}$$

Thus

$$\begin{aligned} &\|\lambda x_1 + (1-\lambda)x_2 - \lambda z_1 - (1-\lambda)z_2\| \\ \leqq\ &\lambda\|x_1 - z_1\| + (1-\lambda)\|x_2 - z_2\| \\ <\ &\|y - z'\|. \end{aligned}$$

Since $F(y)$ is convex set, $\lambda z_1 + (1-\lambda)z_2 \in F(y)$. Hence

$$\begin{aligned} &\min_{z \in F(y)} \|\lambda x_1 + (1-\lambda)x_2 - z\| \\ \leqq\ &\|\lambda x_1 + (1-\lambda)x_2 - (\lambda z_1 + (1-\lambda)z_2)\| \\ <\ &\|y - z'\| \\ =\ &\min_{z \in F(y)} \|y - z\|. \end{aligned}$$

This implies that $\lambda x_1 + (1-\lambda)x_2 \in B(y)$ for all $y \in X$. Therefore for all $y \in X$, $\{x \in X \mid \min_{z \in F(y)} \|x - z\| < \min_{z \in F(y)} \|y - z\|\}$ is convex.

So, the set A satisfies with three properties in Lemma 3.2. Hence we can take a point $y_0 \in X$ such that $X \times \{y_0\} \subset A$. Thus we have $\min_{z \in F(y_0)} \|x - z\| \geqq \min_{z \in F(y_0)} \|y_0 - z\|$ for all $x \in X$. Therefore there exists a point $y_0 \in X$ such that

$$\min_{x \in X, z \in F(y_0)} \|x - z\| = \min_{z \in F(y_0)} \|y_0 - z\|.$$

The proof is completed.

Remark 3.1. In Theorem 3.1, if we remove the condition, convex-valuedness of a set-valued mapping F, then we obtain the following counter-example : Let $E = (\mathbb{R}, |\cdot|)$ and $X = [0, 10] \subset \mathbb{R}$. Define $F : X \to 2^{\mathbb{R}}$ by

$$F(x) = \begin{cases} [x+1, x+2] & \text{if } x \in [0,\ 5], \\ [x-2, x-1] & \text{if } x \in [5, 10]. \end{cases}$$

Then F is upper semi-continuous set-valued mapping with compact-values. But, for all $y_0 \in X$,

$$1 = \min_{z \in F(y_0)} |y_0 - z| \text{ and } \min_{x \in X, z \in F(y_0)} |x - z| = 0.$$

Hence for all $y_0 \in X$, $\min_{z\in F(y_0)} |y_0 - z| \neq \min_{x\in X, z\in F(y_0)} |x - z|$.

Theorem 3.2. Let X be a nonempty compact convex subset of a normed vector space E and $F : X \to 2^E$ be a nonempty compact and convex-valued upper semi-continuous mapping. If for each $y \in X$, $F(y) \cap X \neq \emptyset$, then F has a fixed point in X.

Proof. By Theorem 3.1, there exists a point $y_0 \in X$ such that

$$\min_{z\in F(y_0)} \|y_0 - z\| = \min_{x\in X, z\in F(y_0)} \|x - z\|.$$

By hypothesis, for y_0, $F(y_0) \cap X \neq \emptyset$, i.e., there exists $z' \in X$ such that $z' \in F(y_0) \cap X$. Thus $\min_{x\in X, z\in F(y_0)} \|x - z\| \leq \min_{z\in F(y_0)} \|z' - z\| \leq \|z' - z'\| = 0$. Therefore $\min_{z\in F(y_0)} \|y_0 - z\| = 0$, i.e., $y_0 \in F(y_0)$.

Remark 3.2. In Theorem 3.1, if $F(X) \subset X$ then for each $y \in X$, $F(y) \cap X \neq \emptyset$, since for each $y \in X$, $F(y) \subset X$. Therefore F has a fixed point in X by Theorem 3.2.

Lemma 3.3. Let X be a nonempty compact convex subset of a normed vector space E and $F : X \to 2^E$ be a nonempty set-valued mapping. The following conditions are equivalent ;

(1) There exists a real or complex scalar λ such that $|\lambda| < 1$ and $\lambda y + (1 - \lambda)z \in X$ for all $y \in X$ and some $z \in F(y)$.

(2) For each $y \in X$, $F(y) \cap X \neq \emptyset$.

Proof. For some fixed $y_0 \in X$ and $z \in F(y_0)$ satisfying the condition (1), we can choose a scalar λ_1 such that $|\lambda_1| < 1$ and $\lambda_1 y_0 + (1 - \lambda_1)z \in X$. By taking $y_1 = \lambda_1 y_0 + (1 - \lambda_1)z \in X$, we can also choose a scalar λ_2 such that $|\lambda_2| < 1$ and $\lambda_2 y_1 + (1 - \lambda_2)z \in X$. Put $y_2 = \lambda_2 y_1 + (1 - \lambda_2)z = \lambda_2\lambda_1 y_0 + (1 - \lambda_2\lambda_1)z$. Also we can choose a scalar λ_3 such that $|\lambda_3| < 1$ and $\lambda_3 y_2 + (1 - \lambda_3)z \in X$. It is easily shown that $\lambda_3 y_2 + (1 - \lambda_3)z = \lambda_3\lambda_2\lambda_1 y_0 + (1 - \lambda_3\lambda_2\lambda_1)z$. Continuing this process, we can choose a scalar a_n such that $|a_n| < 1$ and $a_n y_0 + (1 - a_n)z \in X$, where

$a_n = \lambda_n \lambda_{n-1} \cdots \lambda_1$ and $|\lambda_i| < 1$, $1 \le i \le n$. Let $y_n = a_n y_0 + (1 - a_n)z$ for any positive integer n. By the compactness of X, we can take a convergent subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y_{n_k} \to y' \in X$. Since

$$\lim_{k\to\infty} a_{n_k} = \lim_{k\to\infty} \lambda_{n_k}\lambda_{n_k-1}\cdots\lambda_1 = 0,$$
$$y' = \lim_{k\to\infty} y_{n_k} = \lim_{k\to\infty}(a_{n_k}y_0 + (1 - a_{n_k})z) = z.$$

Thus for each $y \in X$, there exists a $z \in F(y)$ such that $z \in X$. This shows that for each $y \in X$, $F(y) \cap X \neq \emptyset$.

Conversely, since for each $y \in X$, $F(y) \cap X \neq \emptyset$, there exists $z \in X$ such that $z \in F(y)$ for each $y \in X$. Since $0 + z \in X$, there exists $\lambda = 0$ such that $|\lambda| < 1$ and $\lambda y + (1 - \lambda)z \in X$, for all $y \in X$.

We can obtain the following Theorem from Theorem 3.2 by using Lemma 3.3.

Theorem 3.4. Let X be a nonempty compact convex subset of a normed vector space E and $F : X \to 2^E$ be a nonempty compact and convex-valued upper semi-continuous mapping. If there exists a real or complex scalar λ such that $|\lambda| < 1$ and $\lambda y + (1 - \lambda)z \in X$ for any $y \in X$ and some $z \in F(y)$, then F has a fixed point in X.

Remark 3.3. Of course we can show the proof of Theorem 3.4 from Theorem 3.1 directly as follows ;

Proof. Suppose that F has not a fixed point in X. By Theorem 3.1, there exists a $y_0 \in X$ such that $0 < \min_{z\in F(y_0)} \|y_0 - z\| = \min_{x\in X, z\in F(y_0)} \|x - z\|$. For y_0 and $z \in F(y_0)$, by hypothesis, there exists λ such that $|\lambda| < 1$ and $x = \lambda y_0 + (1 - \lambda)z \in X$. Then

$$\begin{aligned} 0 < \min_{z\in F(y_0)} \|y_0 - z\| &= \min_{x\in X, z\in F(y_0)} \|x - z\| \\ &\le \min_{z\in F(y_0)} \|\lambda y_0 + (1-\lambda)z - z\| \\ &= |\lambda| \min_{z\in F(y_0)} \|y_0 - z\|, \end{aligned}$$

which contradicts $|\lambda| < 1$.

If F is single-valued in Theorem 3.1, then we obtain the following Ky Fan's result as a corollary.

Corollary 3.1 [4]. Let X be a nonempty compact convex subset of a normed vector space E. For any continuous mapping $f : X \to E$, there exists a point $y_0 \in X$ such that

$$\|y_0 - f(y_0)\| = \min_{x \in X} \|x - f(y_0)\|.$$

In particular, if $f(X) \subset X$, then y_0 is a fixed point of f.

Proof. It can be proved by Remark 2.1 from Theorem 3.1.

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