AN ISOMORPHISM BETWEEN R^2-PLANES

JANG-HWAN IM

Abstract. We define a vertical mapping between two standard R^2-planes. Then we can show that a vertical mapping is an isomorphism between two standard R^2-planes if and only if the induced product space of them has a plane which is neither vertical nor horizontal planes.

1. Introduction

A topological R^3-space is an incidence structure $(\mathcal{P}^3, \mathcal{L})$ which satisfies the following three axioms:

(1) The incidence structure $(\mathcal{P}^3, \mathcal{L})$ is a linear space, i.e., each pair p, q of distinct points is contained in a unique line $p \vee q \in \mathcal{L}$.

(2) The point set $\mathcal{P}^3$ is a topological space homeomorphic to the real line R.

(3) The mapping $\vee : \mathcal{P}^3 \times \mathcal{P}^3 \setminus \triangle \longrightarrow \mathcal{L}$ is continuous, where $\triangle := \{(p,p) : p \in \mathcal{P}^3\}$ denotes the diagonal and $\mathcal{L}$ carries the topology of Hausdorff-convergence.

A plane in a R^3-space $(\mathcal{P}^3, \mathcal{L})$ is a closed subset $E \subseteq \mathcal{P}^3$ which is homeomorphic to R^2 such that $p \vee q \subseteq E$ for each pair of distinct points $p, q \in E$. Obviously, $(E, \mathcal{L}_E)$ is a R^2-plane, where $\mathcal{L}_E := \{l \in \mathcal{L} : l \subseteq E\}$. A R^3-space is called *planar* if each triple $p, q, r \in \mathcal{P}^3$ of non-collinear points is contained in a (unique) plane. The theory of planar R^3-spaces has been developed in the papers [1, 2, 3, 4]. In particular, it is shown that each planar R^3-spaces can be embedded in the ordinary 3-dimensional affine space over the real numbers as a convex subset, and that the group of all collineations is

Key words: topological geometry, space geometry.
1991 AMS Subject Classification: 51H10.

a Lie group. This implies that if we construct another type of R^3-spaces, planar property must be removed. In [3] Betten introduced new models in topological R^3-spaces.

Among them we are interested in the class of product spaces of two standard R^2-planes. A R^2-plan $(R^2, \mathcal{L})$ is called *standard* if all vertical lines $\{x\} \times R$ are in $\mathcal{L}$ and the other lines $l \in \mathcal{L}$ can be written as the graph(f) of a continuous mapping $f : R \longrightarrow R$. Let $E_1 = (R^2, \mathcal{L})$ and $E_2 = (R^2, \Im)$ be two standard R^2-planes. We identify E_1 with the horizontal plane $z = 0$ and E_2 with the vertical plane $y = 0$ in $R^3 = \{(x, y, z) : x, y, z \in R\}$, respectively. We define on R^3 the following curves as lines: $f \times g := \{(x, f(x), g(x)) : x \in R\}$, where f and g lines of E_1 and E_2, respectively. Then it can be shown that for each two points (x_1, y_1, z_1) and (x_2, y_2, z_2) with $x_1 \neq x_2$ there exists a unigue line $f \times g$ which contains the given two points. To be a R^3-space, each vertical plane $\{x\} \times R^2$ with $x \in R$ must be a R^2-plane. In [3] he asked some questions about the geometrical structures. One of them is "can we determine all planes of a given product space?". The already existed planes in a given product space are the planes over the lines on E_1 and over the lines on E_2, respectively.

In [6, 7] R^2-divisible R^3-spaces are studied. A product space of two standard R^2-planes is rather appropriate to a R^2-divisible R^3-space. If we regard these product spaces of two standard R^2-planes as R^2-divisible R^3-spaces, then we prove that the following result which answers one of Betten's questions in [3]: *in a product space of two standard R^2-planes, two given planes are not isomorphic if and only if there exist no further planes.* In the proof of this result we found a nice mapping, so-called a *vertical mapping* .

We start with some basic definitions of R^2-divisible R^3-spaces. Let $\mathcal{P}^n$ denote a topological space which is homeomorphic to R^n. A partition $\Lambda := \{S_i : i \in \mathcal{A}\}$ in $\mathcal{P}^n$ $(n \geq 2)$ is *divisible* if each S_i is closed in $\mathcal{P}^n$ and homeomorphic to $\mathcal{P}^{n-1}$. The topology of Hausdorff-convergence can be defined on the set $\mathcal{U}$ of all non-empty closed subsets of $\mathcal{P}^3$. It is defined by an explicit metric which is described in [5, Chap. 1.3].

0.1 Definition. Let $\mathcal{L}$ be a system of subsets of $\mathcal{P}^3$, and let $\Lambda = \{S_i : i \in \mathcal{A}\}$ be a divisible partition in $\mathcal{P}^3$. The elements of $\mathcal{P}^3$ are called points, and the elements of $\mathcal{L}$ are called lines. We say that $(\mathcal{P}^3, \mathcal{L}, \Lambda)$ is a topological R^2-divisible R^3-space if the following axioms hold:

(1) Each line is closed in the topological space $\mathcal{P}^3$ and is homeomorphic to R.

(2) For all $x \in S_i, y \in S_j$ with $i \neq j$ there is a unique line $l \in \mathcal{L}$ with $x, y \in l$. For $i = j$ there are no lines $l \in \mathcal{L}$ with $x, y \in l$.

(3) The mapping

$$\vee : \mathcal{P}^3 \times \mathcal{P}^3 \setminus \cup_{i \in \mathcal{A}}(S_i \times S_i) \longrightarrow \mathcal{L}$$

is continuous, where $\mathcal{L}$ has the induced topology of Hausdorff-conver -gence.

The joining line in (2) is denoted by $l = x \vee y$. Let $\mathcal{P}^3_{S_i} \times \mathcal{P}^3_{S_j}$ denote the set $\mathcal{P}^3 \times \mathcal{P}^3 \setminus \cup_{i \in \mathcal{A}}(S_i \times S_i)$. If $\Lambda = \{S_i : i \in \mathcal{A}\}$ is a divisible partition in $\mathcal{P}^2$, then we can similarly define a R-divisible R^2-plane $(\mathcal{P}^2, \mathcal{L}, \Lambda)$. Let $(\mathcal{P}^3, \mathcal{L}, \Lambda)$ be a R^2-divisible R^3-space. Then we will consider the following additional axiom:

(Exc) (Continuously existence condition for planes) Given three points $a, b, c \in \mathcal{P}^3$ with $(a, b) \in \mathcal{P}^3_{S_i} \times \mathcal{P}^3_{S_j}$, $c \in S_c$ and $c \notin a \vee b$, then there exists a continuous mapping $\varphi : J \longrightarrow \mathcal{E}$ such that $c \vee z \subseteq \varphi(z)$ for all $z \in J$, where $J = [a, b]$ if $S_c \cap [a, b] = \emptyset$, $J = [a, b] \setminus \{w\}$ if $S_c \cap [a, b] = \{w\}$.

0.2 Definition. Let $(\mathcal{P}^3, \mathcal{L}, \Lambda)$ be a R^2-divisible R^3-space. A subset $E \subseteq \mathcal{P}^3$ is called a plane of $(\mathcal{P}^3, \mathcal{L}, \Lambda)$ if the following conditions hold:

(1) E is closed in $\mathcal{P}^3$ and homeomorphic to R^2,

(2) $(E, \mathcal{L}_E, \Lambda_E)$ is a R-divisible R^2-plane, where $\mathcal{L}_E := \{l \in \mathcal{L} : l \subseteq E\}$ and $\Lambda_E = \{E \cap S_i : i \in \mathcal{A}\}$ is a divisible partition in E.

0.3 Definition. Let $(\mathcal{P}^3, \mathcal{L}, \Lambda)$ be a topological R^2-divisible R^3-space. A subset $E \subseteq \mathcal{P}^3$ is called an *incidence plane* of $(\mathcal{P}^3, \mathcal{L}, \Lambda)$ if it satisfies the following properties:

(1) If $x, y \in E$ with $x \vee y \in \mathcal{L}$, then $x \vee y \subseteq E$.

(2) E is non-trivial, i.e., $E \neq \mathcal{P}^3$, and E is not contained in a line.

If $l, g \in \mathcal{L}$ are two lines with $l \cap g \neq \emptyset$, then their intersection is a point, which will be denoted by $l \wedge g$. Let $(\mathcal{P}^3, \mathcal{L}, \Lambda)$ be a R^2-divisible R^3-space. Since lines are homeomorphic to R, there is a natural notion of intervals in lines. If $l \in \mathcal{L}$ is a line and $p, q \in l$ are two (not necessarily distinct) points on l, then we denote the *interval* which consists of all points on l between p and q by the symbol $[p,q]$. The *open interval* between p and q is defined as $(p,q) := [p,q] \setminus \{p,q\}$. A subset $K \subseteq \mathcal{P}^3$ is called *convex* if it contains with each pair $p,q \in K$ also the interval $[p,q]$. If $a,b,c \in \mathcal{P}^3$ are three non-collinear points, then the *triangle* with vertices a,b,c is the following set: $[c,[a,b]] := \{x \in \mathcal{P}^3 : \exists p \in [a,b] \text{ such that } x \in [c,p]\}$. In R^2-planes we can consider the same definitions. A R^2-plan $(R^2,\mathcal{L})$ is called *standard* if all the vertical lines $\{x\} \times R$ are in $\mathcal{L}$ and the other lines $l \in \mathcal{L}$ can be written as the graph(f) of a continuous mapping $f : R \longrightarrow R$.

(Product spaces of two standard R^2-planes) Let $E_1 = (R^2,\mathcal{L})$ and $E_2 = (R^2,\Im)$ be two standard R^2-planes. We identify E_1 with the horizontal plane $z = 0$ and E_2 with the vertical plane $y = 0$ in $R^3 = \{(x,y,z) : x,y,z \in R\}$, respectively. We define on R^3 the following curves as lines: $f \times g := \{(x,f(x),g(x)) : x \in R\}$, where f and g lines of E_1 and E_2, respectively. Then we can construct on R^3 a R^2-divisible R^3-space.

0.4 Definition. Let $E_1 = (R^2,\mathcal{L})$ and $E_2 = (R^2,\Im)$ be standard R^2-planes. Let $\mathcal{L} \times \Im = \{f \times g : f \in \mathcal{L}, g \in \Im\}$ and let $\Lambda = \{\{x\} \times R^2 : x \in R\}$. The incidence structure $(R^3,\mathcal{L} \times \Im,\Lambda)$ is called the product space of two standard R^2-planes E_1 and E_2 and written by $(R^3,\mathcal{L} \times \Im,\Lambda)_{E_1 \times E_2}$. In a product space $(R^3,\mathcal{L} \times \Im,\Lambda)_{E_1 \times E_2}$ there exist always the planes on the lines of E_1 and the planes on the line of E_2. A plane on a line of E_1 is called a *vertical plane*, and a plane on a line of E_2 is called a *horizontal plane.*

0.5 Lemma. *Let $(R^3,\mathcal{L} \times \Im,\Lambda)_{E_1 \times E_2}$ be the product space of two standard R^2-planes E_1 and E_2. Then $(R^3,\mathcal{L} \times \Im,\Lambda)_{E_1 \times E_2}$ is a topological R^2-divisible R^3-space which satisfies the axiom (Exc).*

Proof. [6, Abschnitt 9] □

0.6 Lemma. *Let $(R^3,\mathcal{L} \times \Im,\Lambda)_{E_1 \times E_2}$ be the product space of two standard R^2-planes E_1 and E_2. Then each incidence plane $E \subseteq R^3$ is a plane of $(R^3,\mathcal{L},\Lambda)$.*

Proof. [7, Th. 4.17] □

0.7 Lemma. *Let $E \subseteq R^3$ be a plane of $(R^3,\mathcal{L} \times \Im,\Lambda)_{E_1 \times E_2}$. Then:*

(1) If E contains two vertical points, i.e., $(x,y,z_1),(x,y,z_2) \in E$ with $z_1 \neq z_2$, then E is a vertical plane.

(2) If E contains two horizontal points, i.e., $(x,y_1,z),(x,y_2,z) \in E$ with $y_1 \neq y_2$, then E is a horizontal plane.

Proof. (1). Let p, q be two vertical points with $p, q \in E$. By [7, lemma 2.2], the joining line $l := p \vee q = \{x\} \times \{y\} \times R$ is contained in E. Let $a \in E \setminus \{x\} \times R^2$, and let V be a vertical plane with $a \in V$ and $l \subseteq V$. Then $a \vee p$ and $a \vee q$ lie on E. Consequently, $E = V$ is a vertical plane.
(2). The assertation can be proved as (1). □

(**Vertical Mappings**) Let $(R^2, \mathcal{L})$ and $(R^2, \Im)$ be two standard R^2-planes. Choose two distinct (not vertical) lines $f_1 = \{(x, f_1(x)) : x \in R\}, f_2 = \{(x, f_2(x)) : x \in R\} \in \mathcal{L}$ and $g_1 = \{(x, g_1(x)) : x \in R\}, g_2 = \{(x, g_2(x)) : x \in R\} \in \Im$, respectively, such that $f_1 \wedge f_2, g_1 \wedge g_2 \in \{c\} \times R$ for some $c \in R$. Then the pair of lines f_1, f_2 and the pair of lines g_1, g_2 determine four convex open sets in R^2, respectively. We denote the four convex open sets by A_i, B_i, $i = 1,2,3,4$, respectively. For $p \in A_i$ let $\mathcal{L}_p^2 = \{f \in \mathcal{L}_p : f \wedge f_1 \neq \emptyset, f \wedge f_2 \neq \emptyset, \text{ f is not vertical }\}$, and for $p \in B_i$ let $\Im_p^2 = \{g \in \Im_p : g \wedge g_1 \neq \emptyset, g \wedge g_2 \neq \emptyset, \text{ g is not vertical }\}$. Then it can be easily shown that $|\mathcal{L}_p^2| \geq 2$ and $|\Im_p^2| \geq 2$. Let $(x,y) \in A_i, i \in \{1,2,3,4\}$. Then we can choose two distinct lines $h_1 = \{(x, h_1(x)) : x \in R\}$ and $h_2 = \{(x, h_2(x)) : x \in R\}$ in $\mathcal{L}_{(x,y)}^2$. Let $h_1 \wedge f_1 := (x_1, y_1), h_1 \wedge f_2 := (x_2, y_2), h_2 \wedge f_1 := (x_3, y_3), h_2 \wedge f_2 := (x_4, y_4)$. Since non-vertical lines can be written as the graph(f) of a continuous function. We have also following intersection points: $\{x_1\} \times R \wedge g_1 := (x_1, z_1), \{x_2\} \times R \wedge g_2 := (x_2, z_2), \{x_3\} \times R \wedge g_1 := (x_3, z_3), \{x_4\} \times R \wedge g_2 := (x_4, z_4)$. Hence there exist two lines $j_1, j_2 \in \Im$ with $j_1 = (x_1, z_1) \vee (x_2, z_2)$ and $j_2 = (x_3, z_3) \vee (x_4, z_4)$. Since each triangle is convex in R^2-planes, it is clear that $j_1 \wedge j_2 (\neq \emptyset) \in B_i, i = 1,2,3,4$,respectively. *If the intersection point $j_1 \wedge j_2$ is uniquely determined by choosing h_1, h_2 independently, and if $j_1 \wedge j_2, h_1 \wedge h_2$ lie on the same vertical line $\{d\} \times R$, then we can consider a mapping in the following manner:*

$$\varphi : (R^2, \mathcal{L}) \longrightarrow (R^2, \Im);$$

$$(x, f_1(x)) \longrightarrow (x, g_1(x)),$$

$$(x, f_2(x)) \longrightarrow (x, g_2(x)),$$

$$\text{for } (x,y) \in A_i, i = 1,2,3,4, \text{ and } h_1, h_2 \in \mathcal{L}^2_{(x,y)}$$

$$(x,y) = h_1 \wedge h_2 \longrightarrow j_1 \wedge j_2 = (x,z), j_1, j_2 \in \Im^2_{(x,z)}$$

The above-defined mapping φ is called a *vertical mapping* between $(R^2, \mathcal{L})$ and $(R^2, \Im)$. For a standard R^2-plane $(R^2, \mathcal{L})$ there exists always a vertical mapping $\varphi : (R^2, \mathcal{L}) \longrightarrow (R^2, \mathcal{L})$ which is identity.

2. Main Results

The main result of this section is the following theorem.

0.8 Theorem. *Let $E_1 = (R^2, \mathcal{L})$ and $E_2 = (R^2, \Im)$ be two standard R^2-planes, and let $(R^3, \mathcal{L} \times \Im, \Lambda)_{E_1 \times E_2}$ be the product space of E_1 and E_2. If there exists a plane which is neither vertical nor horizontal plane if and only if there exists a vertical mapping $\varphi : (R^2, \mathcal{L}) \longrightarrow (R^2, \Im)$ which is an isomorphism.*

Proof. Suppose that E is a plane which is neither vertical nor horizontal plane. Let $p \in E$, and let $l, h \in \mathcal{L}$ with $l, h \subseteq E$ and $l \wedge h = p$. Since E is neither vertical nor horizontal plane, we can denote two lines l, h in the following manner: $l = \{(x, f_1(x), g_1(x)) : x \in R\}$, $h = \{(x, f_2(x), g_2(x)) : x \in R\}$ with $f_1 \neq f_2 \in \mathcal{L}$, $g_1 \neq g_2 \in \Im$ and $p = (x_0, y_0, z_0) = l \wedge h$. The two lines l and g determine in E four convex open sets K_1, K_2, K_3 and K_4. By the projections on $< x, y >$- and $< x, z >$-coordinate planes, we can consider the projections of two lines l and h in the plane E_1 and E_2, respectively. Let $l_1 := \{(x, f_1(x)) : x \in R\}$, $h_1 := \{(x, f_2(x)) : x \in R\}$ in E_1, and let $l_2 := \{(x, g_1(x)) : x \in R\}$, $h_2 := \{(x, g_2(x)) : x \in R\}$ in E_2. Since $p = (x_0, y_0, z_0) \in l \wedge h$, it follows that $l_1 \wedge h_1 = (x_0, y_0)$ and $l_2 \wedge h_2 = (x_0, z_0)$. Through l_1 and h_1 (l_2 and h_2, respectively) E_1 (E_2, respectively) is divided into four convex open sets A_i (B_i, respectively), $i = 1,2,3,4$. For $p \in A_i$ (B_i, respectively) , $i = 1,2,3,4$, let $\mathcal{L}^2_p = \{f \in \mathcal{L}_p : f \wedge l_1 \neq \emptyset, f \wedge h_1 \neq \emptyset, f$ is not vertical $\}$ and $\Im^2_p = \{g \in \Im_p : g \wedge l_2 \neq \emptyset, g \wedge h_2 \neq \emptyset, g$ is not vertical $\}$. We consider the both R^2-planes at the same time on $R^2 = \{(x,y) : x, y \in R\}$. Then it is clear that $l_1 \wedge h_1$, $l_2 \wedge h_2 \in \{y_0\} \times R$. Let $(x,y) \in A_i, i = 1,2,3,4$, and choose $k_1, k_2 \in \mathcal{L}^2_{(x,y)}$ with $k_1 \neq k_2$. Let $k_1 \wedge l_1 = (x_1, y_1)$, $k_1 \wedge h_1 = (x_2, y_2)$, $k_2 \wedge l_1 = (x_3, y_3)$ and $k_2 \wedge h_1 = (x_4, y_4)$. Let $\{x_1\} \times R \wedge l_2 := (x_1, z_1)$, $\{x_2\} \times R \wedge h_2 = (x_2, z_2)$, $\{x_3\} \times R \wedge l_2 := (x_3, z_3)$

and $\{x_4\} \times R \wedge h_2 =: (x_4, z_4)$. Hence let $j_1, j_2 \in \Im$ with $j_1 = (x_1, z_1) \vee (x_2, z_2)$ and $j_2 = (x_3, z_3) \vee (x_4, z_4)$. Since each triangle in E_2 is convex, it is true that $j_1 \wedge j_2 \neq \emptyset \in B_i$, $i = 1, 2, 3, 4$. Let $m_1 := \{x, k_1(x), j_1(x)) : x \in R\}$ and $m_2 := \{(x, k_2(x), j_2(x)) : x \in R\}$. Since each triangle in E is convex, it follows that $m_1 \wedge m_2 (\neq \emptyset) = (x, y, z) \in K_i$, $i = 1, 2, 3, 4$. Consequently we have $j_1 \wedge j_2 = (x, z)$. This shows that $k_1 \wedge k_2, j_1 \wedge j_2 \in \{y\} \times R$. We can define a mapping in the following manner:

$$\varphi : (R^2, \mathcal{L}) \longrightarrow (R^2, \Im);$$

$$(x, f_1(x)) \longrightarrow (x, g_1(x)),$$

$$(x, f_2(x)) \longrightarrow (x, g_2(x)),$$

$$\text{for } (x, y) \in A_i, i = 1, 2, 3, 4, \text{ and } h_1, h_2 \in \mathcal{L}^2_{(x,y)}$$

$$(x, y) = h_1 \wedge h_2 \longrightarrow j_1 \wedge j_2 = (x, z), j_1, j_2 \in \Im^2_{(x,z)}$$

We have to show that φ is well-defined, i.e., the mapping φ is independent of choosing of $k_1, k_2 \in \mathcal{L}^2_{(x,y)}$. Let $\bar{k_1}, \bar{k_2} \in \mathcal{L}^2_{(x,y)}$ with $\bar{k_1} \neq \bar{k_2}$, and let $\bar{k_1} \wedge l_1 = (\bar{x_1}, \bar{y_1})$, $\bar{k_1} \wedge h_1 = (\bar{x_2}, \bar{y_2})$, $\bar{k_2} \wedge l_1 = (\bar{x_3}, \bar{y_3})$ and $\bar{k_2} \wedge h_1 = (\bar{x_4}, \bar{y_4})$. Similarly, let $\bar{j_1} := (\bar{x_1}, \bar{z_1}) \vee (\bar{x_2}, \bar{z_2})$ and $\bar{j_2} := (\bar{x_3}, \bar{z_3}) \vee (\bar{x_4}, \bar{z_4})$ with $(\bar{x_1}, \bar{z_1}), (\bar{x_3}, \bar{z_3}) \in l_2$ and $(\bar{x_2}, \bar{z_2}), (\bar{x_4}, \bar{z_4}) \in h_2$. Then $\bar{m_1} := \{(x, \bar{k_1}(x), \bar{j_1}(x)) : x \in R\}$ and $\bar{m_2} := \{(x, \bar{k_2}(x), \bar{j_2}(x)) : x \in R\}$ lie on E. Since each triangle in E is convex, it follows that $\bar{m_1} \wedge \bar{m_2} = (x, y, \bar{z})$. By lemma 1.7, it follows that $z = \bar{z}$. φ is well-defined. Obviously φ is injective. By lemma 1.7, φ is surjective. By convexity in E, it follows that for $l \in \mathcal{L}$ $l^\varphi \in \Im$. Cnsequently, φ is an isomorphism.

Suppose that there exists a vertical mapping $\varphi : (R^2, \mathcal{L}) \longrightarrow (R^2, \Im)$ which is an isomorphism. Then there exist two distinct (not vertical) lines $f_1 = \{(x, f_1(x)) : x \in R\}, f_2 = \{(x, f_2(x)) : x \in R\} \in \mathcal{L}$ and $g_1 = \{(x, g_1(x)) : x \in R\}, g_2 = \{(x, g_2(x)) : x \in R\} \in \Im$, respectively, such that $f_1 \wedge f_2, g_1 \wedge g_2 \in \{c\} \times R$ for some $c \in R$. Let $l := \{(x, f_1(x), g_1(x)) : x \in R\}$ and $h := \{(x, f_2(x), g_2(x)) : x \in R\}$. Then $l, h \in \mathcal{L}$ and $l \wedge h \neq \emptyset$. Let $E := \{r \in R^3 :$ there exist $p \in l, q \in h$ such that $r \in p \vee q\}$. Since φ is an isomorphism between $(R^2, \mathcal{L})$ and $(R^2, \Im)$, it implies that for the pair of points $r_1, r_2 \in E$, $r_1 \vee r_2$ is contained in E. It is clear that $E \neq R^3$. Therefore, E is an incidence plane. By lemma 1.6, E is a plane of $(R^3, \mathcal{L} \times \Im, \Lambda)_{E_1 \times E_2}$ □

0.9 Corollary. *Let $E_1 = (R^2, \mathcal{L})$ and $E_2 = (R^2, \Im)$ be two standard R^2-planes, and let $(R^3, \mathcal{L} \times \Im, \Lambda)_{E_1 \times E_2}$ be the product space of E_1 and E_2. Then:*

(1) If there exists a plane which is neither vertical nor horizontal plane, then two $E_1 = (R^2, \mathcal{L})$ *and* $E_2 = (R^2, \Im)$ *are isomorphic.*

(2) If $E_1 = E_2 = (R^2, \mathcal{L})$, *then there exist planes which are non-vertical or horizontal.*

References

[1] Betten, D.: Topologische Geometrien auf 3-Mannigfaltigkeiten, Simon Stevin 55(1981), 221-235.

[2] Betten, D.: Flexible Raumgeometrien, Atti. Sem. Mat. Fis. Modena 24(1985), 173-180.

[3] Betten, D.: Einige Klassen topologischer 3-R□ume, Resultate der Math. 12(1987), 37-61.

[4] Betten, D, Horstmann, C.: Einbetting von topologischen Raumgeometrien auf R^3 in den reellen affinen Raum, Resultate der Math. 6(1983), 27-35.

[5] Busemann, H.: The geometry of geodesics, Academic press, New York 1965.

[6] Im, J.-H.: R^2-divisible R^3-R□ume, Dissertation, Univ. Kiel 1996.

[7] Im, J.-H.: A class of topological space geometries (to appear).

[8] Salzmann, H.: Topological planes, Adv. in Math. 2(1967), 1-60.

[9] Salzmann, H.,Betten, D., Grundh□fer, T., H□hl, H., L□wen, R., Stroppel, M.: Compact Projective Planes, De Gruyter. Berin. New York 1995.

Department of Mathematics in Faculty of Natural Science, CHUNG-ANG University, 221 Heuksuk-Dong, Dongjak-Ku, Seoul 156-756, Korea.

e-mail: jhim@unitel.co.kr