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On the Estimation of Reliability Functions for the Freund Model

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Abstract This paper considers the problem of estimating the model parameters and reliability functions for Freund bivariate exponential distribution. Uniformly minimum variance unbiased estimators for model parameters, joint reliability and marginal reliability functions are obtained in the both case of non-identically distributed marginals and identically distributed marginals.

1. Introduction

Several bivariate models based on exponential distributions have been derived in the literature to describe failures of two component system. Freund(1961) considered the bivariate exponential(BVE) model as follows. To start with the failure times of two components are independent exponentials with failure rates α and β respectively and if component 1 fails first, then the failure time distribution of component 2 changes to an exponential distribution with failure rate β'. On the other hand if component 2 fails first, then the failure time distribution of component 1 changes to an exponential distribution with failure rate α'. If X and Y denote the failure times of two components, the joint probability density function(pdf) of X and Y is

$$f(x, y) = \begin{cases} \alpha\beta' \exp\{-\beta'(y-x)-(\alpha+\beta)x\}, & 0 < x < y \\ \alpha'\beta \exp\{-\alpha'(x-y)-(\alpha+\beta)y\}, & 0 < y < x. \end{cases} \tag{1}$$

Block and Basu(1974) and Weier(1981) models are sub-models of Freund. Kunchur and Munoli(1994) obtained the uniformly minimum variance unbaised estimator(UMVUE) of the parallel system reliability $P(\max(X,Y) > t_0)$ for the Freund model in the case of $\alpha \neq \beta$ and $\alpha' \neq \beta'$, where t_0 is mission time.

In the above works, the problem of estimating for the joint reliability and the

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marginal reliability functions of two component system have not been considered. In this paper, the problem of estimating the model parameters, the joint reliability function, and the marginal reliability functions are considered for Freund BVE distribution in the both cases of identically distributed marginals and non-identically distributed marginals.

In Section 2, the UMVUE's of parameters, $P(X>u, Y>v)$ and $P(X>u)=P(Y>u)$, are obtained for the case where $\alpha=\beta$ and $\alpha'=\beta'$. Finally, in Section 3, small sample simulations are made to compare the UMVUE with the maximum likelihood estimator(MLE).

2. Estimation of Reliability Functions

Consider the model (1) with $\alpha=\beta$ and $\alpha'=\beta'$. This case corresponds to Freund BVE distribution which has two identical marginals, and the joint pdf is given by,

$$f(x,y)=\lambda_1\lambda_2\exp\{-\lambda_2|x-y|-2\lambda_1\min(x,y)\},\ 0<x,y<\infty \tag{2}$$

where $\lambda_1=\alpha=\beta$ and $\lambda_2=\alpha'=\beta'$. For a paired sample of size n, the statistics $W_1=2\Sigma_{i=1}^{n}\min(X_i,Y_i)$ and $W_2=\Sigma_{i=1}^{n}|X_i-Y_i|$ are jointly complete sufficient statistics for λ_1 and λ_2, and the distribution of W_i is $G(n,(3-i)\lambda_i)$, $i=1,2$, where $G(n,b)$ denotes the gamma pdf of the form $\gamma(x)=b^n\exp(-bx)x^{n-1}/\Gamma(n)$,
$x>0, b>0, n>0$.

The MLE of λ_i is $\hat{\lambda}_i=n/W_i, i=1,2$. These estimators are biased by a factor of $n/(n-1)$ so the estimator

$$\tilde{\lambda}_i=\frac{n-1}{n}\hat{\lambda}_i,\ i=1,2 \tag{3}$$

is UMVUE of λ_i.

We now find the UMVUE of the joint reliability function, $S(u,v)$;

$$S(u,v)=\frac{\lambda_1}{2\lambda_1-\lambda_2}\exp\{-(2\lambda_1-\lambda_2)\min(u,v)-\lambda_2\max(u,v)\}$$
$$+\frac{\lambda_1-\lambda_2}{2\lambda_1-\lambda_2}\exp\{-2\lambda_1\max(u,v)\},\qquad if\ 0<u,v<\infty. \tag{4}$$

Clearly, $\phi(X_1,Y_1)=I[X_1>u,Y_1>v]$, where $I[A]$ is an indicator function of a set A, is an unbiased estimator of $S(u,v)$ and the estimator $E[\phi(X_1,Y_1)|\omega_1,\omega_2]$ is the

UMVUE of $S(u,v)$. The conditional joint pdf of (X_1,Y_1) given $(W_1,W_2)=(\omega_1,\omega_2)$ is then given by

$$h(x_1,y_1|\omega_1,\omega_2)=M_0(\omega_1-z_1)^{n-2}(\omega_2-z_2+z_1)^{n-2}, if\ z_2<\omega_2+z_1 \quad \text{and} \quad 0<z_1<\omega_1, \tag{5}$$

where $M_0=(n-1)^2/(2\omega_1^{n-1}\omega_2^{n-1})$, $z_1=\min(x_1,y_1)$, and $z_2=\max(x_1,y_1)$.

To find $E[\phi(X_1,Y_1)|\omega_1,\omega_2]$, consider the four cases.

Case 1 : $x_1<y_1$, $u<\omega_1$ and $v<\omega_1$. Here,

$$\tilde{S}(u,v)=\frac{2M_0(\omega_1-v)^{n-1}\omega_2^{n-1}}{(n-1)^2}+\frac{M_0}{n-1}\int_u^{\omega_1}(\omega_1-x_1)^{n-2}(\omega_2-v+x_1)^{n-1}dx_1. \tag{6a}$$

Case 2 : $x_1<y_1$, $u<\omega_1<v$. Here,

$$\tilde{S}(u,v)=\frac{M_0}{n-1}\int_u^{\omega_1}(\omega_1-x_1)^{n-2}(\omega_2-v+x_1)^{n-1}dx_1. \tag{6b}$$

Case 3 : $x_1>y_1$, $u<\omega_1$ and $v<\omega_1$. Here,

$$\tilde{S}(u,v)=\frac{M_0(\omega_1-u)^{n-1}\omega_2^{n-1}}{(n-1)^2}+\frac{M_0(\omega_1-v)^{n-1}\omega_2^{n-1}}{(n-1)^2}$$
$$+\frac{M_0}{n-1}\int_v^{\omega_1}(\omega_1-y_1)^{n-2}(\omega_2-u+y_1)^{n-1}dy_1. \tag{6c}$$

Case 4 : $x_1>y_1$ and $v<\omega_1<u$. Here,

$$\tilde{S}(u,v)=\frac{M_0}{n-1}\int_v^{\omega_1}(\omega_1-y_1)^{n-2}(\omega_2-u+y_1)^{n-1}dy_1. \tag{6d}$$

We next find the UMVUE of the identical marginal reliability function, $S(u)$;

$$S(u)=\frac{\lambda_1-\lambda_2}{2\lambda_1-\lambda_2}e^{-2\lambda_1 u}+\frac{\lambda_1}{2\lambda_1-\lambda_2}e^{-\lambda_2 u},\ if\ 2\lambda_1\neq\lambda_2. \tag{7}$$

From (5), the conditional pdf of X_1 given $(W_1,W_2)=(\omega_1,\omega_2)$ is obtained as

$$h_1(x_1|\omega,\omega_2)=\begin{cases}\frac{M_0}{n-1}(\omega_1-x_1)^{n-2}\omega_2^{n-1}+M_0\int_0^{x_1}(\omega_1-y_1)^{n-2}(\omega_2-x_1+y_1)^{n-2}dy_1, \\ \qquad\qquad if\ 0<x_1<\min(\omega_1,\omega_2), \\ \left(\frac{M_0}{n-1}(\omega_1-x_1)^{n-2}\omega_2^{n-1}+M_0\int_{x_1-\omega_1}^{x_1}(\omega_1-y_1)^{n-2}(\omega_2-x_1+y_1)^{n-2}dy_1\right)I[\omega_2<\omega_1] \\ \quad +M_0\int_0^{x_1}(\omega_1-y_1)^{n-2}(\omega_2-x_1+y_1)^{n-2}dy_1 I[\omega_1<\omega_2], \\ \qquad\qquad if\ \min(\omega_1,\omega_2)<x_1<\max(\omega_1,\omega_2), \\ M_0\int_{x_1-\omega_2}^{\omega_1}(\omega_1-y_1)^{n-2}(\omega_2-x_1+y_1)^{n-2}dy_1,\ \ if\ \max(\omega_1,\omega_2)<x_1<\omega_1+\omega_2. \end{cases} \tag{8}$$

Therefore, the UMVUE of the marginal reliability function is given by

$$\begin{aligned}\widetilde{S}(u)&=\frac{M_0}{(n-1)^2}\omega_2^{n-1}\left((\omega_1-u)^{n-1}-(\omega_1-\min(\omega_1,\omega_2))^{n-1}\right)\mathrm{I}[u\le\min(\omega_1,\omega_2)] \\ &+M_0\int_u^{\min(\omega_1,\omega_2)}\int_0^{x_1}(\omega_1-y_1)^{n-2}(\omega_2-x_1+y_1)^{n-2}dy_1dx_1\mathrm{I}[u\le\min(\omega_1,\omega_2)] \\ &+\left(\frac{M_0}{(n-1)^2}\omega_2^{n-1}(\omega_1-\max(u,\omega_2))^{n-1}\right)\mathrm{I}[\omega_2\le\omega_1] \\ &+M_0\int_{\max(u,\omega_1)}^{\omega_1}\int_{x_1-\omega_1}^{x_1}(\omega_1-y_1)^{n-2}(\omega_2-x_1+y_1)^{n-2}dy_1dx_1\mathrm{I}[\omega_2\le\omega_1] \\ &+M_0\int_{\max(u,\omega_1)}^{\omega_2}\int_0^{\omega_1}(\omega_1-y_1)^{n-2}(\omega_2-x_1+y_1)^{n-2}dy_1dx_1\mathrm{I}[u<\max(\omega_1,\omega_2)] \\ &+M_0\int_{\max(u,\omega_1,\omega_2)}^{\omega_1+\omega_2}\int_{x_1-\omega_2}^{\omega_1}(\omega_1-y_1)^{n-2}(\omega_2-x_1+y_1)^{n-2}dy_1dx_1 I[u<\omega_1+\omega_2].\end{aligned} \tag{9}$$

3. Simulation and Comparisons

In this section we compare the bias and the MSE of the UMVUE and the MLE of each reliability function for finite samples. The behavior is investigated empirically by generating bivariate observations from Freund model. The experiment was repeated 10,000 times with $\lambda_1=0.2$ and $\lambda_2=0.22$ for various values of mission times. For each repetition $\widetilde{S}(u,v)$ was calculated. And the MLE counterparts were obtained by substituting the MLE's of λ_1 and λ_2 in (4). Finally, the estimated bias and MSE were calculated for each estimator. The results are given in Table I . In general, as expected, the estimated bias of the

UMVUE is found to be considerably smaller than the estimated bias of the MLE. The estimated abolute biases of two estimators of each reliability function tend to decrease as n increases, and to increase when reliability increases. The estimated MSE's of two estimators of each reliability function appear to be comparable and tend to decrease as n increases.

Table I. ESTIMATED BIAS AND MSE OF $S(u,v)$ WHEN $\lambda_1 = 0.2, \lambda_2 = 0.22$

				Bias		MSE	
u	v	$S(u,v)$	n	MLE	UMVUE	MLE	UMVUE
0.1	1.12	0.9561	10	-.0326	.0044	.0572	.0519
			20	-.0198	-.0025	.0491	.0463
			30	-.0130	-.0008	.0168	.0157
0.5	0.6	0.7989	10	-.0284	.0039	.0523	.0582
			20	-.0162	-.0020	.0464	.0421
			30	-.0093	-.0007	.0103	.0138
1.0	1.2	0.6383	10	-.0265	.0039	.0523	.0487
			20	-.0132	-.0015	.0328	.0361
			30	-.0076	-.0011	.0090	.0092

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