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Linear estimators in the three-parameter Weibull distribution with an unidentified outlier

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Abstract

We introduce the linear estimators based on order statistics in the three-parameter Weibull distribution and compare the small sample performances of proposed linear estimators in the three- parameter Weibull distribution with an unidentified outlier.

Key Words and Phrases: three parameter Weibull distribution, linear estimator, unidentified outlier,

1. Introduction

The probability density function of the three-parameter Weibull distribution can be written as

$$f(x;\mu,\sigma,\alpha)=\frac{\alpha}{\sigma}\Big(\frac{x-\mu}{\sigma}\Big)^{\alpha-1}\exp\Big\{-(\frac{x-\mu}{\sigma})^{\alpha}\Big\},\ x\geq\mu, \tag{1}$$

where $\mu(\geq 0)$, $\sigma(>0)$, and $\alpha(>0)$ are referred as the location, scale and shape parameters, respectively, denoted by $X\sim WEI(\mu,\ \sigma,\ \alpha)$. Many authors utilized the Weibull distribution because of its wide applicability in statistical inferences(see Bain and Engeldhardt(1987) and Saunders and Woo(1989)). Here we shall introduce the linear estimators for mean based on order statistics in the three parameter Weibull distribution and compare biases and mean squared errors for proposed linear estimators for mean in the three-parameter Weibull distribution with an unidentified outlier.

Linear functions of order statistics, known as L statistics, provide highly efficient estimators in the family of distribution depending on location and scale parameters. Dattatreya Rao and Narasimhan(1989) and Kantan and Narasimhan(1991)

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discussed the linear estimation in double Weibull distribution and reflected gamma distribution with complete and censored samples. They constructed the tables of the location and scale parameters for special value of the shape parameter from complete sample and singly and symmetrically censored samples. Also relative efficiencies of some simplified linear estimators in relation to best linear unbiased estimators were studied. Woo, Lee and Ali(1997) proposed the linear estimators of mean, and several right-tail probability estimators based on the linear estimators of mean in an exponential distribution and compared numerically performances of proposed estimators. Sherman(1997) compared the efficiencies of the sample mean and the sample median in the exponential power family.

In this paper, we shall derive exactly the density functions and joint density functions of some order statistics in the three-parameter Weibull distribution with a scale transited outlier, and shall derive means and variances of several linear estimators for mean as mathematical special function and compare numerically biases and mean squared errors of linear estimators .

2. Linear Estimators

Suppose that are independent random variables where all but one of them are from $WEI(\mu,\ \sigma,\ \alpha)$, but one remaining is from $WEI(\mu,\ b\sigma,\ \alpha)$, where b is a known positive constant. It is well known that if a random variable X follows $WEI(\mu,\ \sigma,\ \alpha)$, then $Y = (X-\mu)^\alpha$ follows exponential distribution with mean σ^α, denoted by $\mathrm{EXP}(\sigma^\alpha)$. So let $Y_i = (X_i - \mu)^\alpha$ for $i = 1, 2, \ldots, n$. Then $Y_1, Y_2, \ldots, Y_n$ are independent random variables where all but one of them are from $\mathrm{EXP}(\sigma^\alpha)$ and a remaining random variable is from $\mathrm{EXP}((b\sigma)^\alpha)$. Assume $Y_{(1)} \le Y_{(2)} \le \cdots \le Y_{(n)}$ are the order statistics for $Y_1, Y_2, \ldots, Y_n$.

From the permanent theory, the density function of $Y_{(i)}, i = 1, 2, \ldots, n$ is

$$\begin{aligned} f_{Y_{(i)}}(y) &= \sum_{p=0}^{i-1}(-1)^p\binom{n-1}{i-1}\binom{i-1}{p}(n-i+b^{-\alpha})\sigma^{-\alpha}e^{-(n-i+p+\frac{1}{b^\alpha})\frac{y}{\sigma^\alpha}} \\ &\quad +(i-1)\sum_{p=0}^{i-2}(-1)^p\binom{n-1}{i-1}\binom{i-2}{p}\frac{1}{\sigma^\alpha}\left[1-e^{-\frac{y}{b^\alpha\sigma^\alpha}}\right]e^{-(n-i+p+1)\frac{y}{\sigma^\alpha}} \\ &\quad ,0<y<\infty. \end{aligned} \tag{2}$$

For $b = 1$, the probability density function of $Y_{(i)}$ for $i = 1, 2, \ldots, n$ is

$$f_{Y_{(i)}}(y) = \frac{n!}{(i-1)!(n-i)!}\sum_{p=0}^{i-1}(-1)^p\binom{i-1}{p}\frac{1}{\sigma^\alpha}e^{\frac{-(n-i+p+1)y}{\sigma^\alpha}}, 0<y<\infty, \tag{3}$$

which is the same result in Johnson et al(1994).

From the permanent theory, the joint density function of $Y_{(i)}$ and $Y_{(j)}$ for $i < j : 1, 2, \ldots, n$ is

$$
\begin{aligned}
f_{Y_{(i)},Y_{(j)}}(x,y) = \; & C(i,j)(i-1)\sum_{p=0}^{i-2}\sum_{q=0}^{j-i-1}(-1)^{p+q}\binom{i-2}{p}\binom{j-i-1}{q}\frac{1}{\sigma^{2\alpha}} \\
& \times(1-e^{-\frac{x}{b^{\alpha}\sigma^{\alpha}}})`e^{-(j-i-1+p-q)\frac{x}{\sigma^{\alpha}}-(n-i+p+1)\frac{y}{b^{\alpha}}} \\
& +C(i,j)(i-1)\sum_{p=0}^{i-1}\sum_{q=0}^{j-i-1}(-1)^{p+q}\binom{i-1}{p}\binom{j-i-1}{q}\frac{1}{b^{a}\sigma^{2\alpha}} \\
& \times e^{-(j-i-1+p-q)\frac{x}{\sigma^{\alpha}}-(n-i+p+1)\frac{y}{b^{\alpha}}} \\
& +C(i,j)(j-i-1)\sum_{p=0}^{i-1}\sum_{q=0}^{j-i-2}(-1)^{p+q}\binom{i-1}{p}\binom{j-i-2}{q}\frac{1}{\sigma^{2\alpha}} \\
& \times(e^{-\frac{y}{b^{\alpha}\sigma^{\alpha}}}-e^{-\frac{x}{b^{\alpha}\sigma^{\alpha}}})e^{-(j-i-1+p-q)\frac{x}{\sigma^{\alpha}}-(n-j+q+1)\frac{y}{b^{\alpha}}} \\
& +C(i,j)\sum_{p=0}^{i-1}\sum_{q=0}^{j-i-1}(-1)^{p+q}\binom{i-1}{p}\binom{j-i-1}{q}\frac{1}{b^{a}\sigma^{2\alpha}} \\
& \times e^{-(j-i-1+p-q)\frac{x}{\sigma^{\alpha}}-(n-j+q+\frac{1}{b^{\alpha}})\frac{y}{\sigma^{\alpha}}} \qquad (4) \\
& +C(i,j)(n-j)\sum_{p=0}^{i-1}\sum_{q=0}^{j-i-1}(-1)^{p+q}\binom{i-1}{p}\binom{j-i-1}{q}\frac{1}{\sigma^{2\alpha}} \\
& \times e^{-(j-i-1+p-q)\frac{x}{\sigma^{\alpha}}-(n-j+q+\frac{1}{b^{\alpha}})\frac{y}{\sigma^{\alpha}}}, 0 < x < y < \infty,
\end{aligned}
$$

where $C(i,j) = \frac{n!}{(i-1)!(j-i-1)!(n-j)!}$.

For $b = 1$, the joint density function of of $Y_{(i)}$ and $Y_{(j)}$ for $i < j : 1, 2, \ldots, n$ is

$$
\begin{aligned}
f_{Y_{(i)},Y_{(j)}}(x,y) = \; & \sum_{p=0}^{i-1}\sum_{r=0}^{j-i-1}C(i,j)(-1)^{p+r}\binom{i-1}{p}\binom{j-i-1}{r}\frac{1}{\sigma^{2\alpha}} \qquad (5) \\
& \times e^{\frac{-(j-i-1+p-q)x-(n-j+r+1)y}{\alpha^{\alpha}}}, 0 < x < y < \infty,
\end{aligned}
$$

which is the same result in Johnson et al(1994).

Let $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ the order statistics for $X_1, X_2, \ldots, X_n$. Then

$$X_{(i)} = \mu + Y_{(i)}^{\frac{1}{\alpha}}, i = 1, \ldots, n. \qquad (6)$$

From (2), the mean and the variance of $X_{(i)}, i = 1, \dots, n$ are

$$\begin{aligned}
E_\sigma(i) \equiv E[X_{(i)}] &= \mu + \sum_{p=0}^{i-1}(-1)^p \binom{n-1}{i-1}\binom{i-1}{p}\sigma\Gamma\left(\frac{1}{\alpha}+1\right) \\
&\times(n-i+b^{-\alpha})(n-i+p+b^{-\alpha}) \\
&+(i-1)\sum_{p=0}^{i-2}(-1)^p \binom{n-1}{i-1}\binom{i-2}{p}\sigma\Gamma\left(\frac{1}{\alpha}+1\right) \\
&\times\left[(n-i+p+1)^{-(\frac{1}{\alpha}+1))} - (n-i+p+1+b^{-\alpha})^{-(\frac{1}{\alpha}+1))}\right],
\end{aligned}$$

$$\begin{aligned}
V_\sigma(i) \equiv Var[X_{(i)}] &= \sum_{p=0}^{i-1}(-1)^p \binom{n-1}{i-1}\binom{i-1}{p}[n-i+b^{-\alpha}]\sigma^2 \\
&\times[n-i+p+b^{-\alpha}]^{-(\frac{2}{\alpha}+1)}\Gamma\left(\frac{2}{\alpha}+1\right) \\
&+(i-1)\sum_{p=0}^{i-2}(-1)^p \binom{n-1}{i-1}\binom{i-2}{p}\sigma^2\Gamma\left(\frac{2}{\alpha}+1\right) \\
&\times\left\{(n-i+p+1)^{-(\frac{2}{\alpha}+1)} - (n-i+p+1+b^{-\alpha})^{-(\frac{2}{\alpha}+1)}\right\} \\
&-\left[\sum_{p=0}^{i-1}(-1)^p \binom{n-1}{i-1}\binom{i-1}{p}[n-i+b^{-\alpha}]\sigma\right. \\
&\times[n-i+p+b^{-\alpha}]^{-(\frac{1}{\alpha}+1)}\Gamma(\frac{1}{\alpha}+1) \\
&+(i-1)\sum_{p=0}^{i-2}(-1)^p \binom{n-1}{i-1}\binom{i-2}{p}\sigma\Gamma\left(\frac{1}{\alpha}+1\right) \\
&\left.\times\left\{(n-i+p+1)^{-(\frac{1}{\alpha}+1)} - (n-i+p+1+b^{-\alpha})^{-(\frac{1}{\alpha}+1)}\right\}\right]^2.
\end{aligned} \tag{7}$$

For $b = 1$, the mean and variance of $X_{(i)}$ for $i = 1, 2, \dots, n$ are

$$E[X_{(i)}] = \mu + \sigma\frac{n!}{(i-1)!(n-i)!}\sum_{p=0}^{i-1}(-1)^p\binom{i-1}{p}(n-i+p+1)^{-(\frac{1}{\alpha}+1)}\Gamma\left(\frac{1}{\alpha}+1\right),$$

$$
\begin{aligned}
Var[X_{(i)}] &= \sigma^2\Big\{\frac{n!}{(i-1)!(n-i)!}\sum_{p=0}^{i-1}(-1)^p\binom{i-1}{p}\Gamma\Big(\frac{1}{\alpha}+1\Big) \qquad (8)\\
&\times(n-i+p+1)^{-(\frac{2}{\alpha}+1)}\Gamma(\frac{2}{\alpha}+1)\\
&-\Big[\frac{n!}{(i-1)!(n-i)!}\sum_{p=0}^{i-1}(-1)^p\binom{i-1}{p}\\
&\times(n-i+p+1)^{-(\frac{1}{\alpha}+1)}\Gamma\Big(\frac{1}{\alpha}+1\Big)\Big]\Big\}.
\end{aligned}
$$

which are the same result in Johnson et al(1994).

From (2), (4), and the formula 6.455(1) in Gradshteyn et al(1965), the covariance between $X_{(i)}$ and $X_{(j)}$ can be obtained as :

$$
\begin{aligned}
Cov(X_{(i)}, X_{(j)}) &= C(i,j)(i-1)\sum_{p=0}^{i-1}\sum_{q=0}^{j-i-1}(-1)^{p+q}\binom{i-2}{p}\binom{j-i-1}{q}\\
&\times\sigma^2\Gamma\Big(\frac{2}{\alpha}+2\Big)(\frac{1}{\alpha}+1)^{-1}\\
&\times\Big\{(n-i+p+1)^{-(\frac{2}{\alpha}+2)}F\Big(1,\frac{2}{\alpha}+2,\frac{1}{\alpha}+2;\frac{j-i+p-q}{n-i+p+1}\Big)\\
&\times(n-i+p+1+b^{-\alpha})^{-(\frac{2}{\alpha}+2)}\\
&\times F\Big(1,\frac{2}{\alpha}+2,\frac{1}{\alpha}+2;\frac{j-i+p-q+b^{-\alpha}}{n-i+p+1+b^{-\alpha}}\Big)\Big\} \qquad (9)\\
&+C(i,j)\sum_{p=0}^{i-1}\sum_{q=0}^{j-i-1}(-1)^{p+q}\binom{i-1}{p}\binom{j-i-1}{q}\frac{\sigma^2}{b^2}\\
&\times\Gamma(\frac{2}{\alpha}+2)(\frac{1}{\alpha}+1)^{-1}(n-i+p+b^{-\alpha})^{-(\frac{2}{\alpha}+2)}\\
&\times F(1,\frac{2}{\alpha}+2,\frac{1}{\alpha}+2;\frac{j-i+p-q+b^{-\alpha}}{n-i+p+1+b^{-\alpha}})\\
&+C(i,j)(j-i-1)\sum_{p=0}^{i-1}\sum_{q=0}^{j-i-2}(-1)^{p+q}\binom{i-1}{p}\binom{j-i-2}{q}\sigma^2\\
&\times\Gamma\Big(\frac{2}{\alpha}+2\Big)(\frac{1}{\alpha}+1)^{-1}(n-i+p+B^{-\alpha})^{-(\frac{2}{\alpha}+2)}
\end{aligned}
$$

$$\times\left\{F\left(1,\frac{2}{\alpha}+2,\frac{1}{\alpha}+2;\frac{j-i-1+p-q+b^{-\alpha}}{n-i+p+b^{-\alpha}}\right)\right.$$

$$\left.-F\left(1,\frac{2}{\alpha}+2,\frac{1}{\alpha}+2;\frac{j-i-1+p-q}{n-i+p+b^{-\alpha}}\right)\right\}$$

$$+C(i,j)\sum_{p=0}^{i-1}\sum_{q=0}^{j-i-1}(-1)^{p+q}\binom{i-1}{p}\binom{j-i-1}{q}\frac{\sigma^2}{b^{\alpha}}$$

$$\times\Gamma\left(\frac{2}{\alpha}+2\right)(\frac{1}{\alpha}+1)^{-1}(n-i+p+b^{-\alpha})^{-(\frac{2}{\alpha}+2)}$$

$$\times F\left(1,\frac{2}{\alpha}+2,\frac{1}{\alpha}+2;\frac{j-i+p-q}{n-i+p+b^{-\alpha}}\right)$$

$$+C(i,j)(n-j)\sum_{p=0}^{i-1}\sum_{q=0}^{j-i-1}(-1)^{p+q}\binom{i-1}{p}\binom{j-i-1}{q}\sigma^2$$

$$\times\Gamma\left(\frac{2}{\alpha}+2\right)(\frac{1}{\alpha}+1)^{-1}(n-i+p+b^{-\alpha})^{-(\frac{2}{\alpha}+2)}$$

$$\times F\left(1,\frac{2}{\alpha}+2,\frac{1}{\alpha}+2;\frac{j-i+p-q}{n-i+p+b^{-\alpha}}\right)$$

$$-\left\{\sum_{p=0}^{i-1}(-1)^{p}\binom{n-1}{i-1}\binom{i-1}{p}\Gamma\left(\frac{1}{\alpha}+1\right)\sigma\right.$$

$$\times(n-i+b^{-\alpha})(n-i+p+b^{-\alpha})^{-(\frac{1}{\alpha}+1)}$$

$$+(i-1)\sum_{p=0}^{i-2}(-1)^{p}\binom{i-1}{i-1}\binom{i-2}{p}\sigma\Gamma\left(\frac{1}{\alpha}+1\right)$$

$$\left.\times\left[(n-i+p+1)^{-(\frac{1}{\alpha}+1)}-(n-i+p+1+b^{-\alpha})^{-(\frac{1}{\alpha}+1)}\right]\right\}$$

$$\times\left\{\sum_{p=0}^{j-1}(-1)^{p}\binom{n-1}{i-1}\binom{j-1}{p}\Gamma\left(\frac{1}{\alpha}+1\right)\sigma\right.$$

$$\times(n-j+b^{-\alpha})(n-j+p+b^{-\alpha})^{-(\frac{1}{\alpha}+1)}$$

$$+(j-1)\sum_{p=0}^{i-2}(-1)^{p}\binom{i-1}{j-1}\binom{j-1}{p}\sigma\Gamma\left(\frac{1}{\alpha}+1\right)$$

$$\times\Big[(n-i+p+1)^{-(\frac{1}{\alpha}+1)}-(n-j+p+1+b^{-\alpha})^{-(\frac{1}{\alpha}+1)}\Big]\Big\}$$
$$\equiv C_\sigma(i,j),$$

where $F(\alpha,\beta;d;z)$ is the Gauss' hypergeometric function.

For $b=1$, the covariance between $X_{(i)}$ and $X_{(j)}$ for $i<j$ is the same result in Johnson et al(1994). Here we propose the following several linear estimators for mean ;

The sample mean : $\overline{X}=\frac{1}{n}\sum_{i=1}^{n}X_i$

The sample midrange : $\overline{X}_{MR}=\frac{1}{2}\{X_{(1)}+X_{(n)}\}$

The sample median :

$$\overline{X}_{MD}=\begin{cases}X_{((n+1)/2)}, & n\text{:odd}\\ \frac{1}{2}\{X_{(n/2)}+X_{((n/2)+1)}\}, & n\text{:even}\end{cases}$$

The sample exterior trimmed mean : $\overline{X}_{EM}=\frac{1}{n-2m}\sum_{i=m+1}^{n-m}X_{(i)}$

The sample interior trimmed mean : $\overline{X}_{IM}=\frac{1}{m}\{\sum_{i=1}^{m}X_{(i)}+\sum_{i=n-m+1}^{n}X_{(i)}\}$

where m is chosen among $\{1,2,\ldots,[n/2]-1\}$ so that mean squared error of $\overline{X}_{EM}$ and $\overline{X}_{IM}$ are minimum respectively.

The sample Winsorized mean :

$$\overline{X}_{WM}=\frac{1}{n}\{(r_0+1)[X_{(r_0+1)}+X_{(n-r_0)}]+\sum_{i=r_0+2}^{n-r_0-1}X_{(j)}\}$$

where r_0 is chosen among $\{1,2,\ldots,[n/2]-1\}$ so that mean squared error of $\overline{X}_{WM}$ is minimum (see David(1981)).

From the results (6) through (9), we can get the means and variances for linear estimators for mean in the three-parameter Weibull distribution with a scale transited outlier.

The mean and the variance of sample mean are

$$E[\overline{X}]=\mu+\frac{[(n-1)+b^{-\alpha}\sigma}{n}\Gamma(\frac{1}{\alpha}+1),$$

$$Var[\overline{X}]=\frac{[(n-1)+b^{-\alpha}]\sigma^2}{n^2}\Big[\Gamma(\frac{2}{\alpha}+1)-\Gamma^2(\frac{1}{\alpha}+1)\Big]. \tag{10}$$

From (7) and (9), the mean and the variance of the sample midrange $\overline{X}_{MR}$ can be obtained as :

$$
\begin{aligned}
E[\overline{X}_{MR}] &= \mu + \frac{\sigma}{2}\Gamma(\frac{1}{\alpha}+1) \\
&\times\Big[(n-i+b^{-\alpha})^{-\frac{1}{\alpha}} + \sum_{p=0}^{n-1}(-1)^p\binom{n-1}{p}b^{-\alpha}[b^{-\alpha}+p]^{(\frac{1}{\alpha}+1)} \\
&+\sum_{p=0}^{n-2}(-1)^p\binom{n-2}{p} \\
&\times\Big\{(p+1)^{(\frac{1}{\alpha}+1)} - (p+1+b^{-\alpha})^{(\frac{1}{\alpha}+1)}\Big\}\Big],
\end{aligned}
$$

$$
\begin{aligned}
Var[\overline{X}_{MR}] &= \frac{[(n-1)+b^{-\alpha}]\sigma^2}{4}[\Gamma(\frac{2}{\alpha}+1) - \Gamma^2(\frac{1}{\alpha}+1)] \qquad (11) \\
&+\frac{1}{4}\sum_{p=0}^{n-1}(-1)^p\binom{n-1}{p}b^{-\alpha}[b^{-\alpha}+p]^{(\frac{1}{\alpha}+1)}\sigma^2\Gamma\Big(\frac{2}{\alpha}+1\Big) \\
&+\frac{1}{4}\sum_{p=0}^{n-2}(-1)^p(n-1)\binom{n-2}{p}\sigma^2\Gamma(\frac{1}{\alpha}+1) \\
&\times\Big\{(p+1)^{(\frac{2}{\alpha}+1)} - (p+1+b^{-\alpha})^{(\frac{2}{\alpha}+1)}\Big\} \\
&-\frac{1}{4}\Big[\sum_{p=0}^{n-1}(-1)^p\binom{n-1}{p}b^{-\alpha}[b^{-\alpha}+p]^{(\frac{1}{\alpha}+1)}\sigma\Gamma\Big(\frac{1}{\alpha}+1\Big) \\
&+\sum_{p=0}^{n-2}(-1)^p(n-1)\binom{n-2}{p}\sigma\Gamma\Big(\frac{1}{\alpha}+1\Big) \\
&\times\Big\{(p+1)^{(\frac{1}{\alpha}+1)} - (p+1+b^{-\alpha})^{(\frac{1}{\alpha}+1)}\Big\}\Big]^2 \\
&+\frac{1}{2}\sum_{p=0}^{n-2}(-1)^p(n-1)\binom{n-2}{p}\frac{1}{b^{\alpha}}\sigma^2\Gamma\Big(\frac{1}{\alpha}+1\Big)(\frac{1}{\alpha}+1)^{-1} \\
&\times(n-1+b^{-\alpha})^{(\frac{2}{\alpha}+2)}F\Big(1,\frac{2}{\alpha}+2,\frac{1}{\alpha}+2;\frac{n-2-p+b^{-\alpha}}{n-1+b^{-\alpha}}\Big) \\
&+\frac{1}{2}\sum_{p=0}^{n-3}(-1)^p(n-1)(n-2)\binom{n-2}{p}\sigma^2\Gamma\Big(\frac{2}{\alpha}+2\Big)(\frac{1}{\alpha}+1)^{-1}
\end{aligned}
$$

$$\times(n-1+b^{-\alpha})^{(\frac{2}{\alpha}+2)}\left\{F\left(1,\frac{2}{\alpha}+2,\frac{1}{\alpha}+2;\frac{n-2-p+b^{-\alpha}}{n-1+b^{-\alpha}}\right)\right.$$
$$\left.-F\left(1,\frac{2}{\alpha}+2,\frac{1}{\alpha}+2;\frac{n-2-p}{n-1+b^{-\alpha}}\right)\right\}$$
$$+\frac{1}{2}\sum_{p=0}^{n-2}(-1)^p(n-1)\binom{n-2}{p}\frac{1}{b^{\alpha}}\sigma^2\Gamma\left(\frac{2}{\alpha}+2\right)\left(\frac{1}{\alpha}+1\right)^{-1}$$
$$\times(n-1+b^{-\alpha})^{(\frac{2}{\alpha}+2)}F\left(1,\frac{2}{\alpha}+2,\frac{1}{\alpha}+2;\frac{n-2-p}{n-1+b^{-\alpha}}\right)$$
$$-\frac{1}{2}\left[(n-1+b^{-\alpha})\sigma\Gamma\left(\frac{1}{\alpha}+1\right)\right.$$
$$\times\left\{\sum_{p=0}^{n-1}(-1)^p\binom{n-1}{p}b^{-\alpha}[b^{-\alpha}+p]^{(\frac{1}{\alpha}+1)}\sigma\Gamma\left(\frac{1}{\alpha}+1\right)\right.$$
$$+\sum_{p=0}^{n-2}(-1)^p(n-1)\binom{n-2}{p}\sigma\Gamma\left(\frac{1}{\alpha}+1\right)$$
$$\left.\left.\times\left\{(p+1)^{(\frac{1}{\alpha}+1)}-(p+1+b^{-\alpha})^{(\frac{1}{\alpha}+1)}\right\}\right\}\right].$$

From (7) and (9), the mean and variance of sample median $\overline{X}_{(MD)}$ are

$$E[\overline{X}_{MD}]=\begin{cases}E_\sigma(m+1), & n=2m+1\\ \frac{1}{2}\{E_\sigma(m)+E_\sigma(m+1)\}, & n=2m,\end{cases}$$

$$Var[\overline{X}_{MD}]=\begin{cases}V_\sigma(m+1), & n=2m+1\\ \frac{1}{4}\{V_\sigma(m)+V_\sigma(m+1)+2C_\sigma(m,m+1)\}, & n=2m.\end{cases}\tag{12}$$

Also, we can get the mean and variance of sample interior trimmed mean $\overline{X}_{(IM)}$ and exterior trimmed mean $\overline{X}_{(EMD)}$ as follows ;

$$E[\overline{X}_{IM}]=\frac{1}{2m}\left\{\sum_{i=0}^{m}E_\sigma(i)+\sum_{i=n-m+1}^{m}E_\sigma(i)\right\},$$

$$E[\overline{X}_{EM}]=\frac{1}{n-2m}\sum_{i=m+1}^{n-m}E_\sigma(i),$$

$$Var[\overline{X}_{IM}] = \frac{1}{4m^2}\Big\{\sum_{i=1}^{m} V_\sigma(i) + \sum_{i=n-m+1}^{m} V_\sigma(i) + \sum_{i=1}^{m} \sum_{i\neq j}\sum_{j=1}^{m} C_\sigma(i,j) \\ + \sum_{i=n-m+1}^{m} \sum_{i\neq j}\sum_{j=n-m+1}^{m} C_\sigma(i,j) + \sum_{i=1}^{m}\sum_{j=n-m+1}^{m} C_\sigma(i,j)\Big\}, \tag{13}$$

and

$$Var[\overline{X}_{EM}] = \frac{1}{(n-2m)^2}\Big\{\sum_{i=m+1}^{n-m} V_\sigma(i) + \sum_{i=m+1}^{n-m} \sum_{i\neq j}\sum_{j=m+1}^{n-m} C_\sigma(i,j)\Big\}.$$

Finally, the mean and variance of sample Winsorized mean $\overline{X}_{WM}$ are

$$E[\overline{X}_{WM}] = \frac{1}{n}\Big\{(r_0+1)[E_\sigma(r_0+1) + E_\sigma(n-r_0)] + \sum_{i=r_0+2}^{n-r_0+1} E_\sigma(i)\Big\},$$

$$Var[\overline{X}_{WM}] = \frac{1}{n^2}\Big\{(r_0+1)^2[V_\sigma(r_0+1) + V_\sigma(n-r_0)] + \sum_{i=r_0+2}^{n-r_0-1} V_\sigma(i) \\ +2(r_0+1)^2 C_\sigma(r_0+1, n-r_0) + 2(r_0+1)\sum_{i=r_0+2}^{n-r_0-1} C_\sigma(r_0+1, i) \\ +2(r_0+1)\sum_{i=r_0+2}^{n-r_0-1} C_\sigma(r_0+1, i) + \sum_{i=r_0+2}^{n-r_0-1} \sum_{i\neq j}\sum_{j=r_0+2}^{n-r_0-1} C_\sigma(i,j)\Big\}. \tag{14}$$

Using (10) through (14), tables show biases and mean squared error of the these linear estimators in the three-parameter Weibull distribution when the scale parameter is an identified outlier for the small sample size $n = 10(5)25$, location parameter $\mu = 0$, scale parameter $\sigma = 1.0, b = 1(1)4$ and shape parameter $\alpha = 0.5$ and 2.

For the shape parameter $\alpha = 2$, sample mean is more efficient than other linear estimator in terms of biases and mean square error. And for the shape parameter $\alpha = 0.5$, sample mean is more useful than other linear estimator in terms of biases but sample exterior trimmed mean is more efficient than other linear estimators in terms of mean squared error.

Table 1. Biases and MSE's for linear estimators of mean in the three-parameter Weibull distribution with an unidentified outlier

n	b	$\overline{X}$	$\overline{X}_{MR}$	$\overline{X}_{MD}$	$\overline{X}_{EM}$	$\overline{X}_{IM}$	$\overline{X}_{WM}$
	1	0.02146 (0.00000)	0.04616 (0.09178)	0.03331 (0.04457)	0.02365(1) (0.02291)(1)	0.02182(4) (0.01100)(4)	0.02484(1) (0.04617)(1)
	2	0.03575 (0.08622)	0.22023 (0.31128)	0.03491 (0.00044)	0.02717(1) (0.03279)(1)	0.03521(4) (0.07266)(4)	0.02799(1) (0.04405)
10	3	0.07004 (0.17724)	0.82699 (0.67377)	0.03599 (0.01159)	0.02967(1) (0.05309)(1)	0.03594(4) (0.04315))	0.03000(2) (0.01159)(4)
	4	0.12433 (0.26586)	1.92621 (1.08299)	0.03646 (0.01620)	0.03087(1) (0.06153)(1)	0.03895(4) (0.00024)(4)	0.03085(2) (0.01620)(4)
	1	0.01431 (0.00000)	0.47282 (0.12392)	0.02547 (0.04821)	0.01540(1) (0.01903)(1)	0.01458(6) (0.01183)(6)	0.01551(1) (0.02723)(1)
	2	0.02065 (0.05908)	0.20681 (0.31428)	0.02530 (0.01959)	0.01675(1) (0.01983)(1)	0.02189(6) (0.05288)(6)	0.01708(2) (0.00393)(3)
15	3	0.03589 (0.11816)	0.79263 (0.66188)	0.02553 (0.01208)	0.01766(2) (0.01552)(2)	0.02363(6) (0.03330)(6)	0.01802(2) (0.00353)(4)
	4	0.06002 (0.17724)	1.86447 (1.06423)	0.02564 (0.00924)	0.01791(2) (0.01995)(2)	0.02310(6) (0.00441)(6)	0.01884(3) (0.00155)(5)
	1	0.01073 (0.00000)	0.05062 (0.14776)	0.01917 (0.04933)	0.01140(1) (0.01638)(1)	0.01075(9) (0.00551)(9)	0.01132(1) (0.01887)(1)
	2	0.01430 (0.04431)	0.20139 (0.32032)	0.01846 (0.02808)	0.01208(1) (0.01366)(1)	0.01478(9) (0.03528)(9)	0.01225(2) (0.00321)(3)
20	3	0.02287 (0.08862)	0.77320 0.65765)	0.01841 (0.02256)	0.01255(2) (0.00940)(2)	0.01554(9) (0.02289)(9)	0.01280(3) (0.00273)(4)
	4	0.036744 (0.13293)	1.83059 (1.05511)	0.01841 (0.02048)	0.01255(2) (0.01303)(2)	0.01554(9) (0.00388)(9)	0.01296(3) (0.00243)(5)
	1	0.00858 (0.00000)	0.05455 (0.16670)	0.01663 (0.05034)	0.00905(1) (0.01446)(1)	0.00863(11) (0.00684)(11)	0.00892(1) (0.01426)(1)
	2	0.01087 (0.03544)	0.19914 (0.32680)	0.01585 (0.03350)	0.00944(1) (0.01013)(1)	0.01198(11) (0.03079)(11)	0.00955(1) (0.00278)(3)
25	3	0.01635 (0.01708)	0.76121 (0.65630)	0.01574 (0.02917)	0.00972(2) (0.00608)(2)	0.01244(11) (0.02056)(11)	0.00988(1) (0.00267)(4)
	4	0.02504 (0.10634)	1080874 (1.04995)	0.01571 (0.02754)	0.00980(2) (0.00919)(2)	0.01220(11) (0.00497)(11)	0.00999(1) (0.00182)(5)

where the numbers in the low parenthesis are biases and the numbers in the right side parenthesis are number of trimmed and winsorized samples.

Table 2. Biases and MSE's for linear estimators of mean in the three-parameter Weibull distribution with an unidentified outlier

n	b	$\overline{X}$	$\overline{X}_{MR}$	$\overline{X}_{MD}$	$\overline{X}_{EM}$	$\overline{X}_{IM}$	$\overline{X}_{WM}$
10	1	2.00000 (0.00000)	32.76273 (3.07881)	2.10040 (1.32793)	1.32782(1) (0.77736)(1)	3.02312(4) (0.33640)(4)	1.58402(1) (0.99788)(1)
	2	2.64000 (0.20000)	47.70828 (3.69829)	2.03148 (1.28306)	1.32422(1) (0.68029)(1)	4.23286(4) (0.57392)(4)	1.53018(1) (0.91159)(1)
	3	3.61600 (0.40000)	70.58502 (4.41174)	1.99537 (1.25527)	1.33805(1) (0.61349)(1)	5.92750(4) (0.81236)(4)	1.53097(2) (0.84944)(1)
	4	5.36000 (0.60000)	110.77702 (5.22178)	1.96816 (1.23276)	1.36499(1) (0.56009)(1)	8.74101(4) (1.06393)(4)	1.55746(1) (0.79587)(1)
15	1	1.33330 (0.00000)	48.86542 (4.45116)	2.17023 (1.40516)	0.99017(1) (0.66258)(1)	2.08720(6) (0.34945)(6)	1.51203(1) (1.07756)(1)
	2	1.61776 (0.13333)	61.40521 (4.96617)	2.11424 (1.37808)	0.96585(1) (0.59058)(1)	2.59962(6) (0.50538)(6)	1.45225(1) (1.02412)(1)
	3	2.11554 (0.26666)	86.60492 (5.65166)	2.08696 (1.36404)	0.96404(2) (0.53889)(2)	3.47898(6) (0.67179)(6)	1.41402(1) (0.98082)(1)
	4	2.38222 (0.40000)	121.19749 (6.39001)	2.06991 (1.35590)	0.97788(2) (0.50209)(2)	4.61571(6) (0.83221)(6)	1.40307(1) (0.94294)(1)
20	1	1.00000 (0.00000)	62.96492 (5.57159)	2.17073 (1.42927)	0.79920(1) (0.58059)(1)	1.21030(9) (0.15787)(9)	1.67978(1) (1.20560)(1)
	2	1.11000 (0.20000)	72.12871 (6.08996)	2.12489 (1.40972)	0.77928(1) (0.52662)(1)	1.45522(9) (0.27065)(9)	1.62315(1) (1.17049)(1)
	3	1.44000 (0.20000)	100.88531 (6.69883)	2.10165 (1.39958)	0.77513(1) (0.48708)(1)	1.81914(9) (0.37615)(9)	1.57811(2) (1.13695)(1)
	4	1.84000 (0.30000)	138.88550 (7.42763)	2.08643 (1.39251)	0.76610(1) (0.45349)(1)	2.33407(9) (0.48995)(9)	1.54587(2) (1.11044)(1)
25	1	0.80000 (0.00000)	76.76559 (6.59970)	2.20518 (1.45044)	0.67263(1) (0.53107)(1)	1.03367(11) (0.19505)(11)	1.73159(1) (1.25025)(1)
	2	0.89510 (0.08000)	92.54499 (7.04829)	2.16745 (1.43527)	0.65482(1) (0.48121)(1)	1.20128(11) (0.28459)(11)	1.67650(1) (1.21951)(1)
	3	1.01816 (0.16000)	118.42013 (7.65502)	2.15165 (1.42919)	0.64491(1) (0.44409)(1)	1.47104(11) (0.37667)(11)	1.63627(2) (1.19518)(1)
	4	1.38240 (0.32000)	156.83854 (8.33202)	2.13960 (1.42408)	0.64154(1) (0.41709)(1)	1.83374(11) (0.46531)(11)	1.60334(2) (1.17396)(1)

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