Journal of Statistical
Theory & Methods
1997, Vol. 8, No. 2, pp. 239 ~ 245

Estimation for Exponential Distribution under General Progressive Type-II Censored Samples

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Abstract

By assuming a general progressive Type-II censored sample, we propose the minimum risk estimator (MRE) and the approximate maximum likelihood estimator (AMLE) of the scale parameter of the one-parameter exponential distribution. An example is given to illustrate the methods of estimation discussed in this paper.

Key Words and Phrases: Approximate maximum likelihood estimator, Exponential distribution, Minimum risk estimator, Progressive Type-II censored sample

1. Introduction

Progressive censoring samples are routinely encountered in life and fatigue testing, where individual observations are time ordered and where, at various times during a test, some of the survivors are removed (i.e. censored) from further observation either by design or by accident. Thus, censoring times are intermixed with failure times. Such samples arise naturally when, at various times during a life test, certain specimems are withdrawn prior to their failure for use at test objects in other experimentation. Its allowance for the removal of live units from the experiment at various stages is an attractive feature as it will potentially save a lot for the experimenters in terms of cost and time.

Progressive censoring samples from the three-parameter log-normal distribution were also considered by Cohen (1976). Cohen and Norgaard (1977) discussed the inference problems for a wide range of distributions under this progressive censoring sampleing scheme. These developments were summarized by Cohen and Whitten (1988), and more by Cohen (1991). Viveros and Balakrishnan (1994) developed

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exact conditional inference based on progressive Type-II censored samples. Balakrishnan and Sandhu (1996) obtained the best liner unbiased and the maximum likelihood estimations for exponential distribution under general progressive Type-II censored sample.

The approximate maximum likelihood estimation method was first developed by Balakrishnan (1989a, b) for the purpose of providing the explicit estimators of the scale parameter in the Rayleigh distribution and the mean and standard deviation in the normal distribution with censoring. Kang (1996) obtained the AMLE for the scale parameter of the double exponential distribution based on Type-II censored samples and he showed that the proposed estimator is generally more efficient than the BLUE and the optimum unbiased absolute estimator.

A Type-II progressively censored life test is conducted as follows; Suppose N randomly selected units were placed on a life test; the failure times of the first r units to fail were not observed; at the time of the $(r+1)$-th failure, R_{r+1} number of surviving units are withdrawn from the test randomly, and so on; at the time of the $(r+i)$-th failure, R_{r+i} number of surviving units are randomly withdrawn from the test; finally, at the time of the m-th failure, the remaining $R_m = N - m - R_{r+1} - R_{r+2} - \cdots - R_{m-1}$ are withdrawn from the test.

Suppose $X_{r+1,N} \leq X_{r+2,N} \leq \ldots \leq X_{m,N}$ are the life-times of the completely observed units to fail, and $R_{r+1},\ R_{r+2},\ \ldots,\ R_m$ are the number of units withdrawn from the test at these failure times, respectively. If the failure times are from a continuous population with cumulative distribution function $F(x)$ and probability density function $f(x)$, the joint probability density function for $X_{r+1,N},\ \ldots,\ X_{m,N}$ is given by

$$f_{X_{r-1,N},\ \ldots\ ,\ X_{m,N}}(x_{r+1},\ \ldots,\ x_m) = c[F(x_{r+1})]^r \prod_{i=r+1}^{m} f(x_i)[1-F(x_i)]^{R_i},$$

where

$$\begin{aligned} c &= \binom{N}{r}(N-r)(N-r-R_{r+1}-1)(N-r-R_{r+1}-R_{r+2}-2)\times\cdots \\ &\quad \times(N-r-R_{r+1}-R_{r+2}-\cdots-R_{m-r}-(m-r)+1) \\ &= \binom{N}{r}(N-r)\prod_{i=r+2}^{m}\left(N-\sum_{i=r+1}^{j-1}R_i-j+1\right) \end{aligned}$$

In this paper, by assuming that such a general progressive Type-II censored sample, we derive the minimum risk estimator and the approximate maximum likelihood estimator of the scale parameter of the one-parameter exponential distribution. The proposed estimators will be compared in terms of the mean squared error (MSE) through Nelson's data.

2. Estimation for the scale parameter

The random variable X has an one-parameter exponential distribution if it has a probability density function (pdf) of forms;

$$f(x;\sigma) = \frac{1}{\sigma}e^{-x/\sigma},\ x > 0,\ \sigma > 0$$

where σ is the scale parameter.

Let $X_{0,N} = 0$,

$$\begin{aligned} D_{i,N} &= (N-i+1)(X_{i,N} - X_{i-1,N}),\ i = 1, 2, \dots, r+1, \\ S_{j,N} &= \Big(N - \sum_{i=r+1}^{j-1} R - i - j + 1\Big)(X_{j,N} - X_{j-1,N}),\ j = r+2, \dots, m. \end{aligned}$$

It is then known that $D_{i,N}/\sigma,\ i = 1, 2, \dots, r+1$ are independent standard exponential random variables. By using similar arguments, Viveros and Balakrishnan (1994) recently established that $S_{j,N}/\sigma,\ j = r+2, \dots, m$ are also independent standard exponential random variables.

Now, by writing

$$X_{r+1,N} = \sum_{i=1}^{r+1} \frac{D_{i,N}}{N-i+1}$$

and

$$X_{j,N} = X_{r+1,N} + \sum_{i=r+2}^{j} \frac{S_{i,N}}{N - \sum_{k=r+1}^{i-1} R_k - i + 1},\ j = r+2, \dots, m,$$

the expectation, the variance, and the covariance of the order statistic (Balakrishnan and Sandhu (1996)) are given by

$$E(X_{r+1,N}) = \sigma \sum_{i=1}^{r+1} \frac{1}{N-i+1} = \sigma\alpha_{r+1},$$

$$E(X_{j,N}) = \sigma\left[\alpha_{r+1} + \sum_{i=r+2}^{j} \frac{1}{N - \sum_{k=r+1}^{i-1} R_k - i + 1}\right],\ j = r+2, \dots, m,$$

$$\mathrm{Var}(X_{r+1,N}) = \sigma^2 \sum_{i=1}^{r+1} \frac{1}{(N-i+1)^2} = \sigma^2\beta_{r+1},$$

$$\mathrm{Var}(X_{j,N}) = \sigma^2\left[\beta_{r+1} + \sum_{i=r+2}^{j} \frac{1}{(N - \sum_{k=r+1}^{i-1} R_k - i + 1)^2}\right],\ j = r+2, \ldots, m,$$

and

$$\mathrm{Cov}(X_{i,N}, X_{j,N}) = \mathrm{Var}(X_{j,N}),\ r+1 \le i < j \le m.$$

Balakrishnan and Sandhu (1996) derived the BLUE of σ to be

$$\widehat{\sigma}_{BLUE} = \frac{1}{m - r - 1 + \frac{\alpha_{r-1}^2}{\beta_{r+1}}}\left[\sum_{j=r+2}^{m} (R_j + 1)(X_{j,N} - X_{r+1,N}) + \left(\frac{\alpha_{r+1}}{\beta_{r+1}}\right)X_{r+1,N}\right] \quad (1)$$

and its variance to be

$$\mathrm{Var}(\widehat{\sigma}_{BLUE}) = \frac{\sigma^2}{m - r - 1 + \frac{\alpha_{r+1}^2}{\beta_{r+1}}}. \quad (2)$$

We obtain the AMLE of parameter in one-parameter distribution with the general progressive Type-II censoring. The likelihood function based on the general progressive Type-II censored sample is given by

$$L = c[F(X_{r+1})]^r \prod_{i=r+1}^{m} f(X_i)[1 - F(X_i)]^{R_i}.$$

In this case, the log-likelihood function based on the general progressive Type-II censored sample is given by

$$\ln(L) = \ln(c) + -(m - r)\ \ln(\sigma) + r\ \ln(1 - e^{-\frac{X_{r+1,N}}{\sigma}}) - \sum_{i+r+1}^{m} (R_i + 1)\frac{X_{i,N}}{\sigma}. \quad (3)$$

The differentiate the logarithm of the likelihood function (3) for σ is given by

$$\begin{aligned} \frac{\partial \ln L}{\partial \sigma} &= r\ X_{r+1,N}\frac{e^{-X_{r+1,N}/\sigma}}{1 - e^{-X_{r+1,N}/\sigma}} + \sigma(m - r) - \sum_{i=r+1}^{m} (R_i + 1)X_{i,N} \\ &= r\ X_{r+1,N}\frac{f(Z_{r+1,N})}{F(Z_{r+1,N})} + \sigma(m - r) - \sum_{i=r+1}^{m} (R_i + 1)X_{i,N} \quad (4) \\ &= 0 \end{aligned}$$

where $Z_{i,N} = X_{i,N}/\sigma$, $f(z)$ and $F(z)$ are the pdf and cdf of the standard exponential distribution, respectively. Balakrishnan and Sandhu (1996) showed the maximum likelihood estimator (MLE) of σ does not exist in an explicit form and has to be determined from the likelihood equation by a numerial method. The equation to be solved numerically for σ is

$$r\ X_{r+1,N}\ \frac{e^{-X_{r+1,N}/\sigma}}{1-e^{-X_{r-1,N}/\sigma}} = -\sigma(m-r) + \sum_{i=r+1}^{m} (R_i+1)X_{i,N}.$$

But we can expand the function $\frac{f(Z_{r-1,N})}{F(Z_{r-1,N})}$, appearing in (4) to Taylor series around the point $\xi_{r+1} = F^{-1}(p_{r+1}) = -\ln(q_{r+1})$ and then approximate it by

$$\frac{f(Z_{r+1,N})}{F(Z_{r+1,N})} \simeq \alpha - \beta Z_{r+1,N} \tag{5}$$

where $p_i = \frac{i}{(n+1)}$, $q_i = 1 - p_i$,

$$\alpha = \frac{f(\xi_{r+1})}{p_{r+1}}\left[1 + \xi_{r+1} + \frac{f(\xi_{r+1})}{p_{r+1}}\xi_{r+1}\right],$$

and

$$\beta = \frac{f(\xi_{r+1})}{p_{r+1}^2}[p_{r+1} + f(\xi_{r+1})].$$

Now making use of the approximate expression in (5), we obtain the approximate likelihood equation of (4) as follows;

$$\begin{aligned} \frac{\partial \ln L}{\partial \sigma} &\simeq \frac{\partial \ln L^*}{\partial \sigma} \\ &= r\ X_{r+1,N}(\alpha - \beta Z_{r+1,N}) + \sigma(m-r) - \sum_{i=r+1}^{m}(R_i+1)X_{i,N} \qquad (6) \\ &= 0. \end{aligned}$$

Upon solving equation (6) for σ, we derive the AMLE of σ as follows;

$$\widehat{\sigma}_{AMLE} = \frac{-A - \sqrt{A^2 + 4(m-r)r\beta X_{r+1,N}}}{2(m-r)}$$

where $A = r\alpha X_{r+1,N} - \sum_{i=r+1}^{m}(R_i+1)X_{i,N}$.

Also we propose the minimum risk estimator, the MRE can be derived by minimizing the mean squared error among the class of estimators of the form $c_1\sum_{j=r+2}^{m}(R_j+1)(X_{j,N} - X_{r+1,N}) + c_2 X_{r+1,N}$ where c_1 and c_2 are constants. We can obtain the MRE as follows;

$$\widehat{\sigma}_{MRE} = c_1 \sum_{j=r+2}^{m}(R_j+1)(X_{j,N} - X_{r+1,N}) + c_2 X_{r+1,N} \tag{7}$$

where

$$c_1 = \frac{1}{m - r + \alpha^2/\beta},$$
$$c_2 = \frac{\alpha/\beta}{m - r + \alpha^2/\beta}.$$

and its the mean squared error (MSE) of the MRE to be

$$\mathrm{MSE}(\hat{\sigma}_{MRE}) = \frac{\sigma^2}{m - r + \frac{\alpha_{r-1}^2}{\beta_{r-1}}}. \tag{8}$$

3. Numerical illustration

For the purpose of illustration, let us consider Nelson's data (1982, p. 228, Table 6.1) which gives data on times to breakdown of an insulating fluid in an accelerated test conducted at various test voltages.

In analyzing the complete data, Nelson assumed a scaled Weibull distribution for the times to breakdown(from the 90% confidence interval [0.459, 1.381] that he determined for the shape parameter, it is quite clear that an exponential model is also appropriate for this data). For the purpose of illustrating the methods of inference presented sample from the $N = 19$ observations recorded at 34 kV in Nelson's Table 6.1 (with one smallest observation censored and three stages of progressive censoring).

i	1	2	3	4	5	6	7	8
$X_{i,N}$	-	0.78	0.96	1.31	2.78	4.85	6.50	7.35
R_i	-	0	3	0	3	0	0	5

Balakrishnan and Sandhu (1996) obtained from (1) and (2) that $\hat{\sigma}_{BLUE} = 9.110$, $\mathrm{RMSE}(\hat{\sigma}_{BLUE}) = \mathrm{MSE}(\hat{\sigma}_{BLUE})/\sigma^2 = 0.125$, and $\hat{\sigma}_{MLE} = 9.111$. We get from (7) and (8) that $\hat{\sigma}_{MRE} = 8.099$, $\mathrm{RMSE}(\hat{\sigma}_{MRE}) = \mathrm{MSE}(\hat{\sigma}_{MRE})/\sigma^2 = 0.111$, and $\hat{\sigma}_{AMLE} = 9.152$.

From (2) and (8), the proposed estimator $\hat{\sigma}_{MRE}$ is more efficient than $\hat{\sigma}_{BLUE}$ in terms of the mean squared error. In the special case when $r = 0$, however, the AMLE of σ becomes identical with the BLUE and the MLE of σ. It is of interest to observe that the precision of the MRE of σ in (7) depends only on r, m and N, and not on the progressive censoring scheme $(R_{r+1}, \ldots, R_m)$.

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