Journal of Statistical
Theory & Methods
1997, Vol. 8, No. 2, pp. 271 ~ 276

A Dynamic Discount Approach to the Poisson Process

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Abstract

A dynamic discount approach is proposed for the estimation of the Poisson parameter and the forecasting of the Poisson random variable, where the parameter of the Poisson distribution varies over time intervals. The recursive estimation procedure of the Poisson parameter is provided. Also the forecasted distribution of the Poisson random variable in the next time interval based on the information gathered until the current time interval is provided.

Key Words and Phrases: Dynamic Linear Models, Discount Model, Recursive Estimation.

1. Introduction

The Poisson parameter is constant over time in the static case, but it is allowed to vary over time in the dynamic case. Harvey and Durbin(1986) took a modified structural approach to the problem of the estimation and the forecasting in the dynamic case. Harvey and Fernandez(1989) proposed a forecasted function, an exponentially weighted moving average of the data in the dynamic case. West et al.(1985) developed a dynamic generalized linear model, which allows the use of one dimensional exponential family distribution. The prior distribution is chosen to be a conjugate family member having the same first two moments as the function of the parameter vector, where the parameter vector evolves over time according to a known linearly additive pattern. Ameen and Harrison(1985) developed the normal discount Bayesian model to overcome some practical disadvantages of the dynamic linear model. They introduced a discount factor δ ($0< \delta <1$) so that the variance of a parameter of the current time is equal to the variance of a parameter of the previous time divided by a discount factor, which implies an increase in variance of $100(1-\delta)/\delta$ percent. Gamerman(1992) presented the dynamic approach to the Poisson process

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under a dynamic generalized linear model, where the parameter vector evolves over time intervals by adding the mean and variance of the evolution error vector of the current time interval on the mean and the variance of the parameter vector obtained in the previous time interval. In this article, we propose a dynamic discount approach to the Poisson process for the estimation of the the Poisson parameter and the forecasting of the Poisson random variable in the next time interval. Here the Poisson parameter evolves over time interval by multiplying parameters of a conjugate gamma distribution by discount factors. This approach provides a quick response to sudden changes of the time-varying Poisson parameter. The proposed model is described in Section 2. The procedures of the recursive estimation of the Poisson parameter and the forecasting of the distribution of the Poisson random variable in the next time interval under the proposed are provided in Section 3. In Section 4 the performances of the proposed model are illustrated via simulation studies.

2. Model Descriptions

Here we assume that the number of occurrences (the Poisson random variable) in the time interval I_i follows the Poisson distribution which has the constant Poisson parameter in each time interval I_i for $i = 1, 2, \cdots$. We denote it by

$$(Y_i|\lambda_i) \sim Poisson(\lambda_i l_i),$$

where Y_i is the number of occurrences in the time interval I_i and l_i is the length of the time interval I_i . Let D_i be a set of information gathered until the end of the time interval I_i , which can be represented as the set of numbers of occurrences in previous time intervals including the time interval I_i. The likelihood function of λ_i in each time interval I_i is obtained as

$$L(\lambda_i|D_i) = \frac{exp(-\lambda_i l_i)(\lambda_i l_i)^{y_i}}{y_i!},$$

where y_i is the observed number of occurrences in the time interval I_i . The likelihood function is used to obtain the posterior distribution of λ_i given D_i from the prior distribution of λ_i given D_{i-1}.

Then the discount Poisson model is defined as follows.

i) observation equation;

$$(Y_i|\lambda_i) \sim Poisson(\lambda_i) \quad \text{for } i = 1, 2, \cdots.$$

where l_i is the length of the time interval I_i .

ii) evolution equation;

$$(\lambda_i|D_{i-1}) \sim Ga(\delta_i\alpha_{i-1}, \delta_i\beta_{i-1}),$$

where δ_i is the discount factor of the time interval I_i , which allows the prior distribution have the same mean but an increase of $100(1\text{-}\delta_i)/\delta_i$ percent in the variance of λ_{i-1} given D_{i-1} , that is, $E(\lambda_i|D_{i-1})$=$E(\lambda_{i-1}|D_{i-1})$ and $V(\lambda_i|D_{i-1})$=$V(\lambda_{i-1}|D_{i-1})/\delta_i$, where α_{i-1} and β_{i-1} parameters of a gamma distribution of λ_{i-1} given D_{i-1}.

iii) initial distribution;

$$(\lambda_0|D_0) \sim Ga(\alpha_0, \beta_0),$$

where λ_0 is the Poisson parameter at time 0, values of α_0 and β_0 are assumed to be given prior to the time interval I_1 .

3. Estimation Procedures

The process is started with initial distribution of the Poisson parameter at time 0, , following a gamma distribution with parameters and , which do not affect distributional behaviors of the Poisson parameter after a certain number of time intervals. At the beginning of the time interval I_i the posterior distribution of λ_i obtained at the end of the previous time interval I_{i-1} ,

$$(\lambda_{i-1}|D_{i-1}) \sim Ga(\alpha_{i-1}, \beta_{i-1}),$$

leads to the prior distribution of λ_i such as

$$(\lambda_i|D_{i-1}) \sim Ga(\delta_i\alpha_{i-1}, \delta_i\beta_{i-1}),$$

where δ_i is the discount factor of the time interval I_i.

With the observation and using the fact that

$$p(\lambda_i|D_i) = p(\lambda_i|y_i, D_{i-1}) \propto L(\lambda_i|y_i, D_{i-1})p(\lambda_i|D_{i-1}),$$

the posterior distribution of λ_i is obtained as

$$(\lambda_i|D_i) \sim Ga(\alpha_i, \beta_i),$$

where

$$\alpha_i = \delta_i\alpha_{i-1} + y_i, \beta_i = \delta_i\beta_{i-1} + l_i.$$

Note that

$$E(Y_i|D_i) = E(E(Y_i|\lambda_i, D_i)|D_i) = l_iE(\lambda_i|D_i)$$

the posterior estimate of the mean of Y_i given D_i is obtained as $l_i\alpha_i/\beta_i$, where l_i is the length of the time interval I_i. The forecasted distribution of the number of occurrences in the next time interval I_{i+1} given D_i is obtained as

$$\begin{aligned} P(Y_{i+1}=y|D_i) &= \int p(y,\lambda_{i+1}|D_i)d\lambda_i \\ &= \int p(y|\lambda_{i+1},D_i)p(\lambda_{i+1}|D_i)d\lambda_{i+1} \\ &= \frac{\Gamma(\delta_{i+1}\alpha_i+y)}{\Gamma(\delta_{i+1}\alpha_i)\Gamma(y+1)}(\frac{\delta_{i+1}\beta_i}{\delta_{i+1}\beta_i+l_{i+1}})^{\delta_{i-1}\alpha_i}(\frac{l_{i+1}}{\delta_{i+1}\beta_i+l_{i+1}})^y . \end{aligned}$$

Thus the mean and the variance of Y_{i+1} given D_i are obtained as, respectively,

$$l_{i+1}\frac{\alpha_i}{\beta_i} \quad \text{and} \quad l_{i+1}\frac{\alpha_i}{\beta_i}+l_{i+1}^2\frac{\alpha_i}{\delta_{i+1}\beta_i^2},$$

where the mean of Y_{i+1} given D_i is used as the forecasted number of occurrences in the time interval I_{i+1} .

4. Illustrations

In this section we consider the performances of the proposed estimation and forecasting procedures in the previous section via simulation studies. The data set, which is assumed to be the set of numbers of occurrences, consists of 100 simulated random samples, from the Poisson population, whose mean is 4 for the first 25 samples, 8 for the next 25 samples, 4 for the next 25 samples, and 6 for the last 25 samples. The length of each time interval I_i is assumed to be 2 for $i=1,2,\cdots,100$.

We assume that the observation distribution of the number of occurrences in the time interval I_i follows the Poisson distribution as,

$$(Y_i|\lambda_i) \sim Poisson(\lambda_i l_i) \quad \text{for} \quad i=1,2,\cdots,100.$$

Figure 1. Posterior Estimate of the Mean of Each Sample

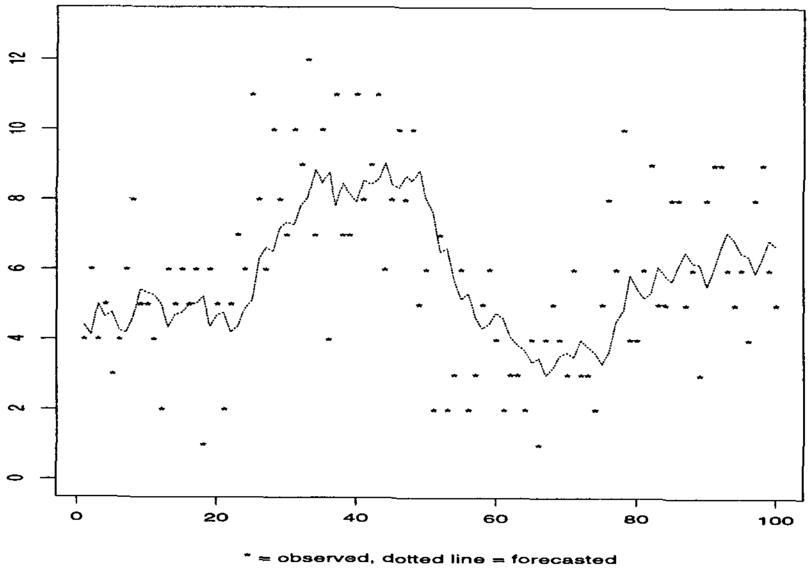

Figure 2. 1-Step Ahead Forecasted Value under the model

We start the process with the initial distribution, $(\lambda_0|D_0) \sim Ga(2.2, 1)$. The ML-estimate of the Poisson parameter from first 5 observations , $\widehat{\lambda}$=2.2, is tentatively used for values of parameters of the initial gamma distribution, 2.2 and 1. And the discount factor δ_i is tentatively obtained to be 0.8 for $i = 1, 2, \cdots, 100$, among 5 values of δ_i - 0.75, 0.8, 0.85, 0.9, 0.95, the value 0.8 gives the closest value of the posterior estimates to $l_i\widehat{\lambda}$=4.4. Figure 1 shows the true mean and the posterior estimate of the mean for each of 100 independent Poisson random variables, where the true mean has step changes over time. Figure 2 shows the observed value and the corresponding forecasted number of occurrences in each time interval.

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