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# Parameter Estimation for Age-Structured Population Dynamics

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**Abstract**

This paper studies parameter estimation for a first-order hyperbolic integro-differential equation modelling one-sex population dynamics. A second-order finite difference scheme is used to estimate parameters such as the age-specific death-rate and the age-specific fertility from fully discrete observations on the population. The function space parameter estimation convergence of this scheme is proved. Also, numerical simulations are performed.

## 0. Introductuon

In this paper we study a parameter estimation problem for population dynamics. The parameter estimation is a process of approximating the unknown parameters in mathematical models by making use of suitable observations on the phenomena. Thus it can be understood as a process of model calibration and hence would be unavoidable to obtain more accurate predictions in the future.

There have been so many studies describing the dynamics of insect or human population, see [1-6] and the references cited there. The first analytical model seems to be that of Malthus [1] which describes the total population by an ordinary differential equation. The modern approaches consider the total population as well as its age structure, which leads to partial differential equations.

The population model adopted in this paper is the following form of McKendrickvon Foerster integro-differential equations [2] :

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = -\mathrm{D}(x)u, \quad (x,t) \in (0,\infty) \times (0,\mathrm{T}), \tag{0.1}$$

$$u(x,0) = u_0(x), \tag{0.2}$$

$$u(0,t) = \mathrm{B}(t) := \int_0^\infty \mathrm{F}(x)u(x,t)\,dx. \tag{0.3}$$

Here $u$ is the age-distribution of the population, $x$ is the age, $t$ is the time, D is the age-specific death-rate, F is the age-specific fertility, and B is the birth-rate.

In this paper we shall assume that $u_0$ is a given smooth nonnegative function which is compactly supported in $[0, \mathrm{M}]$. We also assume that F and D are nonnegative continuous functions on $[0, \mathrm{M}+\mathrm{T}]$. It is shown in [3,4] that the problem (0.1)–(0.3) has a unique solution, which is compactly supported in the age variable for any time. More precisely, $u(x,t)$ vanishes in the region $\{x \geq t + \mathrm{M}\}$, see section 2. Thus we may consider the problem only in the region $[0, \mathrm{M}+\mathrm{T}] \times [0, \mathrm{T}]$. Several numerical schemes had been developed for the model (0.1)–(0.3), see [7-10]. In fact, the algorithms are described in more general situations.

Notice that we must know the function parameters D and F apriori in order to solve (0.1)–(0.3). But, it is not easy to obtain the parameters in precise form. One of the possible approaches is interpolating or extrapolating the previously collected corresponding data obtained from population census. In this paper we estimate them from the observations on the population, not on the parameters themselves. This is so called the parameter estimation problem. The general theory for parameter estimation can be found in [11]. For the size-structured population dispersal models, the parameter estimation problems are considered in several works, see [12-15] and the references cited there. Of particular interest to the development of this paper is the treatment of fully-discrete observations, whereas most of the previous works were concerned on the distributed observations. For that purpose we need $\mathrm{L}^\infty$ estimates regarding the stability with respect to the parameters both for analytical and for numerical solutions.

Let $Q = C[0, \mathrm{M}+\mathrm{T}] \times C[0, \mathrm{M}+\mathrm{T}] = \{(\mathrm{D}, \mathrm{F})\}$ be the parameter space equipped with the usual supremum norm. The solution for (0.1)–(0.3) with a parameter $q \in Q$ will be denoted by $u(q)$ or $u(x,t;q)$.

In this paper three types of observations will be considered. Let $\{\mathrm{a}_i\}_{i=1}^{m_\mathrm{a}}$ be a fixed set of ages contained in the age-interval $[0, \mathrm{M}+\mathrm{T}]$ and let $\{\mathrm{t}_j\}_{j=1}^{m_\mathrm{t}}$ be a fixed set of instances contained in the time-interval $[0, \mathrm{T}]$. We define observation operators $\Gamma_1 : C([0, \mathrm{M}+\mathrm{T}] \times [0, \mathrm{T}]) \to \mathbb{R}^{m_\mathrm{a}}$ and $\Gamma_2 : C([0, \mathrm{M}+\mathrm{T}] \times [0, \mathrm{T}]) \to \mathbb{R}^{m_\mathrm{t}}$ by

$$\begin{aligned} \Gamma_1 v &= (\Gamma_1^1 v, \ldots, \Gamma_1^{m_\mathrm{a}} v), \qquad \Gamma_1^i v = v(\mathrm{a}_i, \mathrm{T}); \\ \Gamma_2 v &= (\Gamma_2^1 v, \ldots, \Gamma_2^{m_\mathrm{t}} v), \qquad \Gamma_2^j v = v(0, \mathrm{t}_j). \end{aligned}$$

We also define $\Gamma_0 : C([0, \mathrm{M}+\mathrm{T}] \times [0, \mathrm{T}]) \to \mathbb{R}^{m_\mathrm{a}+m_\mathrm{t}}$ by $\Gamma_0 v = (\Gamma_1 v, \Gamma_2 v)$. In these cases the observation spaces are $\mathbb{R}^{m_\mathrm{a}}$, $\mathbb{R}^{m_\mathrm{t}}$ and $\mathbb{R}^{m_\mathrm{a}+m_\mathrm{t}}$, respectively, and the parameter-to-output mappings are

$$\Phi_0(q) = \Gamma_0[u(q)]; \quad \Phi_1(q) = \Gamma_1[u(q)]; \quad \Phi_2(q) = \Gamma_2[u(q)].$$

Once an observation operator $\Gamma$ and parameter-to-output mapping $\Phi$ is fixed, the inverse problem or parameter identification problem is to find the inverse mapping of $\Phi$. The parameters are said to be identifiable from an observation process $\Gamma$ if $\Phi$ is injective. Roughly speaking, the identifiability requires sufficiently many observations. But, in most practical situations, it is impossible to obtain such data due to technical or economical reasons. Moreover, due to the presence of modelling error and inaccuracy of collected data, it is naturally suggested to consider an optimization problem. In our case, we choose the output-least-squares(OLS) cost functional and consider the following

PROBLEM (ID) Let $\tilde{Q}$ be an admissible parameter subset of the parameter space $Q$. Given a set of measurements $z = \{z_i\}_{i=1}^m \in Z$, find $q^* \in \tilde{Q}$ that minimizes

$$\mathrm{J}(q) := \|\Phi(q) - z\|^2 = \sum_{k=1}^{m} \left[\Gamma^k [u(q)] - z_k\right]^2,$$

where $\Gamma$ and $\Phi$ are one of those described above.

Notice that both the parameter space and the state space are infinite-dimensional. Thus we need to approximate (ID) by a sequence of finite-dimensional problems for actual computations. The question of convergence called the function space parameter estimation convergence(FSPEC) is to be answered.

This paper is organized as follows. In section 1 we consider the forward problem and prepare various lemmas for the subsequent use. In particular, we show the local Lipschitz continuity with respect to the parameters both of analytical and numerical solutions. In section 2 we address the parameter estimation problem and introduce an approximation scheme. The FSPEC of this scheme is proved. Section 3 is devoted to the numerical simulations which support our result. Finally, in section 4 we make some comments and concluding remarks.

Throughout this paper all the norms of functions are usual supremum norms unless otherwise specified.

## 1. FDM for Forward Problem

It had been shown in [3] that the initial-boundary value problem (0.1)–(0.3) has a unique solution $u$ which is compactly supported in the age variable for any fixed time. In fact, in the characteristic coordinates, the equation (0.1) reduces to an ODE. By integrating and transforming back, we have the following expression for the solution of (0.1)–(0.3) :

$$u(x,t) = \begin{cases} u_0(x-t)\exp\,[-\int_0^t \mathrm{D}(x-t+s)\,ds], & x \geq t, \\ \mathrm{B}(t-x)\exp\,[-\int_0^x \mathrm{D}(s)\,ds], & x < t. \end{cases} \tag{1.1}$$

However, the birth-rate B involves $u$ in its expression (0.3), and hence it is not apriori known. To obtain B we insert (1.1) into (0.3) and get the following integral equation

$$\mathrm{B}(t) = \zeta(t) + \int_0^t \kappa(t-s)\mathrm{B}(s)\,ds, \tag{1.2}$$

where the functions $\zeta$ and $\kappa$ are defined as

$$\zeta(t) = \int_t^\infty \mathrm{F}(x)u_0(x-t)\exp\,[-\int_0^t \mathrm{D}(x-t+s)\,ds]\,dx; \tag{1.3}$$

$$\kappa(x) = \mathrm{F}(x)\exp\,[-\int_0^x \mathrm{D}(s)\,ds]. \tag{1.4}$$

Under our assumptions on $u_0$, F and D, it is clear that $\zeta$ and $\kappa$ are continuous on $[0,\mathrm{T}]$, and hence (1.2) has a unique continuous solution [16, Theorem 3.1, p.30]. Then by inserting it into (1.1) we obtain the solution of (0.1)–(0.3).

**Remark 1.1** The regularity of parameters is inherited to the solution. More precisely, if the parameters are $s$-times continuously differentiable, then (1.2) implies that the birth-rate B is also $s$-times continuously differentiable [16], and then, by (1.1), so does the solution possibly except on the set $\{x = t\}$. Notice that it is needed for the solution of (0.1)–(0.3) to be continuous that

$$u_0(0) = \int_0^\infty \mathrm{F}(x)u_0(x)\,dx$$

holds. See Theorem 5 in [3] for more regularity results. However, we do not need even the continuity of the solution across the characteristic line $\{x = t\}$.

Here, we list some basic properties of solutions which can be directly deduced from the expressions (1.1) and (1.2). By (1.2), the Gronwall inequality and the continuity of B we know that

$$0 \leq \mathrm{B}(t) \leq \|\mathrm{F}\|\|u_0\|\mathrm{M}e^{\|\mathrm{F}\|t}, \tag{1.5}$$

and then, by (1.1) we can see that $u$ is nonincreasing in the characteristic direction and that

$$0 \leq u(x,t) \leq \max\{\|u_0\|, \|\mathrm{F}\|\|u_0\|\mathrm{M}e^{\|\mathrm{F}\|t}\}.$$

Recall that we are assuming that $u_0$ is compactly supported in $[0, \mathrm{M}]$. Then, the expression (1.1) implies that $u(x,t) = 0$ in the region $\{x > t + \mathrm{M}\}$. We also know from (1.1) that the solution of (0.1)–(0.3) is locally Lipschitz with respect to the parameters as we show now.

**Lemma 1.2** *There exists a positive constant* K *such that*

$$\|u(q) - u(\tilde{q})\| \leq \mathrm{K}\|q - \tilde{q}\| \quad \textit{for} \quad q, \tilde{q} \in Q.$$

*More precisely, the constant* K *depends only on* $\|q\|$, $\|\tilde{q}\|$, $\|u_0\|$, M *and* T.

*Proof.* Let us write $q = (\mathrm{D}, \mathrm{F})$ and $\tilde{q} = (\tilde{\mathrm{D}}, \tilde{\mathrm{F}})$. The corresponding functions B, $\kappa$ and $\zeta$ in (1.2)–(1.4) will be denoted by $\tilde{\mathrm{B}}$, $\tilde{\kappa}$, $\tilde{\zeta}$, respectively. From the expression (1.1) for the solutions it suffices to show the same Lipschitz estimate for B, that is,

$$\|\mathrm{B} - \tilde{\mathrm{B}}\| \leq \mathrm{K}\|q - \tilde{q}\| \quad \text{for} \quad q, \tilde{q} \in Q \tag{1.6}$$

for some constant K depending only on $\|q\|$, $\|\tilde{q}\|$, $\|u_0\|$, M and T. First we see that

$$\begin{aligned} |\kappa(x) - \tilde{\kappa}(x)| &\leq |\mathrm{F}(x) - \tilde{\mathrm{F}}(x)| \exp\left[-\int_0^x \mathrm{D}(s)\,ds\right] + \\ &\quad + |\tilde{\mathrm{F}}(x)| \left[\exp\left[-\int_0^x \mathrm{D}(s)\,ds\right] - \exp\left[-\int_0^x \tilde{\mathrm{D}}(s)\,ds\right]\right] \\ &\leq |\mathrm{F}(x) - \tilde{\mathrm{F}}(x)| + \|\tilde{\mathrm{F}}\|(\mathrm{M}+\mathrm{T})\|\mathrm{D} - \tilde{\mathrm{D}}\| \end{aligned}$$

by the Mean Value Theorem, so we get

$$\|\kappa - \tilde{\kappa}\| \leq \|\mathrm{F} - \tilde{\mathrm{F}}\| + \|\tilde{\mathrm{F}}\|(\mathrm{M}+\mathrm{T})\|\mathrm{D} - \tilde{\mathrm{D}}\|. \tag{1.7}$$

Similarly we can prove

$$\|\zeta - \tilde{\zeta}\| \leq \mathrm{M}\|u_0\| \left[\|\mathrm{F} - \tilde{\mathrm{F}}\| + \|\tilde{\mathrm{F}}\|(\mathrm{M}+\mathrm{T})\|\mathrm{D} - \tilde{\mathrm{D}}\|\right], \tag{1.8}$$

where we have used the fact $\operatorname{supp} u_0 \subset [0, \mathrm{M}]$. Now, from the expression (1.2) for the birth-rate, we observe that

$$\begin{aligned} |\mathrm{B}(t) - \tilde{\mathrm{B}}(t)| &\leq |\zeta(t) - \tilde{\zeta}(t)| + \int_0^t \kappa(t-s)|\mathrm{B}(s) - \tilde{\mathrm{B}}(s)|\,ds + \\ &\quad + \int_0^t \tilde{\mathrm{B}}(s)|\kappa(t-s) - \tilde{\kappa}(t-s)|\,ds \\ &\leq \|\zeta - \tilde{\zeta}\| + \|\tilde{\mathrm{B}}\|\,\mathrm{T}\,\|\kappa - \tilde{\kappa}\| + \|\kappa\| \int_0^t |\mathrm{B}(s) - \tilde{\mathrm{B}}(s)|\,ds. \end{aligned}$$

Then, by Gronwall inequality, we obtain

$$\|\mathrm{B} - \tilde{\mathrm{B}}\| \leq \left[\|\zeta - \tilde{\zeta}\| + \|\tilde{\mathrm{B}}\|\,\mathrm{T}\,\|\kappa - \tilde{\kappa}\|\right] e^{\|\kappa\|\mathrm{T}}. \tag{1.9}$$

Combining (1.5), (1.7), (1.8) and (1.9) we get the desired estimate (1.6). ■

Douglas and Milner [7] used the first order backward finite difference in the characteristic age-time direction to approximate the solution and obtained error estimates of order 1. In [9] the authors developed the second order accurate scheme which is described in the following.

In order to approximate the solution of the problem (0.1)–(0.3), we use finite difference method of characteristics over a uniform age-time mesh. Let $N$ be a positive integer and $\Delta t := \mathrm{T}/N$. Let $m = m(N)$ be the smallest integer such that $m[\Delta t] > \mathrm{M}$. Let $t^n := n[\Delta t]$, for $0 \le n \le N$. Let $\delta := \{x_i \,|\, x_i = i[\Delta t],\ 0 \le i \le m\}$. For notational convenience, the grid will be shifted at each time level along the characteristic line $x = t$ as follows:

$$\delta^n := \{x_i^n | x_i^n := x_{n+i} = (n+i)[\Delta t],\ -n \le i \le m\}.$$

We seek a discrete function $w = \{w_i^n\}$, where $w_i^n$ approximates $u(x_i^n, t^n)$ by the method of second order finite difference in the characteristic direction. The values of the parameters D and F at the point $x_i^n$ will be denoted by $\mathrm{D}_i^n$ and $\mathrm{F}_i^n$, respectively. To use the second order finite differences we need to initialize the values $\{w_i^n\}$ at the points

$$\{(x,t)\,|\,t = 0 \text{ or } \Delta t\} \cup \{(x,t)\,|\,x = 0 \text{ or } \Delta t\}.$$

To obtain those initializing values we use the modified first order forward finite difference and the composite trapezoidal rule in approximation of integral (0.3).

ALGORITHM [Second order central FDM scheme]

1. For $i = 0, 1, \cdots, m$ set $w_i^0 := u_0(x_i^0)$.

2. For $i = 0, 1, \cdots, m$ set $w_i^1 := w_i^0 - [\Delta t]\,\mathrm{D}_i^0 w_i^0$;

   Set $w_{-1}^1 := \dfrac{\Delta t}{1 - \frac{1}{2}[\Delta t]\mathrm{F}_{-1}^1}\left[\sum_{i=0}^{m-1} \mathrm{F}_i^1 w_i^1 + \dfrac{1}{2}\mathrm{F}_m^1 w_m^1\right].$

3. For $n = 2, 3, \cdots, N$ do the following steps.

   For $i = 2-n, 3-n, \cdots, m$ set $w_i^n := w_i^{n-2} - 2[\Delta t]\,\mathrm{D}_i^{n-1} w_i^{n-1}$;

   Set $w_{1-n}^n := w_{1-n}^{n-1} - [\Delta t]\,\mathrm{D}_{1-n}^{n-1} w_{1-n}^{n-1}$;

   Set $w_{-n}^n := \dfrac{\Delta t}{1 - \frac{1}{2}[\Delta t]\mathrm{F}_{-n}^n}\left[\sum_{i=1-n}^{m-1} \mathrm{F}_i^n w_i^n + \dfrac{1}{2}\mathrm{F}_m^n w_m^n\right].$

In fact, this algorithm is a simplified version of [9]. Before going further we prepare an algebraic lemma which will be used in subsequent discussions.

**Lemma 1.3** *Suppose that $\{\omega^n\}_{0\le n\le N}$ is a nonnegative sequence such that there exist positive constants $\mathrm{k}_0, \mathrm{k}_1, \mathrm{k}_2$ such that $\mathrm{k}_0 < \mathrm{k}_1$ and the following*

$$(1-\mathrm{k}_0[\Delta t])\omega^n \le \omega^{n-2} + \mathrm{k}_1[\Delta t]\omega^{n-1} + \mathrm{k}_2[\Delta t]\varepsilon, \quad n = 2,\dots,N, \tag{1.10}$$

*holds, where $\Delta t = \mathrm{T}/N$. Then we have for $\Delta t < 1/\mathrm{k}_1$,*

$$\omega^n \le e^{2\mathrm{k}_1\mathrm{T}}\left[\omega^1 + \omega^0 + \frac{\mathrm{k}_2}{2\mathrm{k}_1}\varepsilon\right] \quad \textit{for all } n = 0,\dots,N.$$

This can be easily proved by using the discrete Gronwall argument. The similar estimates can be found in [17].

From physical considerations, it is natural to require for the FDM solutions to be nonnegative, as like the analytical solution. This is indeed true whenever $\Delta t$ is sufficiently small, since the FDM solution is bounded by a constant depending only on the parameters as we now show in the following

**Proposition 1.4** *Let $\{w_i^n\}$ be the FDM solution. Then, for sufficiently small $\Delta t > 0$, we have*

$$\max_{0\le n\le N,\, -n\le i\le m} |w_i^n| \le \mathrm{K} \tag{1.11}$$

*for some positive constant* K *independent of $\Delta t$. More precisely,* K *depends only on $\|u_0\|$, M, T, $\|\mathrm{F}\|$ and $\|\mathrm{D}\|$.*

*Proof.* Let us write $\|w^n\|_1 = [\Delta t]\sum_{i=-n}^{m}|w_i^n|$. First we show that this sequence $\{\|w^n\|_1\}_{0\le n\le N}$ is uniformly bounded by a constant depending only on $\|u_0\|$, M, T, $\|\mathrm{F}\|$ and $\|\mathrm{D}\|$. We see from the algorithm that

$$\begin{aligned}
|w_{-n}^n| &\le \|\mathrm{F}\|\|w^n\|_1, \quad n = 0,\dots,N, && (1.12)\\
|w_{1-n}^n| &\le (1+[\Delta t]\|\mathrm{D}\|)|w_{1-n}^{n-1}|, \quad n = 1,\dots,N, && (1.13)\\
|w_i^1| &\le (1+[\Delta t]\|\mathrm{D}\|)|w_i^0|, \quad i = 0,\dots,m, && (1.14)\\
|w_i^n| &\le |w_i^{n-2}| + 2[\Delta t]\|\mathrm{D}\||w_i^{n-1}|, && (1.15)\\
&\qquad n = 2,\dots,N, \quad i = 2-N,\dots,m.
\end{aligned}$$

By (1.12) and (1.13) we have

$$|w_{1-n}^n| \le \|\mathrm{F}\|(1+[\Delta t]\|\mathrm{D}\|)\|w^{n-1}\|_1. \tag{1.16}$$

Then, from (1.12), (1.15) and (1.16), we get a recurrence relation

$$\begin{aligned}
(1-[\Delta t]\|\mathrm{F}\|)\|w^n\|_1 &\le \|w^{n-2}\|_1 + && (1.17)\\
&+[\Delta t]\left[2\|\mathrm{D}\| + \|\mathrm{F}\|(1+[\Delta t]\|\mathrm{D}\|)\right]\|w^{n-1}\|_1
\end{aligned}$$

for $n = 2, \ldots, N$, which is of the form (1.10) with $\mathrm{k}_0 = \|\mathrm{F}\|$, $\mathrm{k}_1 = 2\|\mathrm{D}\| + \|\mathrm{F}\|(1 + [\Delta t]\|\mathrm{D}\|)$ and $\mathrm{k}_2 = 0$. It is clear that

$$\|w^0\|_1 \le \mathrm{M}\|u_0\|, \tag{1.18}$$

and then, by (1.12), (1.14) and (1.18) we obtain

$$\|w^1\|_1 \le \frac{1}{1 - [\Delta t]\|\mathrm{F}\|}\mathrm{M}(1 + [\Delta t]\|\mathrm{D}\|)\|u_0\|. \tag{1.19}$$

By Lemma 1.3 we conclude that

$$\|w^n\|_1 \le \mathrm{C} \quad \text{for} \quad n = 0, \ldots, N, \tag{1.20}$$

for some positive constant C depending only on $\|u_0\|$, M, T, $\|\mathrm{F}\|$ and $\|\mathrm{D}\|$.

Now, from (1.12), (1.14), (1.16) and (1.20), we know that the terms $|w^n_{-n}|$, $|w^n_{1-n}|$, $|w^0_i|$ and $|w^1_i|$ are bounded by a constant depending only on $\|u_0\|$, M, T, $\|\mathrm{F}\|$ and $\|\mathrm{D}\|$. Applying Lemma 1.3 to the recurrence relation (1.15) on each characteristic line, we obtain (1.11). See the proof of Lemma 1.8, where the same procedure is fully described. ∎

As in [9], we can prove the following second order convergence of the scheme.

**Proposition 1.5** *Let the solution u of (0.1)–(0.3) be three times continuously differentiable with bounded derivatives through third order. Then there exists a positive constant* K *independent of* $\Delta t$ *such that*

$$\max_{0 \le n \le N,\, -n \le i \le m} |u(x^n_i, t^n) - w^n_i| \le \mathrm{K}[\Delta t]^2.$$

**Remark 1.6** In the previous proposition the assumption regarding the regularity of the solution can be weakened as :

1. In the characteristic direction $u$ is three times continuously differentiable with bounded (characteristic directional) derivatives upto third order;

2. In the age direction $u$ is twice continuously differentiable with bounded (age-directional) derivatives upto second order in each subregion $\{x \ge t\}$ and $\{x \le t\}$ of $[0, \mathrm{M} + \mathrm{T}] \times [0, \mathrm{T}]$.

Notice that in this situation the discontinuity across the line $\{x = t\}$ is irrelevant. By Remark 1.1 we see that the above conditions are satisfied if the parameters F and D are three times continuously differentiable on $[0, \mathrm{M} + \mathrm{T}]$.

For $N \in \mathbb{N}$, we define $u^N$ to be the piecewise linear interpolant of the above FDM solution with $\Delta t = \mathrm{T}/N$, that is, $u^N$ is piecewise linear in each $x$ and $t$, and matches with $w$ at the nodes. Then, by Remark 1.6 and the piecewise linearity, we have

**Corollary 1.7** *Let the solution $u$ of (0.1)–(0.3) satisfies the conditions in Remark 1.6. Then there exists a positive constant* K *independent of $N$ such that*

$$\max_{[0,M+T]\times[0,T]} |u(x,t) - u^N(x,t)| \le \mathrm{K}(\mathrm{T}/N)^2.$$

Now we prove the local Lipschitz continuity of the FDM solutions with respect to the parameters which is uniform in the step size. The proof is similar to that of Proposition 1.4 or Proposition 1.5.

**Lemma 1.8** *There exists a positive constant* K *independent of $N$ such that*

$$\max_{[0,M+T]\times[0,T]} |u^N(x,t;\tilde{q}) - u^N(x,t;q)| \le \mathrm{K}\|\tilde{q} - q\| \quad \textit{for } \tilde{q}, q \in Q.$$

*More precisely, the constant* K *depends only on $\|q\|$, $\|\tilde{q}\|$, $\|u_0\|$,* M *and* T.

*Proof.* Let $\|q - \tilde{q}\| = \varepsilon$ and let $N$ be arbitrarily fixed. Let $w = \{w_i^n\}$ and $\tilde{w} = \{\tilde{w}_i^n\}$ be the FDM solutions corresponding parameters $q = (\mathrm{D}, \mathrm{F})$ and $\tilde{q} = (\tilde{\mathrm{D}}, \tilde{\mathrm{F}})$, respectively. Then, by the piecewise linearity of $u^N$, it suffices to show that there exists a constant K independent of $N$ such that

$$\max_{0\le n\le N, -n\le i\le m} |w_i^n - \tilde{w}_i^n| \le \mathrm{K}\varepsilon. \tag{1.21}$$

Let us write $\zeta_i^n = w_i^n - \tilde{w}_i^n$ and $\|\zeta^n\|_1 = [\Delta t]\sum_{i=-n}^m |\zeta_i^n|$. First, we recall that $\max|w_i^n|$ is bounded by a constant depending only on $\|u_0\|$, M, T, $\|\mathrm{F}\|$ and $\|\mathrm{D}\|$, see Proposition 1.4. So, we assume that

$$\max\{\|\mathrm{D}\|, \|\mathrm{F}\|, \|\tilde{\mathrm{D}}\|, \|\tilde{\mathrm{F}}\|\} \le \mathrm{g}_1; \quad \max_{0\le n\le N, -n\le i\le m} |w_i^n| \le \mathrm{g}_0, \tag{1.22}$$

for some positive constants $\mathrm{g}_0$ and $\mathrm{g}_1$ depending only on $\|q\|$, $\|\tilde{q}\|$, $\|u_0\|$, M and T. For $n = 1, \dots, N$, we have

$$\begin{aligned} |\zeta_{-n}^n| &\le [\Delta t]\sum_{i=-n}^{m} \left\{ |w_i^n||\mathrm{F}_i^n - \tilde{\mathrm{F}}_i^n| + \tilde{\mathrm{F}}_i^n|w_i^n - \tilde{w}_i^n| \right\} \\ &\le \mathrm{g}_1\|\zeta^n\|_1 + (\mathrm{M} + \mathrm{T})\mathrm{g}_0\varepsilon. \end{aligned} \tag{1.23}$$

For $n = 1, \dots, N$, we have

$$\begin{aligned} |\zeta_{1-n}^n| &\le |w_{1-n}^{n-1} - \tilde{w}_{1-n}^{n-1}| + [\Delta t]|\mathrm{D}_{1-n}^{n-1}w_{1-n}^{n-1} - \tilde{\mathrm{D}}_{1-n}^{n-1}\tilde{w}_{1-n}^{n-1}| \\ &\le |w_{1-n}^{n-1} - \tilde{w}_{1-n}^{n-1}| + [\Delta t]\left\{ |w_{1-n}^{n-1}||\mathrm{D}_{1-n}^{n-1} - \tilde{\mathrm{D}}_{1-n}^{n-1}| + \right. \\ &\qquad \left. + \tilde{\mathrm{D}}_{1-n}^{n-1}|w_{1-n}^{n-1} - \tilde{w}_{1-n}^{n-1}| \right\} \\ &\le \mathrm{g}_0[\Delta t]\varepsilon + (1 + \mathrm{g}_1[\Delta t])\left\{ \mathrm{g}_1\|\zeta^{n-1}\|_1 + (\mathrm{M} + \mathrm{T})\mathrm{g}_0\varepsilon \right\} \end{aligned} \tag{1.24}$$

by (1.23). Similarly, for $i = 0, 1, 2, \dots, m$, we have

$$|\zeta_i^1| \le (1 + \mathrm{g}_1[\Delta t])|\zeta_i^0| + \mathrm{g}_0[\Delta t]\varepsilon. \tag{1.25}$$

For $n \ge 2$ and $i \ge 2 - n$, we have from the algorithm that

$$\begin{aligned} |\zeta_i^n| &\le |w_i^{n-2} - \tilde{w}_i^{n-2}| + 2[\Delta t]|\mathrm{D}_i^{n-1} w_i^{n-1} - \tilde{\mathrm{D}}_i^{n-1} \tilde{w}_i^{n-1}| \\ &\le |w_i^{n-2} - \tilde{w}_i^{n-2}| + 2[\Delta t]\left\{|w_i^{n-1}||\mathrm{D}_i^{n-1} - \tilde{\mathrm{D}}_i^{n-1}| + \right. \\ &\qquad \left. + \tilde{\mathrm{D}}_i^{n-1}|w_i^{n-1} - \tilde{w}_i^{n-1}|\right\} \\ &\le |\zeta_i^{n-2}| + 2[\Delta t]\mathrm{g}_1|\zeta_i^{n-1}| + 2[\Delta t]\mathrm{g}_0\varepsilon. \end{aligned} \tag{1.26}$$

It follows from (1.23), (1.24) and (1.26) that for $n \ge 2$, we obtain

$$\begin{aligned} \|\zeta^n\|_1 \le\ & [\Delta t]\{\mathrm{g}_1\|\zeta^n\|_1 + (\mathrm{M}+\mathrm{T})\mathrm{g}_0\varepsilon + \mathrm{g}_0[\Delta t]\varepsilon + \\ & \quad +(1 + \mathrm{g}_1[\Delta t])\left[\mathrm{g}_1\|\zeta^{n-1}\|_1 + (\mathrm{M}+\mathrm{T})\mathrm{g}_0\varepsilon\right]\} + \\ & + \|\zeta^{n-2}\|_1 + 2(\mathrm{M}+\mathrm{T})[\Delta t]\mathrm{g}_0\varepsilon + 2\mathrm{g}_1[\Delta t]\|\zeta^{n-1}\|_1. \end{aligned}$$

Consequently, we have

$$\begin{aligned} (1 - \mathrm{k}_0[\Delta t])\|\zeta^n\|_1 \le\ & \|\zeta^{n-2}\|_1 + \mathrm{k}_1[\Delta t]\|\zeta^{n-1}\|_1 + \\ & + \mathrm{k}_2[\Delta t]\varepsilon, \quad 2 \le n \le N, \end{aligned} \tag{1.27}$$

where we set

$$\mathrm{k}_0 = \mathrm{g}_1; \quad \mathrm{k}_1 = (3 + \mathrm{g}_1)\mathrm{g}_1; \quad \mathrm{k}_2 = 4(\mathrm{M}+\mathrm{T})\mathrm{g}_0 + [1 + (\mathrm{M}+\mathrm{T})\mathrm{g}_1]\,\mathrm{g}_0.$$

Then, by Lemma 1.3, we obtain for all $n = 0, \dots, N$,

$$\|\zeta^n\|_1 \le e^{2\mathrm{k}_1\mathrm{T}}\left[\|\zeta^0\|_1 + \|\zeta^1\|_1 + \frac{\mathrm{k}_2}{2\mathrm{k}_1}\varepsilon\right]. \tag{1.28}$$

Note that the constants $\mathrm{k}_0$, $\mathrm{k}_1$ and $\mathrm{k}_2$ are depending only on $\|q\|$, $\|\tilde{q}\|$, $\|u_0\|$, M and T. It is clear that $\zeta_i^0 = 0$ for all $i$ and hence that

$$\|\zeta^0\|_1 = 0. \tag{1.29}$$

On the other hand, from (1.23) and (1.25), we know

$$\begin{aligned} &|\zeta_{-1}^1| \le \mathrm{g}_1\|\zeta^1\|_1 + (\mathrm{M}+\mathrm{T})\mathrm{g}_0\varepsilon, \\ &|\zeta_i^1| \le \mathrm{g}_0[\Delta t]\varepsilon, \quad i = 0, \dots, m, \end{aligned} \tag{1.30}$$

since $\|\zeta^0\|_1 = 0$, from which we deduce

$$(1 - \mathrm{g}_1[\Delta t])\|\zeta^1\|_1 \le 2(\mathrm{M}+\mathrm{T})\mathrm{g}_0[\Delta t]\varepsilon. \tag{1.31}$$

So, by (1.28), (1.29) and (1.31) we get for all $n = 0, \dots, N$,

$$\|\zeta^n\|_1 \le e^{2k_1T}\left[\frac{2(M+T)g_0[\Delta t]}{1-g_1[\Delta t]} + \frac{k_2}{2k_1}\right]\varepsilon. \tag{1.32}$$

Now we will prove (1.21) on each characteristic line. First consider the characteristic lines starting from the age axis. Let $i \ge 0$ be arbitrarily fixed. It is clear that $|\zeta_i^0| = 0$. At the time level $t = [\Delta t]$, we have shown in (1.30) that $|\zeta_i^1| \le g_0[\Delta t]\varepsilon$. Thus, applying Lemma 1.3 to the sequence $\{|\zeta_i^n|\}_{0\le n\le N}$ satisfying the recurrence relation (1.26), we obtain

$$|\zeta_i^n| \le e^{4g_1T}g_0\left[[\Delta t] + \frac{1}{2g_1}\right]\varepsilon \quad \text{for all } n = 0, \dots, N. \tag{1.33}$$

Now we consider the characteristic lines starting from the time axis. Let $n \ge 1$ be fixed and consider the sequence $\{|\zeta_{-n}^{n+k}|\}_{0\le k\le N-n}$. Notice that this sequence satisfies also the recurrence relation (1.26). By (1.23) and (1.32) we have on the time axis

$$|\zeta_{-n}^n| \le g_1\|\zeta^n\|_1 + (M+T)g_0\varepsilon \le C_1\varepsilon,$$

for some positive constant $C_1$ independent of $N$. At the age $x = [\Delta t]$ we have from (1.24) and (1.32) that

$$|\zeta_{-n}^{n+1}| \le g_0[\Delta t]\varepsilon + (1 + g_1[\Delta t])\{g_1\|\zeta^n\|_1 + (M+T)g_0\varepsilon\} \le C_2\varepsilon,$$

for some positive constant $C_2$ independent of $N$. Applying Lemma 1.3 we obtain

$$|\zeta_n^{n+k}| \le C\varepsilon, \quad \text{for all } k = 0, \dots, N-n, \tag{1.34}$$

for some positive constant C depending only on $\|q\|$, $\|\tilde{q}\|$, $\|u_0\|$, M and T. By (1.33) and (1.34) we get the whole lemma. ■

**Theorem 1.9** *Let $q^0$ be an element of $Q$ such that the solution $u(q^0)$ of (0.1)–(0.3) satisfies the conditions in Remark 1.6. Suppose that $\{q^k\}$ is a sequence in $Q$ satisfying $q^k \to q^0$ in $Q$. Then we have*

$$\|u^{N_k}(q^k) - u(q^0)\| \to 0,$$

*for any increasing sequence $\{N_k\}$ in* $\mathbb{N}$.

*Proof.* By triangle inequality we have for each $k$,

$$\|u^{N_k}(q^k) - u(q^0)\| \le \|u^{N_k}(q^k) - u^{N_k}(q^0)\| + \|u^{N_k}(q^0) - u(q^0)\|.$$

Since $\{q^k\}$ converges to $q^0$, the sequence is bounded. So, by Corollary 1.7 and Lemma 1.8, we know that

$$\|u^{N_k}(q^k) - u(q^0)\| \le K_1\|q^k - q^0\| + K_2(T/N_k)^2,$$

for some positive constants $K_1$ and $K_2$ independent of $k$. This proves the desired convergence. ∎

## 2. Parameter Estimation

Recall that the parameters D and F are assumed to be nonnegative and continuous on $[0, M+T]$. This is natural from biological considerations. To construct an admissible parameter space $\tilde{Q}$ of problem (ID) as a compact subset of $Q$ we impose further regularity assumptions. Let $k_D$ and $k_F$ be fixed positive constants and set

$$\begin{aligned} Q^0 &= \{(D,F) \in W^{4,\infty}(0,M+T) \times W^{4,\infty}(0,M+T) \mid \\ &\qquad D \geq 0,\ \|D\|_{4,\infty} \leq k_D,\ F \geq 0,\ \|F\|_{4,\infty} \leq k_F\}, \end{aligned}$$

where $W^{4,\infty}$ denotes the usual Sobolev space equipped with the norm $\|\cdot\|_{4,\infty}$. By the Rellich-Kondrachov compactness Theorem [19, p.12], $Q^0$ forms a precompact subset of $C^3[0,M+T] \times C^3[0,M+T]$. Define $\tilde{Q}$ to be the closure of $Q^0$ in $C^3[0,M+T] \times C^3[0,M+T]$. Then, it is clear that $\tilde{Q}$ is compact as a subset of $Q$, and

$$\begin{aligned} \tilde{Q} &\subset \{(D,F) \in C^3[0,M+T] \times C^3[0,M+T] \mid \\ &\qquad D \geq 0,\ \|D\|_{3,\infty} \leq k_D,\ F \geq 0,\ \|F\|_{3,\infty} \leq k_F\}. \end{aligned} \tag{2.1}$$

It is easy to see that Lemma 1.2 implies the continuity of the cost functional J. Since the admissible parameter subset $\tilde{Q}$ is compact we have

**Theorem 2.1** *The problem (ID) has a solution.*

Now we construct an approximation scheme for the parameter estimation problem (ID).

First, we construct a sequence $\{Q_{M,K}\}_{(M,K)\in\mathbb{N}^2}$ of finite dimensional subspaces of the parameter space $Q$. For $s \in \mathbb{N}$, let $L^s$ be the set of all piecewise linear spline functions on $[0,M+T]$ with respect to the nodes $\{(i/s)(M+T)\}_{0\leq i\leq s}$, and let $I^s$ be the linear spline interpolating operator on $[0,M+T]$ at the nodes. Define a map $P^{M,K} : \tilde{Q} \to L^M \times L^K$ by $P^{M,K}(D,F) = (I^M D, I^K F)$ and set $Q_{M,K} = P^{M,K}(\tilde{Q})$. Then, it is clear that $Q_{M,K}$ is a compact subset of $Q$.

Next, we construct a sequence $\{H^N\}_{N\in\mathbb{N}}$ of finite dimensional subspaces of the state space $C([0,M+T]\times[0,T])$. Define $H^N$ to be the set of all continuous functions on $[0,M+T] \times [0,T]$ which is piecewise linear in each age and time variable with respect to the nodes $\{(iT/N, jT/N) \mid 0 \leq i \leq (M/T+1)N,\ 0 \leq j \leq N\}$.

Finally we approximate the problem (ID) by a sequence of finite dimensional problems. Let $\Gamma$ be one of the observation operators described in section 1 and let $\Phi$ be the corresponding parameter-to-output mapping.

PROBLEM $(\mathrm{ID}^{\mathrm{N}}_{\mathrm{M,K}})$ Given a set of measurements $z = \{z_i\}_{i=1}^m$, find $q^* \in Q_{M,K}$ that minimizes

$$\mathrm{J}^N(q) := \|\Phi^N(q) - z\|_2^2 = \sum_{k=1}^{m} \left[z_k - \Gamma^k \left[u^N(q)\right]\right]^2,$$

where $u^N \in H^N$ is the unique linear spline function on $[0, \mathrm{M}+\mathrm{T}] \times [0, \mathrm{T}]$ matching the FDM solution with step size $\Delta t = \mathrm{T}/N$ as we described in section 1.

We hope that each problem $(\mathrm{ID}^{\mathrm{N}}_{\mathrm{M,K}})$ has a solution $q^N_{M,K}$ and that the sequence $\{q^N_{M,K}\}$ converges to a solution of the original problem (ID). More precisely, we will show in the following that

(H1) For each $N, M, K$, there exists a solution $q^N_{M,K} \in Q_{M,K}$ of $(\mathrm{ID}^{\mathrm{N}}_{\mathrm{M,K}})$ and there exists a subsequence of $\{q^N_{M,K}\}$ converging to an element in $Q$.

(H2) For every convergent subsequence $\{q^{N_k}_{M_k,K_k}\}$ of $\{q^N_{M,K}\}$ the following properties hold :

(a) its limit is a solution $q^* \in \tilde{Q}$ of (ID).

(b) $\|u^{N_k}(q^{N_k}_{M_k,K_k}) - u(q^*)\| \to 0$ as $k \to \infty$.

(c) $|\mathrm{J}^{N_k}(q^{N_k}_{M_k,K_k}) - \mathrm{J}(q^*)| \to 0$ as $k \to \infty$.

A parameter estimation scheme satisfying the above conditions (H1) and (H2) is called function space parameter estimation convergent (FSPEC) [11, p.61].

To prove FSPEC of our finite difference approximation scheme we prepare the following three lemmas.

**Lemma 2.2** *For each $\tilde{q} \in \tilde{Q}$, we have $\|P^{M,K}\tilde{q} - \tilde{q}\| \to 0$ as $M, K \to \infty$, where the convergence is uniform on $\tilde{Q}$.*

*Proof.* Recalling the definitions of $\tilde{Q}$, $Q_{M,K}$ and $P^{M,K}$ the lemma is the direct consequence of the well-known estimate [18] for the interplating operators. In fact, we know that there exists a positive constant K independent of $s$ such that

$$\|h - I^s h\|_\infty \le \mathrm{K} s^{-2} \|D^2 h\|_\infty$$

for all $h \in W^{2,\infty}(0, \mathrm{M}+\mathrm{T})$. By (2.1) we obtain the result. ■

**Lemma 2.3** *For any sequence $\{q^N_{M,K}\}$ in $Q$, $q^N_{M,K} \in Q_{M,K}$, there exists a convergent subsequence $\{q^{N_k}_{M_k,K_k}\}$ of $\{q^N_{M,K}\}$ in $Q$. Moreover, any subsequential limit of the sequence $\{q^N_{M,K}\}$ is contained in $\tilde{Q}$.*

*Proof.* First, we construct a sequence $\{\tilde{q}^N_{M,K}\}$ in $\tilde{Q}$ by arbitrary choosing $\tilde{q}^N_{M,K}$ in $(P^{M,K})^{-1}(q^N_{M,K})$, so that $P^{M,K}(\tilde{q}^N_{M,K}) = q^N_{M,K}$ holds. This is possible since each map

$P^{M,K} : \tilde{Q} \to Q_{M,K}$ is surjective. The compactness of $\tilde{Q}$ implies that there exists a subsequence $\{\tilde{q}^{N_k}_{M_k,K_k}\}$ of $\{\tilde{q}^{N}_{M,K}\}$ which converges to an element in $\tilde{Q}$. Let

$$\tilde{q}^{N_k}_{M_k,K_k} \to q^* \in \tilde{Q} \quad \text{as } k \to \infty. \tag{2.2}$$

We will show that the corresponding subsequence $\{q^{N_k}_{M_k,K_k}\}$ of the original sequence $\{q^{N}_{M,K}\}$ converges to the same $q^*$. In fact, by the triangle inequality, we have

$$\|q^{N_k}_{M_k,K_k} - q^*\| \le \|q^{N_k}_{M_k,K_k} - \tilde{q}^{N_k}_{M_k,K_k}\| + \|\tilde{q}^{N_k}_{M_k,K_k} - q^*\|.$$

By (2.2) the second term on the RHS goes to zero as $k$ goes to infinity. On the other hand the first term also goes to zero by Lemma 2.2 since $q^{N_k}_{M_k,K_k} = P^{M_k,K_k}(\tilde{q}^{N_k}_{M_k,K_k})$.

Now, to prove the second statement, let $\{q^{N_k}_{M_k,K_k}\}$ be a convergent subsequence of $\{q^{N}_{M,K}\}$. Applying the same process to this subsequence we know that the subsequential limit is contained in $\tilde{Q}$. The proof is completed. ∎

**Lemma 2.4** *Let $\{q^k\}$ be a sequence in $Q$ which converges to an element $q^* \in \tilde{Q}$. Then, for any increasing sequence $\{N_k\}$ in $\mathbb{N}$, we have*

$$\|u^{N_k}(q^k) - u(q^*)\| \to 0 \tag{2.3}$$

$$|\mathrm{J}^{N_k}(q^k) - \mathrm{J}(q^*)| \to 0 \tag{2.4}$$

*as $k$ goes to infinity.*

*Proof.* By (2.1) we know that the solution $u(q^*)$ of (0.1)–(0.3) satisfies the conditions in Remark 1.6. So we obtain (2.3) by Theorem 1.9. Recalling the definition of cost functionals, (2.4) is the consequence of (2.3). ∎

Now we can prove our main result.

**Theorem 2.5** *The parameter estimation scheme is FSPEC.*

*Proof.* First, we verify (H1). Note that Lemma 1.8 implies the continuity of $\mathrm{J}^N$. Then, from the compactness of $Q_{M,K}$ the existence of a solution $q^N_{M,K}$ of $(\mathrm{ID}^{\mathrm{N}}_{\mathrm{M,K}})$ follows. The existence of a convergent subsequence of $\{q^N_{M,K}\}$ is shown in Lemma 2.3.

To verify (H2)-(b) and (H2)-(c) let $\{q^{N_k}_{M_k,K_k}\}$ be a convergent subsequence of $\{q^N_{M,K}\}$ and let $q^*$ be its limit point. By Lemma 2.3 we know that $q^* \in \tilde{Q}$. Then, by Lemma 2.4, we obtain (H2)-(b) and (H2)-(c). To show (H2)-(a) let $\{q^{N_k}_{M_k,K_k}\}$ be a subsequence of $\{q^N_{M,K}\}$ which converges to $q^* \in Q$. We already know that $q^* \in \tilde{Q}$. Thus it is only remained to show that this limit $q^*$ is a solution to the original problem (ID). Let $\tilde{q} \in \tilde{Q}$ be arbitrarily fixed. Define a sequence $\{q^k\}$ by

$q^k = P^{M_k,K_k}\tilde{q} \in Q_{M_k,K_k}$. Then, by Lemma 2.2, we know that the sequence $\{q^k\}$ converges to $\tilde{q} \in \tilde{Q}$. Now by Lemma 2.4 we deduce that

$$|\mathrm{J}^{N_k}(q^k) - \mathrm{J}(\tilde{q})| \to 0. \tag{2.5}$$

On the other hand, since each $q^{N_k}_{M_k,K_k}$ is a solution to $(\mathrm{ID}^{\mathrm{N_k}}_{\mathrm{M_k,K_k}})$ and $q^k \in Q_{M_k,K_k}$, we know that

$$\mathrm{J}^{N_k}(q^{N_k}_{M_k,K_k}) \le \mathrm{J}^{N_k}(q^k) \quad \text{for each } k \in \mathbb{N}. \tag{2.6}$$

Sending $k$ to infinity in (2.6) we obtain from the condition (H2)-(c) and (2.5) that $\mathrm{J}(q^*) \le \mathrm{J}(\tilde{q})$ for all $\tilde{q} \in \tilde{Q}$, which states that $q^*$ is a solution of the problem (ID). The proof is completed. ■

## 3. Numberical Results

To illustrate our theory we present an example. All the numerical computations were excuted on a SUN-SPARC 20 workstation using MATLAB.

The true functions for the death-rate D and the fertility F are assumed as follows:

$$\mathrm{D}(x) = \frac{e^{x+1}}{1+5(x+1)^2}; \tag{3.1}$$

$$\mathrm{F}(x) = \begin{cases} \dfrac{\pi^3}{16}x(2-x)\sin(\frac{1}{2}\pi x) & \text{if } 0 \le x \le 2, \\ 0 & \text{elsewhere.} \end{cases} \tag{3.2}$$

We set M = 2, T = 2 and the initial age-distribution is assumed as

$$u_0(x) = \begin{cases} 2-x & \text{if } 0 \le x < 2, \\ 0 & \text{elsewhere.} \end{cases} \tag{3.3}$$

The second order convergence of FDM solutions for this model is shown in Table 1. Figure 1 shows the FDM solutions for various step sizes at the time T = 2.

**Example 3.1** [Estimation of the age-specific death-rate D]. In this example, we estimate the death-rate under the assumption that the fertility F is known as in (3.2). The observation operator is $\Gamma_1$ : Uniformly distributed 33 observation points $\{\mathrm{a}_i\}_{i=0}^{32}$ are selected in the age-interval $[0, \mathrm{M}+\mathrm{T}]$, where $\mathrm{a}_i = i/8$, $i = 0, \dots, 32$. The data $\{z_i^{\mathrm{a}}\}_{i=0}^{32}$ are collected by calculating the FDM solution at the points $\{(\mathrm{a}_i, 2)\}$ with true parameters D and F and with step-size $\Delta t = 1/1024$. We did not optimize the step size since the computation was done in a minute.

We started with the constant function 0 as an initial guess. Table 2 shows the FSPEC. The OLS-Error in Table 2 means the output-least-squared error

$$\sum_{i=0}^{32}\left\{z_i^{\mathrm{a}} - \Gamma_1^i[u^N(\mathrm{D}^{\mathrm{N}}_{\mathrm{M}})]\right\}^2 = \sum_{i=0}^{32}[z_i^{\mathrm{a}} - u^N(\mathrm{a}_i, 2; \mathrm{D}^{\mathrm{N}}_{\mathrm{M}})]^2,$$

Table 1: The convergence of FDM solutions

| N | $\Delta t$ | $\|w(N) - w(2N)\|_\infty$ | $\frac{\|w(N)-w(2N)\|_\infty}{\|w(2N)-w(4N)\|_\infty}$ |
|---|---|---|---|
| 16 | 1/8 | 8.3244e-02 | 3.5078 |
| 32 | 1/16 | 2.2976e-02 | 3.6231 |
| 64 | 1/32 | 6.0669e-03 | 3.7871 |
| 128 | 1/64 | 1.5607e-03 | 3.8874 |
| 256 | 1/128 | 3.9589e-04 | 3.9421 |
| 512 | 1/256 | 9.9705e-05 | 3.9707 |
| 1024 | 1/512 | 2.5019e-05 | 3.9852 |
| 2048 | 1/1024 | 6.2662e-06 | 3.9926 |

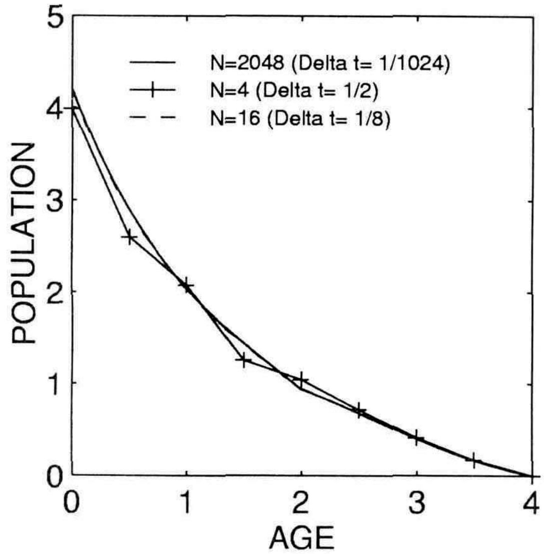


Figure 1: The convergence of FDM solutions

where $D_M^N$ is the estimated approximation for D, N is the dimension of the approximated state space and M is the dimension of the approximated parameter space. Figure 2 shows the comparison of the true and estimated death-rates for various M and N. Observe that the estimated parameters converges to the true D as M and N increase.

Table 2: Death-rate D estimation

| M | N | OLS-Error |
|---|---|---|
| 3 | 16 | 6.6495e-02 |
| | 32 | 6.2660e-02 |
| | 64 | 6.3306e-02 |
| 5 | 16 | 2.3795e-02 |
| | 32 | 1.5083e-02 |
| | 64 | 1.5518e-02 |
| 9 | 16 | 2.1609e-02 |
| | 32 | 3.0701e-03 |
| | 64 | 2.4022e-03 |

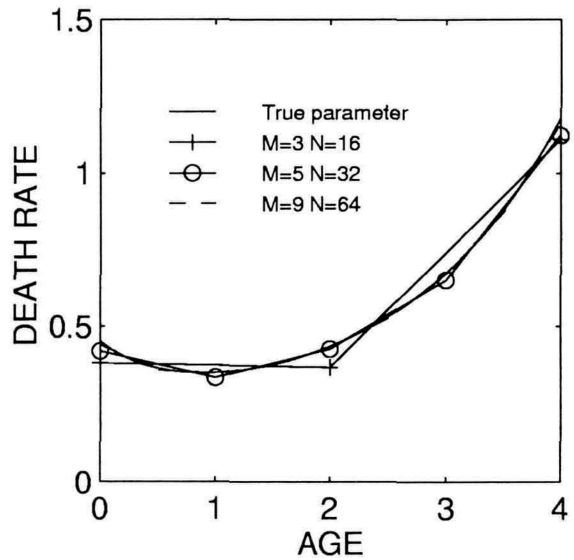


Figure 2: Death-rate D estimation

**Example 3.2** [Estimation of the age-specific fertility F] In this example, we estimate the fertility function under the assumption that the death-rate D is known as in (3.1). The observation operator is $\Gamma_2$ : Uniformly distributed 17 observation points $\{t_j\}_{j=0}^{16}$ are selected in the time-interval $[0, T]$, where $t_j = j/8$, $j = 0, \ldots, 16$. The data $\{z_j^t\}_{j=0}^{16}$ are collected by calculating the FDM solution at the points $\{(0, t_j)\}$ with true parameters D and F and with step-size $\Delta t = 1/1024$.

It is assumed that the fertility is known to be zero at the age 0 and after the age 2. Thus we have estimated the fertility as a function defined on the age-interval $[0, 2]$. We started with the constant function 0 as an initial guess for the fertility F.

Table 3: Fertility F estimation

| K | N | OLS-Error |
|---|---|---|
| 3 | 16 | 2.2594e-02 |
| | 32 | 2.0919e-02 |
| | 64 | 2.2041e-02 |
| 5 | 16 | 1.1428e-02 |
| | 32 | 6.1997e-03 |
| | 64 | 6.6859e-03 |
| 9 | 16 | 8.6978e-03 |
| | 32 | 6.2797e-04 |
| | 64 | 2.7813e-04 |

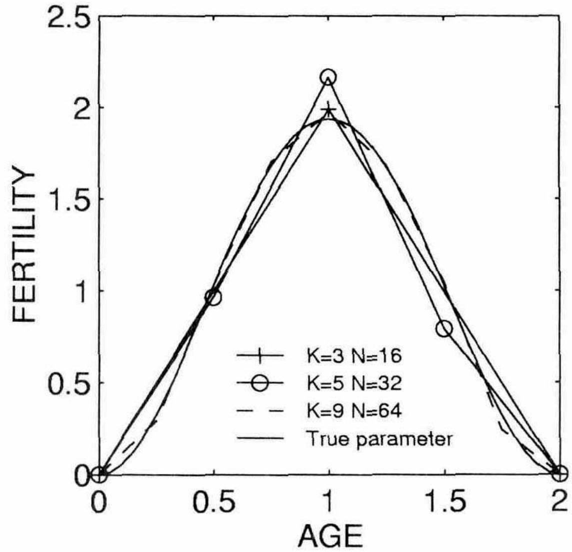


Figure 3: Fertility F estimation

Table 3 shows the FSPEC. The OLS-Error means the output-least-squared error

$$\sum_{j=0}^{16}\left\{z_j^{\mathrm{t}} - \Gamma_2^j[u^N(\mathrm{F}_{\mathrm{K}}^{\mathrm{N}})]\right\}^2 = \sum_{j=0}^{16}[z_j^{\mathrm{t}} - u^N(0,\mathrm{t}_j;\mathrm{F}_{\mathrm{K}}^{\mathrm{N}})]^2,$$

where $\mathrm{F}_{\mathrm{K}}^{\mathrm{N}}$ is the estimated approximation for F, N is the dimension of the approximated state space and K is the dimension of the approximated parameter space. Figure 3 shows the comparison of the true and estimated fertility for various K and N. Observe that the estimated parameters converges to the true F as K and N increase.

Table 4: D–F estimation

| K | M | N | OLS-Error |
|---|---|---|---|
| 3 | 3 | 16 | 8.2695e-02 |
| | | 32 | 7.7511e-02 |
| | | 64 | 7.8473e-02 |
| 5 | 5 | 16 | 2.6467e-02 |
| | | 32 | 1.5894e-02 |
| | | 64 | 1.6158e-02 |
| 9 | 9 | 16 | 2.3045e-02 |
| | | 32 | 2.2773e-03 |
| | | 64 | 2.3004e-03 |

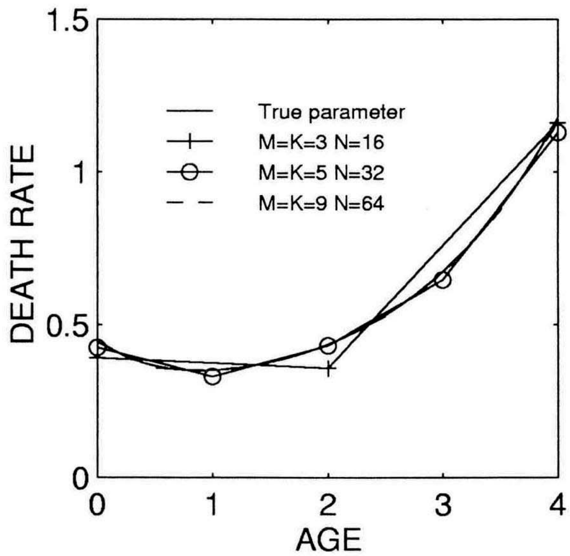


Figure 4: D-F estimation (death-rate)

**Example 3.3** [Estimation of F and D] Now we estimate both F and D simultaneously. The observation operator is $\Gamma_0$ : The observations are made on the set of points $\{(\mathrm{a}_i, 2)\}_{i=0}^{32} \cup \{(0, \mathrm{t}_j)\}_{j=0}^{16}$, where $\{\mathrm{a}_i\}$ and $\{\mathrm{t}_j\}$ are the same as in previous examples. The data $(z^{\mathrm{a}}, z^{\mathrm{t}}) = (z_0^{\mathrm{a}}, \ldots, z_{32}^{\mathrm{a}}, z_0^{\mathrm{t}}, \ldots, z_{16}^{\mathrm{t}})$ is calculated by the FDM solution with true parameters and with step-size $\Delta t = 1/1024$.
As in Example 3.2 it is assumed that the fertility is known to be zero at the age 0 and

after the age 2. We started with the constant function 0 both for the death-rate D and for the fertility F as initial guesses. Table 4 shows the FSPEC. The OLS-Error means the output-least-squared error

$$\sum_{i=0}^{32}[z_i^{\mathrm{a}} - u^N(\mathrm{a}_i, 2; \mathrm{D}_{\mathrm{K,M}}^{\mathrm{N}}, \mathrm{F}_{\mathrm{K,M}}^{\mathrm{N}})]^2 + \sum_{j=0}^{16}[z_j^{\mathrm{t}} - u^N(0, \mathrm{t}_j; \mathrm{D}_{\mathrm{K,M}}^{\mathrm{N}}, \mathrm{F}_{\mathrm{K,M}}^{\mathrm{N}})]^2,$$

where $\mathrm{F}_{\mathrm{K,M}}^{\mathrm{N}}$ and $\mathrm{D}_{\mathrm{K,M}}^{\mathrm{N}}$ are the estimated approximations for F and D, respectively, N is the dimension of the approximated state space, M is the dimension of the approximated parameter space for D, and K is the dimension of the approximated parameter space for F. Figure 4 and Figure 5 show the comparison of the true and estimated parameters for various K,M and N. It shows that the estimated parameters converges to the true parameters as K,M and N increase.

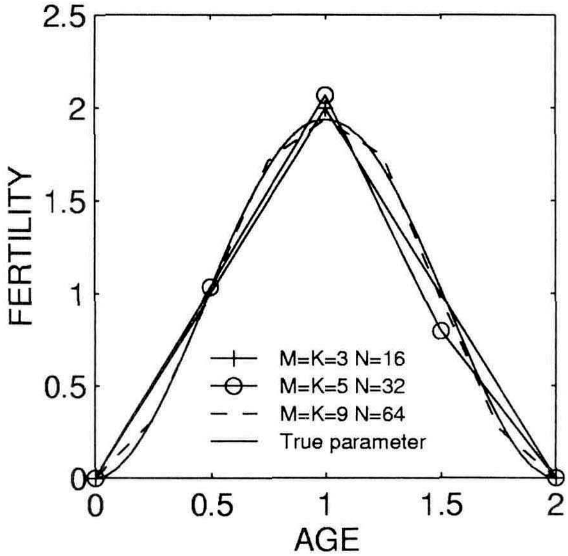


Figure 5: D-F estimation (fertility)

In the above examples, for the minimization process, we adopted the Levenberg-Marquardt method [20], which has been commonly used for the least squared error functional. We used finite differences to approximate the derivatives of cost functionals.

## 4. Concluding Remarks

A simple McKendric type integro-differential equation is considered as a mathematical model for the dynamics of one-sex population. We developed an approximation scheme (FDM scheme) for estimating distributed parameters such as the age-specific death-rate and the age-specific fertility. Numerical experiments support the function space parameter estimation convergence of our scheme.

In this paper we have assumed the pointwise observations. In practical situations the observations might be possible only for the integrals on several subintervals. It is obvious that the approximation scheme can be adjusted for that case by changing observation operators.

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