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# ON THE NORMAL BUNDLE OF A SUBMANIFOLD IN A KÄHLER MANIFOLD

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Abstract. We show that the normal bundle of a Lagrangian submanifold in a Kähler manifold has a symplectic structure and provide the equivalent conditions for the normal bundle of such to be Kähler.

## 1. Preliminaries

We consider a submanifold $\tilde{M}$ of a Riemannian manifold $(M^{2n}, g)$. A Riemannian metric $G$ is induced on $\tilde{M}$ and there is also a metric $G^{\perp}$ induced on each fiber of the normal bundle $N\tilde{M}$ of $\tilde{M}$. We call by $D$ the Riemannian connection of $(\tilde{M}, G)$. The normal connection $D^{\perp}$ and its curvature tensor $R^{\perp}$ are defined as usual (in the sense of [3]). It is well known (Refer to [1]) that on the normal bundle $N\tilde{M}$, there is a naturally induced metric $\tilde{g}$, called the Sasaki metric. This metric structure was determined, by Recziegel [4], in an invariant manner:

$$\tilde{g}(\tilde{X}, \tilde{Y}) = G(\pi_* \tilde{X}, \pi_* \tilde{Y}) + G^{\perp}(K\tilde{X}, K\tilde{Y})$$

where $\pi_*$ is the differential of the projection map $\pi : N\tilde{M} \to \tilde{M}$ of the normal bundle $N\tilde{M}$ and $K : TN\tilde{M} \to N\tilde{M}$ is the connection map. Note that both the mappings $\pi_*$ and $K$ are onto and fiber-preserving linear transformations.

We call the kernels of the mappings $\pi_*$ and $K$ the vertical subspace $VN\tilde{M}$ and the horizontal subspace $HN\tilde{M}$, respectively. Then, the

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vertical subspace and the horizontal subspace are orthogonal in the Sasaki metric and the vertical space is tangent to the fiber. Moreover, there is a decomposition:

$$T_{\tilde{x}}N\tilde{M} = H_{\tilde{x}}N\tilde{M} \oplus V_{\tilde{x}}N\tilde{M}$$

for each $\tilde{x} = (x, \xi) \in N\tilde{M}$.

Given a tangent vector field $X$ and a normal vector field $\eta$ to $\tilde{M}$, there are defined the horizontal lift $X^H \in HN\tilde{M}$ and the vertical lift $\eta^V \in VN\tilde{M}$ such that

$$\pi_* X^H = X, KX^H = 0, \pi_* \eta^V = 0, \text{ and } K\eta^V = \eta.$$

Thus, we have that at the point $\tilde{x} = (x, \xi)$

$$\begin{aligned} \tilde{g}(X^H, Y^H)_{\tilde{x}} &= G(\pi_* X^H, \pi_* Y^H)_x = G(X, Y)_x \\ \tilde{g}(X^H, \eta^V)_{\tilde{x}} &= G(\pi_* X^H, \pi_* \eta^V)_x + G^\perp(KX^H, K\eta^V)_\xi = 0 \\ \tilde{g}(\eta^V, \zeta^V)_{\tilde{x}} &= G^\perp(K\eta^V, K\zeta^V)_\xi = G^\perp(\eta^V, \zeta^V)_\xi \end{aligned} \tag{1}$$

Now, since we can write

$$\tilde{X} = (\pi_* \tilde{X})^H + (K\tilde{X})^V$$

for a tangent vector $\tilde{X}$ to $N\tilde{M}$, it is enough to consider various combinations of horizontally and vertically lifted vector fields.

We will need the following lemmas. Proofs are routine and we omit them here.

Lemma 1.1 ([1]). *Let $X$ and $Y$ be tangent vector fields, and $\eta$ and $\zeta$ normal vector fields of $\tilde{M}$. Then, at each point $(x, \xi)$ of the normal bundle $N\tilde{M}$, the Lie brackets are:*

$$[\eta^V, \zeta^V] = 0, \qquad [X^H, \eta^V] = (D^\perp_X \eta)^V,$$
$$\pi_*[X^H, Y^H] = [X, Y], \qquad K[X^H, Y^H] = -R^\perp_{XY}\xi.$$

By definition, $R^\perp_{XY}\eta$ is a normal vector field of $\tilde{M}$. For any normal vector field $\zeta$, we may compute the inner product $g(R^\perp_{XY}\eta, \zeta)$. We define the adjoint $\hat{R}_{\eta\zeta}X$ by the equality $G(\hat{R}_{\eta\zeta}X, Y) = g(R^\perp_{XY}\eta, \zeta)$. The covariant derivatives with respect to the Riemannian connection $\tilde{\nabla}$ of the Sasaki metric $\tilde{g}$ on $N\tilde{M}$ are easily computed. And, we have

Lemma 1.2 ([1]). *Let $X$ and $Y$ be tangent vector fields, and $\eta$ and $\zeta$ normal vector fields of $\tilde{M}$. Then, at each point $(x, \xi)$ of the normal bundle $N\tilde{M}$,*

$$\tilde{\nabla}_{\eta^V}\zeta^V = 0\,, \qquad \tilde{\nabla}_{X^H}\zeta^V = (D^{\perp}_X\zeta)^V + \frac{1}{2}(\hat{R}_{\xi\zeta}X)^H\,,$$
$$\tilde{\nabla}_{\eta^V}Y^H = \frac{1}{2}(\hat{R}_{\xi\eta}Y)^H\,, \qquad \tilde{\nabla}_{X^H}Y^H = (D_XY)^H - \frac{1}{2}(R^{\perp}_{XY}\xi)^V\,.$$

## 2. Main results

Let us consider a Lagrangian submanifold $L$ of a Kähler manifold $(M^{2n}, J, g)$. On the normal bundle $NL$ of $L$, we define $\tilde{J}$ by

$$\tilde{J}X^H = (JX)^V$$

and

$$\tilde{J}\xi^V = (J\xi)^H$$

for any tangent vector field $X$ and normal vector field $\xi$ to $L$. Then, it is easy to see that $\tilde{J}$ is an almost complex structure on $NL$.

Thus, we have

Proposition 2.1. *$(NL, \tilde{J}, \tilde{g})$ is an almost Hermitian manifold.*

*Proof.* By the argument above, it remains to show that $\tilde{J}$ is compatible with the Sasaki metric $\tilde{g}$ on $NL$. And, this follows immediately from the compatibility of $J$ with $g$ and (1). □ □

Let us denote by $\tilde{\nabla}$ and $\nabla$ the Riemannian connection of $\tilde{g}$ and $g$, respectively. Let $G$ be the induced metric on $L$, $D$ its Riemannian connection, and $\underline{R}$ the curvature tensor on $L$.

We now present our main theorems.

Theorem 2.2. *Let $L$ be a Lagrangian submanifold of a Kähler manifold $(M^{2n}, J, g)$. Then, $(NL, \tilde{J}, \tilde{g})$ is a symplectic manifold.*

*Proof.* By Proposition 2.1, it remains to show that the fundamental 2-form $\Omega$ defined by $\Omega(\tilde{X}, \tilde{Y}) = \tilde{g}(\tilde{X}, \tilde{J}\tilde{Y})$ is non-degenerate and closed. The non-degeneracy of $\Omega$ is immediate since $\tilde{g}$ is positive definite and $\tilde{J}$ is non-singular at each point.

In order to show that $\Omega$ is closed, we first observe that

$$\begin{aligned} \Omega(X^H, Y^H) &= \tilde{g}(X^H, \tilde{J}Y^H) \\ &= g(\pi_* X^H, \pi_* \tilde{J}Y^H) + g(KX^H, K\tilde{J}Y^H) \\ &= 0\,, \end{aligned} \tag{2}$$

$$\begin{aligned} \Omega(X^H, \eta^V) &= \tilde{g}(X^H, \tilde{J}\eta^V) \\ &= g(\pi_* X^H, \pi_* \tilde{J}\eta^V) + g(KX^H, K\tilde{J}\eta^V) \\ &= g(X, J\eta), \end{aligned} \tag{3}$$

and likewise,

$$\Omega(\eta^V, \zeta^V) = 0. \tag{4}$$

We recall here the coboundary formula

$$\begin{aligned} 3d\Phi(X, Y, Z) = X\Phi(Y, Z) + Y\Phi(Z, X) + Z\Phi(X, Y) \\ - \Phi([X, Y], Z) - \Phi([Y, Z], X) - \Phi([Z, X], Y). \end{aligned}$$

Using Lemma 1.1 and (2) - (4), we compute

$$\begin{aligned} 3d\Omega(X^H, Y^H, Z^H) &= -\,\Omega([X^H, Y^H], Z^H) - \Omega([Y^H, Z^H], X^H) \\ &\quad - \Omega([Z^H, X^H], Y^H) \\ &= \Omega((R^{\perp}_{XY}\xi)^V, Z^H) + \Omega((R^{\perp}_{YZ}\xi)^V, X^H) \\ &\quad + \Omega((R^{\perp}_{ZX}\xi)^V, Y^H) \end{aligned}$$

But, from the Gauss-Weingarten equations, we have

$$\nabla_X J\xi = D_X J\xi + \sigma(X, J\xi)$$

and

$$J\nabla_X\xi = -JA_\xi X + JD^\perp_X\xi\,.$$

We compare tangential parts of these using Kähler condition and get

$$JD^\perp_X\xi = D_X J\xi\,. \tag{5}$$

Continuing our computation, using this and the Bianchi identity, we get

$$\begin{aligned}
3d\Omega(X^H,Y^H,Z^H) &= -\,g(\underline{R}_{XY}J\xi,Z) - g(\underline{R}_{YZ}J\xi,X) - g(\underline{R}_{ZX}J\xi,Y)\\
&= g(\underline{R}_{XY}Z,J\xi) + g(\underline{R}_{YZ}X,J\xi) + g(\underline{R}_{ZX}Y,J\xi)\\
&= 0 \qquad (6)
\end{aligned}$$

Likewise, we compute, using Lemma 1.2 and (2) - (4),

$$\begin{aligned}
3d\Omega(X^H,Y^H,\eta^V) &= X^H\Omega(Y^H,\eta^V) + Y^H\Omega(\eta^V,X^H)\\
&\quad - \Omega([X^H,Y^H],\eta^V) - \Omega([Y^H,\eta^V],X^H)\\
&\quad - \Omega([\eta^V,X^H],Y^H)\\
&= \tilde g(\tilde\nabla_{X^H}Y^H,(J\eta^V)^H) + \tilde g(Y^H,\tilde\nabla_{X^H}(J\eta)^H)\\
&\quad + \tilde g(\tilde\nabla_{Y^H}\eta^V,(JX)^V) + \tilde g(\eta^V,\tilde\nabla_{Y^H}(JX)^V)\\
&\quad - \Omega([X,Y]^H - (R^\perp_{XY}\xi)^V,\eta^V)\\
&\quad - \Omega((D^\perp_Y\eta)^V,X^H) + \Omega((D^\perp_X\eta)^V,Y^H)\\
&= \tilde g((D_XY)^H,(J\eta)^H) + \tilde g(Y^H,(D_XJ\eta)^H)\\
&\quad + \tilde g((D^\perp_Y\eta)^V,(JX)^V) + \tilde g(\eta^V,(D^\perp_YJX)^V)\\
&\quad - \Omega([X,Y]^H,\eta^V) - \Omega((D^\perp_Y\eta)^V,X^H)\\
&\quad + \Omega((D^\perp_X\eta)^V,Y^H)\\
&= \tilde g((D_XY)^H,(J\eta)^H) + \tilde g(\eta^V,(D^\perp_YJX)^V)\\
&\quad - \tilde g([X,Y]^H,(J\eta)^H)\\
&= \tilde g((D_XY)^H,\tilde J\eta^V) + \tilde g(\tilde J\eta^V,-(D_XY)^H)\\
&\quad - \tilde g([X,Y]^H,\tilde J\eta^V)\\
&= 0 \qquad (7)
\end{aligned}$$

and

$$
\begin{aligned}
3d\Omega(X^H,\eta^V,\zeta^V) &= \eta^V\Omega(\zeta^V,X^H)+\zeta^V\Omega(X^H,\eta^V)\\
&\quad -\Omega([X^H,\eta^V],\zeta^V)-\Omega([\eta^V,\zeta^V],X^H)\\
&\quad -\Omega([\zeta^V,X^H],\eta^V)\\
&= \tilde{g}(\tilde{\nabla}_{\eta^V}\zeta^V,\tilde{J}X^H)+\tilde{g}(\zeta^V,\tilde{\nabla}_{\eta^V}(JX)^V)\\
&\quad +\tilde{g}(\tilde{\nabla}_{\zeta^V}X^H,(J\eta)^H)+\tilde{g}(X^H,\tilde{\nabla}_{\zeta^V}(J\eta)^H)\\
&\quad -\Omega((D^\perp_X\eta)^V,\zeta^V)+\Omega((D^\perp_X\zeta)^V,\eta^V)\\
&= \frac{1}{2}\,\tilde{g}((\hat{R}_{\xi\zeta}X)^H,(J\eta)^H)+\frac{1}{2}\,\tilde{g}((\hat{R}_{\xi\zeta}J\eta)^H,X^H)\\
&= 0 \qquad (8)
\end{aligned}
$$

Finally,

$$3d\Omega(\eta^V,\zeta^V,\delta^V)=0 \qquad (9)$$

is trivial. This completes our proof. □ □

THEOREM 2.3. *Let $L$ be a Lagrangian submanifold of a Kähler manifold $(M^{2n},J,g)$. Then, the followings are equivalent:*

(1) *$NL$ is Kähler.*
(2) *$L$ has flat normal connection.*
(3) *$L$ is flat.*

*Proof.* We compute the Nijenhuis torsion.

$$
\begin{aligned}
[\tilde{J},\tilde{J}](X^H,Y^H) &= -[X^H,Y^H]+[\tilde{J}X^H,\tilde{J}Y^H]\\
&\quad -\tilde{J}[(JX)^V,Y^H]-\tilde{J}[X^H,(JY)^V]\\
&= -[X^H,Y^H]+(JD^\perp_Y JX)^H-(JD^\perp_X JY)^H\\
&= -[X,Y]^H+(R^\perp_{XY}\xi)^V+(JD^\perp_Y JX-JD^\perp_X JY)^H
\end{aligned}
$$

So, using (5), we have

$$
\begin{aligned}
[\tilde{J},\tilde{J}](X^H,Y^H) &= -[X^H,Y^H]+(R^\perp_{XY}\xi)^V-(D_YX-D_XY)^H\\
&= -[X,Y]^H+(R^\perp_{XY}\xi)^V-[Y,X]^H\\
&= (R^\perp_{XY}\xi)^V \qquad (10)
\end{aligned}
$$

$$\begin{aligned}[\tilde{J},\tilde{J}](X^H,\zeta^V) &= -[X^H,\zeta^V]+[(JX)^V,(J\zeta)^H] \\ &\quad -\tilde{J}[(JX)^V,\zeta^V]-\tilde{J}[X^H,(J\zeta)^H] \\ &= -(D^\perp_X\zeta)^V-(D^\perp_{J\zeta}JX)^V-\tilde{J}([X,J\zeta]^H-(R^\perp_{XJ\zeta}\xi)^V) \\ &= -(D^\perp_X\zeta)^V-(D^\perp_{J\zeta}JX)^V-(J[X,J\zeta])^V+(JR^\perp_{XJ\zeta}\xi)^H \\ &= -(\nabla_X\zeta)^V-(A_\zeta X)^H-(\nabla_{J\zeta}JX)^V-(A_{JX}J\zeta)^V \\ &\quad -(J\nabla_X J\zeta)^V+(J\nabla_{J\zeta}X)^V+(JR^\perp_{XJ\zeta}\xi)^H \\ &= -(\nabla_X\zeta)^V-(A_\zeta X)^H-(\nabla_{J\zeta}JX)^V-(A_{JX}J\zeta)^V \\ &\quad +(\nabla_X\zeta)^V+(\nabla_{J\zeta}JX)^V+(JR^\perp_{XJ\zeta}\xi)^H \\ &= -(A_\zeta X)^V-(A_{JX}J\zeta)^V+(JR^\perp_{XJ\zeta}\xi)^H \qquad (11)\end{aligned}$$

Again, using (5) and the Kähler condition, we see that

$$\begin{aligned}J[X,J\zeta] &= -\nabla_X\zeta-\nabla_{J\zeta}JX \\ &= -D^\perp_X\zeta-D^\perp_{J\zeta}JX+A_\zeta X+A_{JX}J\zeta.\end{aligned}$$

But, since $J[X,J\zeta]$ is normal to $L$, we have

$$A_\zeta X+A_{JX}J\zeta=0.$$

Thus, we see, from (11), that

$$\begin{aligned}[\tilde{J},\tilde{J}](X^H,\zeta^V) &= (JR^\perp_{XJ\zeta}\xi)^H \\ &= (\underline{R}_{XJ\zeta}J\xi)^H \qquad (12)\end{aligned}$$

and

$$\begin{aligned}[\tilde{J},\tilde{J}](\eta^V,\zeta^V) &= -[\eta^V,\zeta^V]+[(J\eta)^H,(J\zeta)^H] \\ &\quad -\tilde{J}[(J\eta)^H,\zeta^V]-\tilde{J}[\eta^V,(J\zeta)^H] \\ &= [(J\eta)^H,(J\zeta)^H]-\tilde{J}(D^\perp_{J\eta}\zeta)^V+\tilde{J}(D^\perp_{J\zeta}\eta)^V \\ &= [J\eta,J\zeta]^H-(R^\perp_{J\eta J\zeta}\xi)^V-(JD^\perp_{J\eta}\zeta)^H+(JD^\perp_{J\zeta}\eta)^H \\ &= [J\eta,J\zeta]^H-(R^\perp_{J\eta J\zeta}\xi)^V-(D_{J\eta}J\zeta)^H+(D_{J\zeta}J\eta)^V \\ &= -(R^\perp_{J\eta J\zeta}\xi)^V \qquad (13)\end{aligned}$$

In view of (10), (12), and (13), we conclude that $[\tilde{J}, \tilde{J}]$ vanishes if and only if $L$ has flat normal connection.

This together with the equations (6) - (9) shows that $NL$ is Kähler if and only if $L$ has flat normal connection.

Moreover, using (11), we have

$$R_{XY} J\xi = JR^{\perp}_{XY}\xi,$$

from which we easily see that $L$ is flat if and only if $NL$ has flat normal connection. □ □

In the proof of the previous theorem, we have also shown

Corollary 2.4. *$\tilde{J}$ on $NL$ is integrable if and only if $L$ has flat connection.*

## References

[1] A. Borisenko and A. Yampol'skii, *On the Sasaki metric of the normal bundle of a submanifold in a Riemannian space*, Mat. Sb. **134(176)** (1987), 157–175; English transl. in Math. USSR Sb. **61** (1988).

[2] Cartan, *Geometry of Riemannian spaces*, Math. Sci. Press, Brookline, Mass., 1983.

[3] B.-Y. Chen, *Geometry of Submanifolds*, Marcel Dekker, New York, 1995.

[4] H. Reckziegel, *On the eigenvalues of the shape operator of an isometric immersion into a space of constant curvature*, Math. Ann. **243** (1979), 71–82.

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