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NORMALITY CONDITION FOR MULTIOBJECTIVE OPTIMIZATION WITH SET FUNCTIONS

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Abstract. In this paper we generalize the concept of normality for the problem involving set functions in a view of Jahn. The main result is that the problem with Slater's constraint qualification is normal and stable.

1. Introduction

The idea of normality was applied to duality in the ordinary convex optimization program by Van Slyke and Wets [9]. In the case of a convex programming, one can find that normality is equivalently described by the closure of a perturbation function [7]. Ponstein described the relationship between the Van Slyke and Wets approach and the Rockafellar approach in the finite dimensional case [6].

For a multiobjective program, normality idea was extended to vector supremization problems by Nieuwenhuis[5] and Borwein[1]. Jahn extended related notions to Pareto-minimization problems[3]. Jahn's normality is defined through all the optimization problems scalarized by the positive cone whereas Nieuwenhuis' involves the primal problem in a direct way.

In this paper we consider normality for the problem involving set functions in a view of Jahn. The main result is that the problem with Slater's constraint qualification is normal and stable.

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2. Multiobjective Dual Programming Problem with Set Functions

In this section, we specify multiobjective programming with set functions and its dual problem through scalarization of Lagrangean function.

Let $(X, \mathcal{A}, \mu)$ be a finite, atomless measure space and $L^1(X, \mathcal{A}, \mu)$ be separable. Then, by considering characteristic function χ_Ω of Ω in $\mathcal{A}$, we can embed $\mathcal{A}$ into $L^\infty(X, \mathcal{A}, \mu)$. In this setting for $\Omega, \Lambda \in \mathcal{A}$, and $\alpha \in I = [0, 1]$, there exists a sequence, called *a Morris sequence*, $\{\Gamma_n\} \subset \mathcal{A}$ such that

$$\chi_{\Gamma_n} \xrightarrow{w^*} \alpha\chi_\Omega + (1-\alpha)\chi_\Lambda,$$

where $\xrightarrow{w^*}$ denotes the *weak**– convergence of elements in $L^\infty(X, \mathcal{A}, \mu)$ [4].

A subfamily $\mathcal{S}$ is said to be *convex* if for every $(\alpha, \Omega, \Lambda) \in I \times \mathcal{S} \times \mathcal{S}$ and every Morris sequence $\{\Gamma_n\}$ associated with $(\alpha, \Omega, \Lambda)$ in $\mathcal{A}$, there exists a subsequence $\{\Gamma_{n_k}\}$ of $\{\Gamma_n\}$ in $\mathcal{S}$. If $\mathcal{S} \subseteq \mathcal{A}$ is convex, then the *weak**–closure $cl(\mathcal{S})$ of $\chi_\mathcal{S}$ in $L^\infty(X, \mathcal{A}, \mu)$ is the *weak**–closed convex hull of $\chi_\mathcal{S}$, and $\overline{\mathcal{A}} = \{f \in L^\infty : 0 \le f \le 1\}$.

DEFINITION 2.1. Let $\mathcal{S}$ be a convex subfamily of $\mathcal{A}$. Let K be a convex cone of R^n. A set function $H : \mathcal{S} \to R^n$ is called K-convex, if given $(\alpha, \Omega_1, \Omega_2) \in I \times \mathcal{S} \times \mathcal{S}$ and Morris-sequence $\{\Gamma_n\}$ in $\mathcal{A}$ associated with $(\alpha, \Omega_1, \Omega_2)$, there exists a subsequence $\{\Gamma_{n_k}\}$ of $\{\Gamma_n\}$ in $\mathcal{S}$ such that

$$\limsup_{k\to\infty} H(\Gamma_{n_k}) \leqq_K \alpha H(\Omega_1) + (1-\alpha) H(\Omega_2),$$

where lim sup is taken over each component. And $x <_K y$ denotes $y - x \in int(K)$,$x \le_K y$ denotes $y - x \in K \setminus \{\mathbf{0}\}$, and $x \leqq_K y$ denotes $y - x \in K$.

DEFINITION 2.2. A set function $H = (H_1, H_2, \ldots, H_n) : \mathcal{S} \to R^n$ is called *weak**-continuous on $\mathcal{S}$ if for each $f \in cl(\mathcal{S})$ and for each $j = 1, 2, \ldots, n$, the sequence $\{H_j(\Omega_k)\}$ converges to the same limit for all $\{\Omega_k\}$ with $\chi_{\Omega_k} \xrightarrow{w^*} f$.

Now multiobjective programming problem with set functions can be described as follows:

$$\text{(P)} \qquad \begin{array}{c} Min_D \quad F(\Omega) \\ \text{subject to } \Omega \in \mathcal{S} \\ \text{and } G(\Omega) \leqq_Q \mathbf{0}, \end{array}$$

which has been defined as the problem finding all feasible efficient $D-$ or properly efficient $D-$solution with respect to the pointed closed convex cones D and Q of Euclidean spaces R^p and R^m with nonempty interiors, D^o and Q^o, respectively. That is, letting $\mathcal{S}' = \{\Omega \in \mathcal{S} \colon G(\Omega) \leqq_Q \mathbf{0}\}$, we want to find $\Omega^* \in \mathcal{S}'$ such that

$$(F(\mathcal{S}') - F(\Omega^*)) \cap (-D) = \{\mathbf{0}\}, \quad \emptyset \text{ if } \mathbf{0} \notin D$$

or

$$cl(p(F(\mathcal{S}') + D - F(\Omega^*))) \cap (-D) = \{\mathbf{0}\}, \emptyset \text{ if } \mathbf{0} \notin D,$$

where the set $p(S) = \{\alpha y \colon \alpha > 0, y \in S\}$ is the projecting cone for a set $S \subset R^p$.

For the primal problem (P), we assume that $F : \mathcal{S} \to R^p, \quad G : \mathcal{S} \to R^m$ are $D-$convex, $Q-$convex, respectively and $weak^*$–continuous. Under these assumptions and Slater's constraint qualification , we have the Lagrange multiplier theorem for a properly efficient D-solution as in usual multiobjective optimization problems [2].

The generalized Slater's constraint qualification that there exists $\Omega_o \in \mathcal{S}$ such that $G(\Omega_o) <_Q \mathbf{0}$ is assumed in the sequel.

We define a dual programming problem based on the scalarization of the vector-valued Lagrangian function. Let $< x, y >$ denote the inner product of two vectors.

Definition 2.3. The dual problem to the problem (P) is defined as

$$Max_D \bigcup_{\mu \in D^o, \lambda \in Q^o} Y_{H-(\mu,\lambda)},$$

where $Y_{H-(\mu,\lambda)}$ is the set

$$\{y \in R^p :< \mu, F(\Omega) > + < \lambda, G(\Omega) > \geq < \mu, y > \text{ for } \Omega \in \mathcal{S}\}.$$

Weak duality theorem for the above dual program follows.

Theorem 2.4. *For any* $y \in \bigcup_{\mu \in D^o, \lambda \in Q^o} Y_{H^-(\mu,\lambda)}$ *and for any* $\Omega \in \mathcal{S}'$, $F(\Omega) \not\leq_D y$.

Proof. Let $\Omega \in \mathcal{S}$ and $y \in Y_{H^-(\mu,\lambda)}$ some $\mu \in D^o$, $\lambda \in Q^o$. Then $< \mu, F(\Omega) > + < \lambda, G(\Omega) > \geq < \mu, y >$. Since $< \lambda, G(\Omega) > \leqq 0$, $< \mu, F(\Omega) > \geq < \mu, y >$. Then $F(\Omega) \not\leq_D y$ for $\mu \in D^o$. □ □

3. Normality and Stability of Multiobjective Programming Problem with Set Functions

In this section, we generalize Jahn's normality and stability concepts for the primal problem (P) with set functions. Let us introduce following sets.

$$\mathcal{B} = \{(u, y) \in R^m \times R^p : F(\Omega) \leqq_D y, G(\Omega) \leqq_Q u \text{ for some } \Omega \in \mathcal{S}\},$$
$$\mathcal{Y}_{\mathcal{B}} = \{y \in R^p : (\mathbf{0}, y) \in \mathcal{B}\}.$$

Furthermore, for $u \in R^p$, we denote the set $\{(u, \alpha) \in R^m \times R : \alpha =< \mu, y > \text{ for } (u, y) \in \mathcal{B}\}$ by $\mathcal{B}(\mu)$ and the set $\{\alpha \in R : (\mathbf{0}, \alpha) \in \mathcal{B}(\mu), \mathbf{0} \in R^m\}$ by $\mathcal{A}_{\mathcal{B}(\mu)}$, respectively.

Note that $cl(\mathcal{B})$ is convex and that for the ordinary convex scalar optimization problem $\inf\{f(x) : g(x) \leqq_Q \mathbf{0}, x \in X \subset R^n\}$, $\mathcal{B} \subset$ epigraph of $w \subset cl(\mathcal{B})$ where $w(u) = \inf\{f(x) : g(x) \leqq_Q u\}$.

Definition 3.1. The primal problem (P) is said to be normal if for every $\mu \in D^o$,

$$cl(\mathcal{A}_{\mathcal{B}(\mu)}) = \mathcal{A}_{cl(\mathcal{B}(\mu))}.$$

Note that $cl(\mathcal{A}_{\mathcal{B}(\mu)}) \subset \mathcal{A}_{cl(\mathcal{B}(\mu))}$ is always true.

Our main result is that Slater's constraint qualification is a necessary condition for the primal problem (P) to be normal.

Theorem 3.2. *Suppose that interiors of D and Q are nonempty. Then Slater's constraint qualification in the problem* (P) *with set functions implies normality.*

Proof. Let $\mu \in D^o$. Then $< \mu, F(\cdot) >: \mathcal{S} \to R$ is a convex set function. For $(\alpha, \Omega_1, \Omega_2) \in [0,1] \times \mathcal{S} \times \mathcal{S}$ and $\{\Gamma_n\}$ a Morris sequence associated with $(\alpha, \Omega_1, \Omega_2)$, there is a subsequence $\{\Gamma_{n_k}\}$ of $\{\Gamma_n\}$ in $\mathcal{S}$ such that

$$\alpha F(\Omega_1) + (1-\alpha)F(\Omega_2) - \limsup_{k\to\infty} F(\Gamma_{n_k}) \in D,$$

since F is D-convex over $\mathcal{S}$. Also since F is w^*−continuous and $\chi_{\Gamma_n} \overset{w^*}{\longrightarrow} \alpha\chi_{\Omega_1} + (1-\alpha)\chi_{\Omega_2}$, it follows that $\limsup_{k\to\infty} F(\Gamma_{n_k}) = \limsup_{n\to\infty} F(\Gamma_n) = \lim_{n\to\infty} F(\Gamma_n)$. Now $\mu \in D^o$ implies that

$$< \mu, \alpha F(\Omega_1) + (1-\alpha)F(\Omega_2) - \limsup_{k\to\infty} F(\Gamma_{n_k}) > \geq\ 0.$$

Hence,

$$\begin{aligned}&< \mu, \alpha F(\Omega_1) + (1-\alpha)F(\Omega_2) > \\ &\qquad\qquad \geq < \mu, \limsup_{n\to\infty} F(\Gamma_n) > = < \mu, \lim_{n\to\infty} F(\Gamma_n) > .\end{aligned}$$

Therefore, $< \mu, F(\Omega) >$ is a convex function.

Now it suffices to show that $cl(\mathcal{A}_{\mathcal{B}(\mu)}) \supset \mathcal{A}_{cl(\mathcal{B}(\mu))}$. Let $a > 0$. Consider an element $(\mathbf{0}, y') = (\mathbf{0}, < \mu, F(\Omega_o) > +a)$ with $G(\Omega_o) <_Q \mathbf{0}$, $\Omega_o \in \mathcal{S}$ from the Slater's constraint qualification. Clearly $(\mathbf{0}, y') \in \mathcal{B}^o$, where

$$\mathcal{B}^o = \{(u,y) :< \mu, F(\Omega) > \quad < y, \quad G(\Omega) <_Q u \text{ for some } \Omega \in \mathcal{S}\}.$$

Moreover, $(\mathbf{0}, y') \in int\mathcal{B}^o$. For each u with $G(\Omega) <_Q u <_Q \mathbf{0}$, the origin $\mathbf{0} \in int(u+Q)$. Let $D = R_+ \cup \{0\}$. Take a neighborhood V of $\mathbf{0}$ in $u+Q$ and U of $< \mu, F(\Omega_o) > +a$ in $int(< \mu, F(\Omega_o) > +D)$. Then $V \times U$ is a neighborhood of $(\mathbf{0}, < \mu, F(\Omega_o) > +a)$ and $V \times U \subset \mathcal{B}^o$.

Now let $\widehat{y} \in \mathcal{A}_{cl(\mathcal{B}(\mu))}$. Then $(\mathbf{0}, \widehat{y}) \in cl(\mathcal{B}(\mu))$ so that there exists a sequence $(u_n, y_n) \in \mathcal{B}(\mu)$ converging to $(\mathbf{0}, \widehat{y})$. Since $(u_n, y_n) \in \mathcal{B}(\mu)$ for each n, there exists $\{\Omega_n\} \subset \mathcal{S}$ such that

$$G(\Omega_n) \leqq_Q u_n, \quad < \mu, F(\Omega_n) > \leqq_D y_n.$$

Choose $q \in intQ$ and $p > 0$. Then we have sequences $\{u'_n = u_n + \frac{1}{n} \cdot q\}$ and $\{y'_n = y_n + \frac{1}{n} \cdot p\}$ such that $G(\Omega_n) <_Q u'_n$ and $< \mu, F(\Omega_n) ><_D y'_n$. Hence, $(u'_n, y'_n) \in \mathcal{B}^o$ for each n. But then

$$\lim_{n\to\infty} u'_n = \mathbf{0}, \quad and \quad \lim_{n\to\infty} y'_n = \widehat{y}.$$

Therefore, $(\mathbf{0}, \widehat{y}) \in cl(\mathcal{B}^o)$. Now since $(\mathbf{0}, y') \in int\mathcal{B}^o$ and $(\mathbf{0}, \widehat{y}) \in cl(\mathcal{B}^o)$ and $\mathcal{B}^o$ is convex, considering the affine set $M = \{(\mathbf{0}, y) : y \in R\}$ in $R^m \times R$ and applying Corollary 6.5.1[7], we have $\widehat{y}$ in $cl(\mathcal{A}_{\mathcal{B}^o(\mu)}) \subset cl(\mathcal{A}_{\mathcal{B}(\mu)})$. Consequently, (P) is normal □ □

Now we give a definition of stability for multiobjective optimization with set functions.

Definition 3.3. The primal problem (P) is said to be stable if it is normal and for arbitrary $\mu \in D^o$, the problem (D_μ)

$$\sup_{\lambda \in Q^o} \inf_{\Omega \in \mathcal{S}} \{< \mu, F(\Omega) > + < \lambda, G(\Omega) >\}$$

has at least one solution.

We also show that Slater's constraint qualification implies stability of the primal problem (P).

Theorem 3.4. *Suppose that, for each $\mu \in D^o$, the problem* (P_μ)

$$\inf\{< \mu, F(\Omega) >: G(\Omega) \leqq_Q \mathbf{0}, \Omega \in \mathcal{S}\}$$

has a solution whose value is finite. If Slater's constraint qualification holds, then the problem (P) *is stable.*

Proof. Applying Theorem 3.2, the problem (P_μ) is normal. Since $< \mu, F(\Omega) >$ is a convex set function, it follows, by Theorem 3.1[2], that there exists a $\lambda^* \geq_{Q^o} \mathbf{0}$ such that $\mu_o = \inf\{< \mu, F(\Omega) > + < \lambda^*, G(\Omega) >: G \in \mathcal{S}\}$, so that the infimum is obtained at some $\Omega^* \in \mathcal{S}$. Then, for any $\lambda \in Q^o$,

$$\begin{aligned} &\inf\{< \mu, F(\Omega) > + < \lambda, G(\Omega) >: \Omega \in \mathcal{S}\} \\ &\quad \leq \inf\{< \mu, F(\Omega) > + < \lambda^*, G(\Omega) >: \Omega \in \mathcal{S}\} \\ &\quad \leq \inf\{< \mu, F(\Omega) > + < \lambda^*, G(\Omega) >: \Omega \in \mathcal{S}, G(\Omega) \leqq_Q \mathbf{0}\} \\ &\quad \leq \mu_o. \end{aligned}$$

Thus $\mu_o = \sup_{\lambda \in Q^o} \inf_{\Omega \in \mathcal{S}} \{< \mu, F(\Omega) > + < \lambda, G(\Omega) >\}$. Therefore, (P) is stable. $\square$ $\square$

References

1. J.M.Borwein, and J.W.Nieuwenhuis, *Two kinds of Normality in Vector Optimization*, Math. Programming **28** (1984), 185–191.
2. W.S.Hsia, T.Y.Lee, and J.Y.Lee, *Lagrange Multiplier Theorem of Multiobjective Programming Problems with Set Functions*, Journal of Optimization Theory and Applications **70** (1991), 137–155.
3. J. Jahn, *Duality in Vector Optimization*, Math. Programming **25** (1983), 343–353.
4. R.J.T.Morris, *Optimal Constrained Selection of a Measurable Subset*, Journal of Mathematical Analysis and Applications **70** (1979), 546–562.
5. J.W.Nieuwenhuis, *Supremal Points and Generalized Duality*, Math. Operationforsch. Statist. Ser. Optimization **11** (1980), 41–59.
6. J. Ponstein, *Approaches to the Theory of Optimization*, Cambridge Univ. Press, London and New York, 1980.
7. R.T.Rockafellar, *Convex Analysis*, Princeton Univ. Press, Princeton, New Jersey, 1970.
8. Y.Sawaragi, H.Nakayama and T.Tanino, *Theory of Multiobjective Optimization*, Academic Press, Orlando, 1985.
9. R.M. Van Slyke and R.J.-B. Wets, *A Duality Theory for Abstract Mathematical Programs with Application to Optimal Control Theory*, Journal of Math. Anal. and Appl. **22** (1968), 679–706.

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