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# COMPLEX BORDISM OF CLASSIFYING SPACES OF THE DIHEDRAL GROUP

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Abstract. In this paper, we study the $BP_*$-module structure of $BP_*(BG)$ mod $(p, v_1, \cdots)^2$ for non abelian groups of the order $p^3$. We know $grBP_*(BG) = BP_* \otimes H(H_*(BG); Q_1) \oplus BP^*/(p, v_1) \otimes ImQ_1$. The similar fact occurs for $BP_*$-homology $grBP_*(BG) = BP_* s^{-1} H(H_*(BG); Q_1) \oplus BP_*/(p, v) s^{-1} H^{odd}(BG)$ by using the spectral sequence $E_2^{*,*} = Ext_{BP^*}(BP_*(BG), BP^*) \Rightarrow BP^*(BG)$.

## 0. Introduction

Let $G$ be a finite group. By $G$-$U$-manifold we mean a weakly complex manifold with a free $G$-action preserving its weakly complex structure. The group of bordism classes of closed $G$-$U$-manifolds is isomorphic to the complex bordism group $M_*(BG)$ of the classifying spaces $BG$. If $S$ is a Sylow $p$-subgroup of $G$, the inclusion map induces a splitting epimorphism $MU_*(BS) \Rightarrow MU_*(BG)$. Hence we need know first for $p$-group $G$. Moreover Quillen isomorphism $MU_*(-)_{(p)} \cong MU_{*(p)} \otimes_{BP_*} BP_*(-)$ shows that we need to know only $BP_*(BG)$. When $G$ is a cyclic or quaternion group, giving dimensional filtration the graded group $grBP_*(BG) = BP_* \otimes H_*(BG)$ since $H_{even}(BG) = 0$ [M]. By Johnson-Wilson [3], $grBP_*(BG)$ is given for an elementary abelian $p$-groups using arguments to generalize Kunneth formula. In this paper we determine $BP_*$-module structure of $grBP_*(BG)$ mod $(p, v_1, \cdots)^2$ for non abelian groups of the order $p^3$. For $p = 2$, the new

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group is the dihedral group $D_4$. The bordism group $grBP_*(BD_{2q})$, $q :\neq 2$, was studied by Kamata-Minami [5]. Recall the Milnor primitive operation $Q_0 = \beta, Q = p^1\beta - \beta p^1 (= S_q^2 S_q^1 - S_q^1 S_q^2$ for $p = 2)$. For the above group, we can extend the operation $Q_1$ on $H_*(BG)$ so that $Q_1|H^{even}(BG) = 0$. By Tezuka-Tagita [7], we know $grBP_*(BG) = BP^* \otimes H(H_*(BG); Q_1) \oplus BP^*/(p, v_1) \otimes ImQ_1$ since $d_{2p-1} = v_1 \otimes Q_1$ is the only non zero differential in the Atiyah-Hirzebruch spectral sequence. The similar fact occurs for $BP_*$-homology $grBP_*(BG) = BP_* s^{-1} \otimes H(H^*(BG); Q_1) \oplus BP_*/(p, v_1)s^{-1}H^{odd}(BG)$ where $s^{-1}$ is the descending degree one map, by using the spectral sequence $E_2^{*,*} = Ext_{BP^*}(BP_*(BG), BP^*) \Rightarrow BP^*(BG)$. In particular, generators and relations are given explicitly for $BP_*(BD)$ in the last section.

## 1. Bordism and cobordism

Assume always that $G$ is a $p$-group. Let us write by $H^*$ (resp. $H\mathbb{Z}/p^*$, $H^{even}$, $H^{odd}$) the cohomology $H^*(BG)$ (resp. $H^*(BG; \mathbb{Z}/p)$, $H^{even}(BG)$, $H^{odd}(BG)$). In this section we consider only groups which satisfy the following assumption.

Assumption 1.1. $pH\mathbb{Z}/P^{odd} = 0$ hence $H^{odd} \subset H\mathbb{Z}/P^{odd}$, moreover $Q_1/H^{odd}$ is injective.

Since $Q_1|H^{odd}$ is injective, we can define $Q_1|H^{even} = 0$.

Lemma 1.2. $grBP^*(BG) \cong BP^* \otimes H(H^*; Q_1) + BP^*/(p, v_1) \otimes ImQ_1$.

*Proof.* Consider Atiyah-Hirzebruch spectral sequence

$$E_2^{*,*} = H^*(BG; BP^*) = BP^* \otimes H \Longrightarrow BP^*(BG).$$

The first nonzero differential is $d_{2p-1} = v_1 \otimes Q_1$, hence we get that $E_{2p}$ is isomorphic the righthandside of the module in the lemma. Since $KerQ_1 = ImQ_1 + H(H^*; Q_1)$ is even dimensionally generated, and so is $E_{2p}^{*,*}$. Therefore $E_{2p} \cong E_\infty$. □ □

Given $\mathbb{Z}_{(p)}$-module $A$, let us write by $FA$ the $\mathbb{Z}_{(p)}$-free module generated by $\mathbb{Z}_{(p)}$-module generators of $A$. Let $F(x)$ be a generator which corresponds $x$ in $A$.

THEOREM 1.3. *There is a $BP^*$-module isomorphism*

$$BP^*(BG) \cong BP^* \otimes (FH(H^*;Q_1) + FImQ_1)/R$$

*where $R$ is generated, modulo $(p, v_1, \cdots)^2$, $\sum_{n=0} v_n F(Q_n Q_i^{-1}(x)) = 0$ for $i = 0, 1$, and $x \in KerQ_1$.*

*Proof.* If $\overline{x_1} \in ImQ_1$, then there is a relation $v_1 x_1 + v_2 x_2 + \cdots = 0$ form Lemma 1.2, for $\rho(x_1) = \overline{x_1}$ where $\rho : BP \to HZ_{(p)}$ is the Thom map. From Lemma 2.1 there is $y \in HZ/p^*$ such that $Q_n(y) = \rho(X_n)$, and $y = Q_1^{-1} x_1$. Since $BP^*(BG) \otimes_{BP^*} \mathbb{Z}_{(p)} = H^{even}$ we have the relation in the lemma. For $x_0 \in ImQ_0$, we also have the relation by the same arguments. □ □

Now we consider the bordism theory. We also write by $H_*$ the homology $H_*(BG)$. Since $H_*$ is torsion module, there is an isomorphism

$$H_{*-1} \cong s^{-1} H^*, \qquad \text{for } * \geqq 2.$$

where $s^{-1}$ is the operation descending degree one. Note that if $px = 0$, $s^{-1}x = Q_0^{-1} x$ for $x \in H^*$.

Consider the spectral sequence

$$E^2_{*,*} = H_*(BG; BP_*) \Longrightarrow BP_*(BG) \tag{1.4}$$

LEMMA 1.5. $E^{2p}_{*,*} \cong BP_* s^{-1} H(H^*; Q_1) + BP_*/(p, v_1) s^{-1} H^{odd}$.

*Proof.* First note $H\mathbb{Z}/P_* = Hom(H\mathbb{Z}/p^*; \mathbb{Z}/p)$. Hence we can define the dual operation $Q_{1*}$ in $H\mathbb{Z}/p_*$. Since $Q_1 Q_0 = -Q_0 Q_1$. We see easily

$$Q_{1*} s^{-1}(ImQ_1) = s^{-1} H^{odd}.$$

The first non zero differential in (1.4) is $d_{2p-1} = v_1 \otimes Q_{1*}$. Hence we get the lemma. □ □

We use here arguments by Ravenel and Johnson-Wilson [3]. Recall the universal coefficient spectral sequence

$$E_2^{*,*} = Ext_{BP_*}(BP_*(BG), BP^*) \Longrightarrow BP^*(BG). \tag{1.6}$$

Given $BP_*$-filtration in $BP_*(BG)$, we can construct spectral sequence

$$G_2^{*,*} = Ext_{BP_*}(grBP_*(BG), BP^*) \Longrightarrow E_2^{*,*}. \tag{1.7}$$

Then it is easily seen

LEMMA 1.8 [3, LEMMA 6.5]. $Ext_{BP_*}(BP_*/(p^k), BP^*) \cong sBP^*/(p^k)$, $Ext_{BP_*}(BP_*/(p, v_1), BP^*) \cong s^{2p}BP_*/(p, v_1)$.

Therefore from Lemma 1.2, Lemma 1.5 and Lemma 1.8, we get

$$Ext_{BP_*}((E^{2p}_{*,*} \text{ in Lemma 1.5}), BP^*) \cong grBP^*(BG). \tag{1.9}$$

If $E^{2p} \neq E_\infty = grBP_*(BG)$ in (1.4), there is an element in $grBP^*(BG)$ which does not correspond $G_\infty$ and $G_2$, and this makes a contradiction to (1.6). Hence (1.6), (1.7) and $E^{2p}$ in (1.4) all collapse.

THEOREM 1.10. *There is a $BP_*$-module isomorphism*

$$BP_*(BG) \cong BP_* \otimes Fs^{-1}(H(H^*; Q_1) + H^{odd})/R$$

*where the relation $R$ is generated, modulo $(p, v_1, \cdots)^2$, by*

$$\sum v_n s^{-1} Q_0 F(Q_{n*} Q_{i*}^{-1} s^{-1}(x)) = 0$$

*for $i = 0, 1, x \in (H(H^*; Q_1) + H^{odd})$.*

## 2. $Q_2$-operation

We give examples 2.1 – 2.3 satisfying Assumption 1.1.

2.1. $G = \mathbb{Z}/p \times \mathbb{Z}/p$. The cohomology $H^{even} = \mathbb{Z}/p[y_1, y_2]$ and $H^{odd} = H^{even}e$ where $|y_i| = 2, |e| = 3$ and $Q_1 e = y_1^p y_2 - y_1 y_2^p$.

2.2. $G$ is a non abelian $p$-group of the order $p^3$. Then $G$ is isomorphic to one of $D, Q, E, M$ (see Lewis [6] or [7]). The cohomology $H^{even}$ is generated by elements $c_1, \cdots, c_2, y_1, y_2$, and $H^{odd}$ is generated as an $H^{even}$-module by $e$ (resp. $0, d_1$ and $d_2, e$) for $D$ (resp. $Q, E, M$). Then we can take ring generators such that the $Q_1$-operation is given by $Q_1 e = c_2 y_2$ (resp. $0, Q_1 d_i = c_2 y_i, Q_1 e = c_p y_2^p$). Hence Assumption 1.1 is satisfied for these cases.

2.3. The semi-dihedral groups $SD_2$. $H\mathbb{Z}/2^*$ is detected by $(D, Q)$ (see [2]). Hence we get the assumption.

## 3. Description of $BP_*(BD)$

In this section we write down $BP_*(BD)$ more explicitly. Recall $D =< a, b|a^4 = b^2 = 1, [a, b] = a^2 >$. The cohomology is given (1, 6, 8).

$$H^{even} = (\widetilde{\mathbb{Z}}/2[y_1, y_2]/(y_1^2 + y_1y_2)) \otimes \widetilde{\mathbb{Z}}/4[C_2] \tag{3.1}$$
$$H^{odd} = (\mathbb{Z}/2[y_1, y_2, c_2]/(y_1^2 + y_1y_2)e$$
$$H\mathbb{Z}/2^* = \mathbb{Z}/2[x_1, x_2, u]/(x_1^2 + x_1x_2)$$

Where $\widetilde{\mathbb{Z}}/s[x]$ means $\mathbb{Z}[x]/(sx)$ and where $x_i^2 = y_i, C = u^2$ and $e = x_2u$ in $H\mathbb{Z}/2^*$. Since $Q_0u = ux_2, Q_1e = y_2c_2$. Hence we get

$$H(H^*; Q_1) = (\widetilde{\mathbb{Z}}/2[y_2] \oplus \widetilde{\mathbb{Z}}/4[c_2]) \otimes \wedge(y_1). \tag{3.2}$$

From Lemma 1.5 and Theorem 1.10, we have

$$grBP_*(BD) = BP_*\{1\} \oplus BP_*/2s^{-1}\{y_1^i, y_2^i, y_1c_2^j\} \tag{3.3}$$
$$+ BP/4s^{-1}\{c_2^j\} \oplus BP_*/(2, v_1)s^{-1}\{y_1^is_2^je, y_2^ic_2^je\}.$$

We will construct $D$-$U$-manifolds which represent elements in (3.3). Before doing this, we see how these generators in $H\mathbb{Z}_*$ are defined. Consider the extension

$$0 \to< a >= \mathbb{Z}/4 \to D \to< b >= \mathbb{Z}/2 \to 0 \tag{3.4}$$

and induced spectral sequence (see Lewis p.510 [6]). The action $b^*$ on $H^*(B\mathbb{Z}/4) \cong \widetilde{\mathbb{Z}}/4[u]$ is given by $b^*u = 3u = -u$. Let us write $T = (1 - b^*)$ and $N = (1 + b^*)$. Then

$$E^2_{0,*} = H_*/ImT = \begin{cases} \mathbb{Z}/4\{s^{-1}u^i\} & \text{if} \quad i|2 \\ \mathbb{Z}/2\{s^{-1}u^i\} & \text{otherwise} \end{cases}$$

(3.5)

$$E^2_{2j+1,*} = KerT/ImN = \begin{cases} \mathbb{Z}/2\{s^{-1}u^i\} & \text{if} \quad i|2 \\ \mathbb{Z}/2\{s^{-1}2u^i\} & \text{otherwise} \end{cases}$$

$$E^2_{2j+2,*} = KerN/ImT = \begin{cases} \mathbb{Z}/2\{s^{-1}2u^i\} & \text{if} \quad i|2 \\ \mathbb{Z}/2\{s^{-1}u^i\} & \text{otherwise} \end{cases}$$

By the universal coefficient theorem and (3.1) this spectral sequence collapses (confer Lewis p.510).

The elements $s^{-1}u,\ s^{-1}u^2 \in E^2_{0,*}$ corresponds $s^1y_1,\ s^{-1}c_2$, the element $s^{-1}2u \in E^2_{1,1}$ corresponds $s^{-1}e$, and $s^{-1}u \in E^2_{2j,2}$ corresponds $s^{-1}(y_1y_2^j)$. Moreover $1 \in E^2_{2j-1,0}$ corresponds $s^{-1}y_2^j$.

We define a $D$-$U$-manifold

$$X(j,i) = (S^{2j-1} \times D/<a>) \times_{<b>} S^{2i-1} \tag{3.7}$$

where $D$ acts on $S^{2j-1} \times D/<a>$ by

$$\begin{cases} a(z,0) = (iz,0) \\ a(z,1) = (-iz,1) \end{cases} \qquad \begin{cases} b(z,0) = (z,1) \\ b(z,1) = (z,0) \end{cases}$$

identifying $(z,n) \in S^{2j-1} \times \mathbb{Z}/2 \subset C^j \times \mathbb{Z}/2$, and where $b$ acts on $S^{2i-1}$ by $b(z) = (-z)$ in $C^i$. Then we get the map

$$\xi : X(j,i)/D \longrightarrow BD. \tag{3.8}$$

The fibering

$$S^{2j-1}/<a> \longrightarrow X(j,i)/D \longrightarrow S^{2i-1}/<b>$$

induces the spectral sequence

$$H_*(S^{2i-1}/<b>; H_*(S^{2j-1}/<a>)) \Longrightarrow H_*(X(j,i)/D). \tag{3.9}$$

The map $\xi$ in (4.8) induces the map of spectral sequences (3.9) to (3.5). Then the fundamental class of $X(j,i)$ is represented in $E_\infty$ in (3.9) by the nonzero element of right up side. Hence we know that $X(2j,0) = s^{-1}c^j$, $X(2j-1,0) = s^{-1}y_1c_2^{j-1}$, $X(0,i) = s^{-1}y_2^i$, and for $ij > 0$, $X(2j,i) = s^{-1}ec_2^{j-1}y_1y_2^{i-1} = s^{-1}ec_2^{j-1}y_1^i$, $X(2j-1,i) = s^{-1}ec_2^{j-1}y_2^{i-1}$.

The only element which is not expressed by $X(j,i)$ is $s^{-1}y_1^j$ for $j \geqq 2$. Note that there is a homomorphism $\lambda$ in $D$ such that $\lambda : b \leftrightarrow ab$, $\lambda : a \leftrightarrow a^3$. Then $s^{-1}y_2 = s^{-1}y_2 + s^{-1}y_1$. Take $X'(0,i) = M/<ab> \times S^{2i-1}$ and this manifold represents $s^{-1}y_1^i + s^{-1}y_2^i$.

Next consider relations $\sum v_n Q_{n*} Q_{k*}^{-1}(x) = 0$. First consider the case $x = X(0,i)$. Since $s^{-1}y_2 = Q_{0*}y_2$, we see $Q_{0*}^{-1}s^{-1}y_2^i = y_2^i$. The $Q_{n*}$-operation acts

$$\begin{aligned} Q_{n*}y_2^i &= \sum < y_2^i, Q_n x_2 y_2^k > x_2 y_2^k, \qquad \text{where recall} \quad x_2 = y_2 \\ &= \sum < y_2^i, y_2^{p^n+k} > x_2 y_2^k = x_2 y_2^{1-p^n}. \end{aligned}$$

Therefore we have

$$(3.10) \quad \sum v_n X(0, i-p^n+1) = 0, \qquad \sum v_n X'(0, i-p^n+1) = 0$$

This relation is well known and also given by the relation in $BP_*(BZ/2)$ and [2] the product of the formal group law in $BP_*$-theory (for example, see [4], [5]).

When $x = X(2j, 0)$, the fact $Q_{0*}^{-1}s^{-1}(c_2^j) = 0$ induces only trivial relation. As for $x = X(2j-1, 0)$, the formula

$$Q_{n*}c_2^j y_1 = \sum < c_2^j y_1, Q_n c_2^k x_1 > c_2^k x_1 = 0 \qquad \text{for} \quad n \geqq 1$$

follows the relation

$$(3.11) \qquad 2X(2j-1, 0) = 0.$$

At last we consider the case $ij > 0$. Since $s^{-1}y_2^i c_2^j e = c_2^j y_2^i u$ (see (3.1)), we get

$$(3.12) \qquad \begin{aligned} Q_{n*}c_2^j y_2^i e &= \sum < c_2^j y_2^i e, Q_n c_2^k y_2^l u > c_2^k y_2^l u \\ &= \sum < c_2^j y_2^i e, c_2^k y_2^l Q_n u > c_2^k y_2^l u \\ &= \sum < c_2^{j-k} y_2^{i-l} e, Q_n u > c_2^k y_2^l u. \end{aligned}$$

Lemma 3.13. *There are polynomials* $F_n(u, y_2)$ *such that* $Q_n u = f_n(u, y_2) u x_2$ *and* $f_{n+1} = u f_n^2 + y_2 f_n^2 + (\partial f_n/\partial u)^2 y_2 u^2$.

*Proof.* At first recall $Q_0 u = ux_2$. $Q_1$-action is

$$Q_1 u = Sq^2 Q_0 u + Q_0 Sq^2 u = Sq^2(ux_2) = u^2 x_2 + ux_2^3 = ux(u + x_2^2).$$

By the induction on $n \geqq 1$, we see

$$\begin{aligned} Q_{n+1} u &= (Sq^{2^{n+1}} Q_n + Q_n Sq^{2^{n+1}})u \\ &= Sq^{2^{n+1}} Q_n u = Sq^{2^{n+1}}(xuf), \qquad \text{where} \quad |xuf| = 2^{n+1} + 1, \\ &= xu^2 f^2 + x^3 u f^2 + x^2 u^2 Sq^{|f|-1} f. \end{aligned}$$

If $f_n = \sum \lambda_i u^i y_2^j$, then

$$Sq^{|f|-1} f_n = \sum \lambda_i i(ux_2) u^{2(i-1)} y_2^{2j} = ux_2(\partial f_n / \partial u).$$

Therefore $Q_{n+1} u = ux_2(uf_n^2 + x_2^2 f_n^2 + x_2^2 u^2 (\partial f/\partial u)^2)$. □ □

Let us write $f_n = \sum f_{n,i} u^i y^j$. Then we get

$$\begin{aligned} Q_{n*} c_2^j y_2^i e &= \sum < c_2^k y_2^\ell e, \sum f_{n,t} u^t y_2^{2^n - 1 - t} e > c_2^{j-k} y_2^{i-1} u \\ &= \sum f_{n,2t} c_2^{j-t} y_2^{i-(2^n - 1 - 2t)} u. \end{aligned}$$

Hence we have the relation

$$\sum_n v_n \Big(\sum_t f_{n,2t} X(j - t, i + 2t + 1 - 2^n)\Big) = 0. \tag{3.14}$$

Next consider the relation such that $v_1 X(j,i) + \cdots = 0$. If $Q_{1*} w = c_2^j y_2^i u$, then

$$\begin{aligned} c_2^j y_2^i u &= \sum < w, Q_1 c_2^k y_2^\ell u > c_2^k y_2^\ell u \\ &= \sum < w, c_2^k y_2^\ell e(u + y_2) > c_2^k y_2^\ell u \end{aligned}$$

shows $w = c_2^j y_2^{i+1} e$ or $w = c_2^j y_2^i eu$. Since $Q_{0*} c_2^j y_2^{i+1} e = c_2^j y_2^{i+1} u$, the case $w = c_2^j y_2^{i+1} e$ gives a relation such that $2x(j, i+1) + \cdots = 0$, which is contained in (3.14). Hence we need only the case $w = c_2^j y_2^i eu$,

$$\begin{aligned} Q_{n*} w &= \sum < c_2^j y_2^i eu, Q_n c_2^k y_2^\ell u > c_2^k y_2^\ell u \\ &= \sum < c_2^j y_2^i eu, f_{n,t} u^t y_2^{2^n - 1 - t} e > c_2^{j-k} y_2^{i-1} u \\ &= f_{n,2t+1} c_2^{j-t} y_2^{i-(2^n - 1 - 2t - 1)} u. \end{aligned}$$

Therefore we get

$$\sum_n v_n(\sum_t f_{n,2t+1}X(j-t, i-2^n+2t+2)) = 0. \tag{3.15}$$

THEOREM 3.10. *There is a $BP_*$-module isomorphism*

$$BP_*(BD) = BP_*\{\ X(j,i), X'(0,i') \mid j,i \geqq 0, i \geqq 2\ \}/R$$

*where* $R = ((3.10), (3.11), (3.14), (3.15))$ *mod* $(2, v_1, \cdots)^2$.

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