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SEQUENTIAL COMPACTNESS AND SEMICOMPACTNESS

Jae Deuk Myung and Hee Chan Choi

Abstract. In this paper, we introduce two notions of compactness defined by sequential convergence and compare them.

1. Preliminaries

Let X be a set and I be the closed unit interval. Then a function F from X into I is called a *fuzzy set* in X. For any fuzzy set F, $\{x \in X \mid F(x) > 0\}$ is called the support of F and denoted by supp$F$, i.e., supp$F = \{x \in X \mid F(x) > 0\}$. And for any $\alpha \in (0,1]$, a fuzzy set x_α in X is called a *fuzzy point* if its support is a singleton $\{x\}$ and its value is α on its support. That is,

$$x_\alpha(y) = \begin{cases} \alpha & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}$$

Definition 1.1. Let X be a nonempty set and I be the closed unit interval. A family δ of functions from X into I is called a *fuzzy topology* on X if

(1) $\emptyset,\ X \in \delta$
(2) for all $U_i \in \delta$, $\cup U_i \in \delta$
(3) if $U_1,\ U_2 \in \delta$, then $U_1 \cap U_2 \in \delta$.

The pair (X, δ) is called a *fuzzy topological space*. A member of δ is called an *open set*. And a fuzzy set F in X is said to be *closed* if $F^c = X - F$ is open in X, i.e., $F^c \in \delta$.

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Definition 1.2. Let $\{P_n\}$ be a sequence of fuzzy points and P a fuzzy point in a fuzzy topological space (X, δ). We say that $\{P_n\}$ *converges* to P, or P is a *limit* of the sequence $\{P_n\}$ and write $P_n \to P$ if for every Q-neighborhood A of P there is a natural number m such that $P_n Q A$ for all $n \geq m$.

Remarks. Note that, given any fuzzy point P in X, every sequence $\{P_n\}$ of fuzzy points such that $P_n Q(1-P)$ for all $n \geq m$ converges to P.

Definition 1.3. Let $\{P_n\}$ be a sequence of fuzzy points and P a fuzzy point in a fuzzy topological space $(X, \Im)$. Then P is said to be a *limit value* of the sequence $\{P_n\}$ if there is a subsequence of $\{P_n\}$ converging to P.

One has that every limit of a sequence is one of its limit values.

Definition 1.4. Let $\{P_n\}$ be a sequence of fuzzy points and P a fuzzy point in a fuzzy topological space (X, δ). Then P is said to be a *cluster* for the sequence $\{P_n\}$ if for every Q-neighborhood A of P and for every natural number m there is a natural number $n \geq m$ such that $P_n Q A$.

Remarks. It is easy to see that every limit value of a sequence is a cluster of the sequence.

Definition 1.5. A fuzzy topological space (X, δ) is said to be C_1 if every fuzzy point P in X has a *countable fundamental Q-neighborhood system* (briefly C.F.Q.N.S.).

Lemma 1.6. *If (X, δ) is C_1 fuzzy topological space, then for every fuzzy point in X there exists a C.F.Q.N.S. $\{A_i\}$ such that $A_1 \supset A_2 \supset \cdots \supset A_i \supset \cdots$.*

Proof. By assumption, there exists a C.F.Q.N.S. $B = \{B_i\}$ of P. Define $A_1 = B_1,\ A_2 = B_1 \cap B_2, \ldots, A_n = \cap_{i=1}^n B_i, \ldots$. Clearly, $A_1 \supset A_2 \supset \cdots \supset A_i \supset \cdots$. In order to prove that these Q-neighborhood of P form an F.Q.N.S. of P, let A be a Q-neighborhood of P. There exists $B_i \in B$ such that $B_i \subset A$. Since PQB_i for every $i = 1, 2, \ldots, n$, $PQ(\cap_{i=1}^n B_i) = A_n \subset A$. □ □

THEOREM 1.7. *Let $(X, \Im)$ be a C_1 fuzzy topological space, $\{P_n\}$ be a sequence of fuzzy points and P a fuzzy point in X. If P is a cluster for the sequence $\{P_n\}$, then P is one of its limit value.*

Proof. By Lemma 1.6, there exists a C.F.Q.N.S. $\{A_i\}$ such that $A_1 \supset A_2 \supset \cdots \supset A_i \supset \cdots$. Since P is a cluster for $\{P_n\}$, for every $n \in \mathbb{N}$ there is $k(n) \in \mathbb{N}$ such that $P_{k(n)} Q A_n$. We define, in this way, a sequence of natural numbers $\{k(n)\}$ which can be taken to be strictly increasing. To show that the subsequence $\{P_{k(n)}\}$ converges to P, let A be Q-neighborhood of P. There exists $n_0 \in \mathbb{N}$ such that $PQA_n \subset A$; but $A_n \subset A_{n_0}$ for each $n \geq n_0$, and this implies $P_{k(n)} Q A_n \subset A_{n_0} \subset A$ for each $n \geq n_0$. □ □

2. semicompact and sequential compact

DEFINITION 2.1. A fuzzy topological space (X, δ) is said to be *semicompact* if every sequence of fuzzy points in X has a cluster.

DEFINITION 2.2. A fuzzy topological space (X, δ) is said to be *sequentially compact* if every sequence of fuzzy points in X has a limit value.

PROPOSITION 2.3. *Every fuzzy sequentially compact space is semicompact.*

PROPOSITION 2.4. *If every C_1 fuzzy topological space is semicompact, then it is also sequentially compact.*

In the following theorems we give some characterizations of sequentially compact space.

THEOREM 2.5. *Let $f : X \to Y$ be any function and P be any fuzzy point in X. Then*

(1) *For $A \in I^X$ and PQA, we have $f(P)Qf(A)$.*
(2) *For $B \in I^Y$ and $f(P)QB$, we have $PQf^{-1}(B)$.*

THEOREM 2.6. *Let X and Y be fuzzy topological spaces. Let $\{P_n\}$ be a sequence of fuzzy points in X and P fuzzy point in X. If $f : X \to Y$ is Q-continuous, then $f(P)$ is a limit of $f(P_n)$ whenever P is a limit of sequence $\{P_n\}$.*

Proof. Since f is Q-continuous, $f^{-1}(A)$ is a Q-neighborhood of P for every Q-neighborhood A of $f(P)$. Since P is a limit of sequence $\{P_n\}$, for every Q-neighborhood $f^{-1}(A)$ of P, there is a natural number m such that $P_n Q f^{-1}(A)$ for all $n \geq m$. Then $f(P_n) Q f(f^{-1}(A)) = A$, since f is onto. Hence $f(P)$ is a limit of $f(P_n)$. □ □

THEOREM 2.7. *Let X and Y be fuzzy topological spaces and $f : X \to Y$ be a Q-continuous function from X to Y which is onto. If X is sequentially compact, then Y is also sequentially compact.*

Proof. Let P be a fuzzy point in X and $\{P_n\}$ a sequence of fuzzy points. For every Q-neighborhood A of $f(P)$, since f is Q-continuous, $f^{-1}(A)$ is a Q-neighborhood of P. Since X is sequentially compact there exists a subsequence $\{P_{n(k)}\}$ of $\{P_n\}$. For every Q-neighborhood $f^{-1}(A)$ of P, there is a natural number m such that $P_{n(k)} Q f^{-1}(A)$ for all $n \geq m$. Then $f(P_{n(k)}) Q f(f^{-1}(A)) = A$, since f is onto. Hence for every Q-neighborhood A of $f(P)$, there is a natural number m such that $f(P_{n(k)}) Q A$ for all $n \geq m$. Hence there is a subsequence $f(P_{n(k)})$ of $f(P_n)$ that converges to $f(P)$. □ □

THEOREM 2.8. *Let $(X_i)_{i \in J}$ be a family of fuzzy topological spaces and P be fuzzy point in X and let $X = \prod_{i \in J} X_i$ with product fuzzy topology $\Im$. For each $i \in J$, let π_i denote the canonical projection of X onto X_i and $\{P_n\}$ sequence of fuzzy points in X. Then P is a limit of $\{P_n\}$ if and only if $\pi_i(P)$ is a limit of $\pi_i(P_n)$.*

Proof. Let $\pi_i : X \to X_i$ be the canonical projection mapping. Since π_i is Q-continuous and onto, X_i is sequentially compact when X is sequentially compact by Theorem 2.6.

Conversely, suppose that X_i is sequentially compact, let $\{P_n\}$ be a sequence of fuzzy points in X, and A be Q-neighborhood of P. Then there exists $B \in \Im$ such that $PQB \subset A$. By the definition of the defined base for the product space $\prod_i X_i$, $B = \pi_{j_1}^{-1}(E_{j_1}) \cap \cdots \cap \pi_{j_m}^{-1}(E_{j_m})$ where

E_{j_k} is an open subset of the coordinate space X_{j_k}. Recall that PQB; hence $\pi_{j_1}(P)Q\pi_{j_1}(B), \ldots, \pi_{j_m}(P)Q\pi_{j_m}(B) = E_{j_m}$. By hypothesis, $\pi_{j_i}(P)$ is a limit of $\pi_{j_i}(P_n)$. $\square$ $\square$

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Department of Mathematics
Kyung Hee University
Suwon 449-701, Korea