NOTES ON THE ANALYTIC FEYNMAN INTEGRAL

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Abstract In this paper, we prove a translation theorem for the analytic Feynman integral for functions in \( \mathcal{F}_{A_1, A_2}(B) \) and show how this integral can be modified so as to be translation invariant.

1. Introduction

Let \( H \) be a real separable, infinite dimensional Hilbert space and let \( M(H) \) denote the space of all \( C \)-valued countably additive measures on \( B(H) \), the Borel class of \( H \). The Fresnel class \( \mathcal{F}(H) \) is defined as the space of all Fourier transforms of elements of \( M(H) \) \cite{2}; that is, \( \mathcal{F}(H) = \{ f = \hat{\mu} : \mu \in M(H) \} \) where

\[
(1.1) \quad f(h) = \int_{H} \exp \{ i \langle h, h_1 \rangle \} d\mu(h_1).
\]

Also if we define \( \| f \| = \| \mu \| \), then \( \mathcal{F}(H) \) becomes a Banach algebra isomorphic to \( M(H) \). Finally, the Fresnel integral \( \mathcal{F}(f) \) is defined by

\[
(1.2) \quad \mathcal{F}(f) = \int_{H} \exp \{ -\frac{i}{2} \| h \|^2 \} d\mu(h).
\]

When appropriate choices are made for the Hilbert space \( H \) and the function \( f \), the Fresnel integral provides one means of giving a rigorous definition of Feynman's path integral from quantum theory \cite{2,8}. The Fresnel integral has defined on Hilbert space \cite{2}, classical Wiener space \cite{3} and abstract Wiener space \( (H, B, \nu) \) \cite{11,12} settings, and used as an approach to the Feynman integral.

Recently, Kalianpur and Bromley \cite{11} introduced the analytic Feynman integral on abstract Wiener space, and established existence theorems of this integral for functions in the Fresnel class \( \mathcal{F}(B) \). And

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Kallianpur, Kannan and Karandikar [12] also introduced the sequential Feynman integral on abstract Wiener and Hilbert spaces, and obtained existence theorems of the analytic and the sequential Feynman integrals for functions in larger classes $\mathcal{G}^q(B)$ and $\mathcal{G}^q(H)$ than Fresnel classes $\mathcal{F}(B)$ and $\mathcal{F}(H)$ considered in [2,11]. Moreover, Chung and Kang [7] also proved the translation theorem of the above integrals for functions in $\mathcal{G}^q(B)$ and $\mathcal{G}^q(H)$.

In this paper, we introduce the analytic Feynman integral on $B \times B$ discussed by Kallianpur and Bromley [11], and prove a translation theorem of this integral for functions in larger classes $\mathcal{F}_{A_1,A_2}(B)$ and $\mathcal{F}_{A_1,A_2}(H)$ than $\mathcal{F}(B)$ and $\mathcal{F}(H)$, which differ from the classes $\mathcal{G}^q(B)$ and $\mathcal{G}^q(H)$. Moreover we show how the integral can be modified so as to be translation invariant.

2. Preliminaries

Let $H$ be a real separable infinite dimensional Hilbert space with inner product $\langle \cdot , \cdot \rangle$ and norm $|\cdot|$. Let $m$ be the Gauss measure on $H$ defined by

$$m(E) = (2\pi)^{-\frac{n}{2}} \int_F \exp\left\{-\frac{|x|^2}{2}\right\} dx$$

where $E = P^{-1}(F)$, $F$ is a Borel set in the image of an $n$-dimensional projection $P$ in $H$ and $dx$ is Lebesgue measure in $PH$. Let $\| \cdot \|$ be a measurable norm on $H$ with respect to $m$ on $H$. Let $B$ denote the completion of $H$ with respect to $\| \cdot \|$ and let $i$ denote the natural injection from $H$ to $B$. The adjoint operator $i^*$ of $i$ is one-to-one and maps $B^*$ continuously onto a dense subset of $H^*$. By identifying $H$ with $H^*$ and $B^*$ with $i^*(B^*)$, we have a triple $B^* \subset H^* \equiv H \subset B$ and $\langle h , x \rangle = (h , x)$ for all $h \in H$ and $x \in B^*$, where $(\cdot , \cdot)$ denotes the natural dual pairing between $B$ and $B^*$. By a well known result of Gross [9], $m \circ i^{-1}$ has a unique countably additive extension $\nu$ to the Borel $\sigma$-algebra $\mathcal{B}(B)$ of $B$. The triple $(H,B,\nu)$ is called an abstract Wiener space and the Hilbert space $H$ is called the generator of $(H,B,\nu)$. For more detail, see [9,10,11,12,13].

Let $\{e_n\}$ denote a complete orthogonal system on $H$ such that the $e_n$'s are in $B^*$. For each $h \in H$ and $x \in B^*$, we introduce a stochastic
inner product on $H \times B$ defined by

$$
(2.1) \quad (h, x)^\sim = \begin{cases} 
\lim_{n \to \infty} \sum_{k=1}^{n} <h, e_k > (e_k, x), & \text{if the limit exists} \\
0, & \text{otherwise.}
\end{cases}
$$

It is well known that for every $h \in H$, $(h, x)^\sim$ exists for $\nu$-a.e. $x \in B$ and is a Borel measurable function of $B$ having a Gaussian distribution with mean zero and variance $\|h\|^2$. Furthermore, it is easy to show that $(h, x)^\sim = (h, x)$ $\nu$-a.e. on $B$ if $h \in B^*$. Note that if both $h$ and $x$ are in $H$ then $(h, x)^\sim = <h, x>.$

Given two $C$-valued functions $F$ and $G$ on $B \times B$, we say that $F = G$ s-a.e. if $F(\alpha x_1, \beta x_2) = G(\alpha x_1, \beta x_2)$ for $\nu \times \nu$-a.e. $(x_1, x_2) \in B \times B$ for all $\alpha > 0$ and $\beta > 0$. For a function $F$ on $B \times B$, we will denote by $[F]$ the equivalence class of functions which equal s-a.e.

Let $M(H)$ denote the class of $C$-valued countably additive measures defined on $B(H)$. $M(H)$ is a Banach algebra under the total variation norm and with convolution as multiplication.

**Definition 2.1.** Let $A_1$ and $A_2$ be two bounded, non-negative self-adjoint operators on $H$. Let $\mathcal{F}_{A_1, A_2}(B)$ be the space of all $s$-equivalence classes of functions $F$ on $B \times B$ which, for some $\mu \in M(H)$, have the form

$$
(2.2) \quad F(x_1, x_2) = \int_H \exp\{\nu((A_1^{1/2}h, x_1)^\sim + (A_2^{1/2}h, x_2)^\sim)]\}d\mu(h).
$$

The map $\mu \rightarrow [F]$ defined by (2.2) sets up an algebra isomorphism between $M(H)$ and $\mathcal{F}_{A_1, A_2}(B)$ if $\text{Ran}(A_1 + A_2)$ is dense in $H$ [11,p241] where Ran indicates the range of an operator. In this case, $\mathcal{F}_{A_1, A_2}(B)$ becomes a Banach algebra under the norm $\|F\| = \|\mu\|.$

**Definition 2.2.** Let $F$ be a functional on $B \times B$ such that the integral

$$
(2.3) \quad J_F(\lambda_1, \lambda_2) = \int_{B \times B} F(\lambda_1^{-\frac{1}{2}}x_1, \lambda_2^{-\frac{1}{2}}x_2)d(\nu \times \nu)(x_1, x_2)
$$

exists for all $\lambda_1 > 0$ and $\lambda_2 > 0$. If there exists an analytic function $J^*_F(z)$ on $\Omega = \{z = (z_1, z_2) \in C^2 : \text{Re} z_k > 0 \text{ for } k = 1, 2\}$ such that
Let $q_1$ and $q_2$ be non-zero real numbers. If the following limit (2.5) exists, we call it the analytic Feynman integral of $F$ over $B \times B$ with parameter $q = (q_1, q_2)$ and we write
\begin{equation}
I_q^a[F(\cdot, \cdot)] = \lim_{z \to -(\frac{i}{q_1}, \frac{i}{q_2})} I_a^z[F(\cdot, \cdot)]
\end{equation}
where \( z = (z_1, z_2) \) approaches through $\Omega$. In particular, if $q_1 = q_2 = q$, we write
\begin{equation}
I_q^a[F(\cdot, \cdot)] = I_{a(q,q)}^q[F(\cdot, \cdot)].
\end{equation}

The following theorem plays an important role in this paper. We state it without proof [11].

\textbf{THEOREM 2.3.} Let $F \in \mathcal{F}_{A_1, A_2}(B)$ be given by (2.2). Then the analytic Feynman integral $I_q^a[F(\cdot, \cdot)]$ exists for $q = (q_1, q_2)$ where $q_1 \neq 0$ and $q_2 \neq 0$, and
\begin{equation}
I_q^a[F(\cdot, \cdot)] = \int_H \exp\left\{ -\frac{i}{2} \sum_{k=1}^{2} q_k^{-1} \langle A_k h, h \rangle \right\} d\mu(h).
\end{equation}
In particular,
\begin{equation}
I_{a(1,1)}^1[F(\cdot, \cdot)] = \int_H \exp\left\{ -\frac{i}{2} \langle (A_1 - A_2) h, h \rangle \right\} d\mu(h).
\end{equation}

\textbf{REMARK 2.4.} Let $A$ be a bounded self-adjoint operator on $H$. Then we may write $A = A^+ - A^-$ where $A^+$ and $A^-$ are each bounded, non-negative and self-adjoint. Take $A_1 = A^+$ and $A_2 = A^-$ in the definition above. For any $F \in \mathcal{F}_{A_1, A_2}(B)$, (2.8) becomes
\begin{equation}
I_{a(1,1)}^1[F(\cdot, \cdot)] = \int_H \exp\left\{ -\frac{i}{2} \langle A h, h \rangle \right\} d\mu(h).
\end{equation}
Also, in this case, if $A^+$ is the identity and $A^-$ is zero, then $\mathcal{F}_{A_1, A_2}(B)$ is essentially $\mathcal{F}(H)$ and $\mathcal{F}(B)$ in Hilbert and abstract Wiener space settings, respectively, and also $I_{a(1,1)}^1[F(\cdot, \cdot)] = \mathcal{F}(F_0)$ where $F_0(x_1) = F(x_1, x_2)$ for all $(x_1, x_2) \in B \times B$. In this sense, Definition 2.1 includes the definition of $\mathcal{F}(H)$ and $\mathcal{F}(B)$ as special cases.
3. The analytic Feynman integral

In this section, we establish a translation theorem for the analytic Feynman integral of functions in \( F_{A_1,A_2}(B) \), and show how this integral can be modified so as to be translation invariant.

**Theorem 3.1.** Let \( F \in F_{A_1,A_2}(B) \) be given by (2.2). Then the analytic Feynman integral

\[
G(y_1,y_2) = I_a^q[F((\cdot,\cdot) + (y_1,y_2))]
\]

exists for every \((y_1,y_2) \in B \times B\), and \( G \) also belongs to \( F_{A_1,A_2}(B) \).

**Proof.** Let the measure \( \mu^* \) be defined by

\[
\mu^*(E) = \int_E \exp\{i[(A_{\frac{1}{2}}h,x_1)\sim + (A_{\frac{3}{2}}h,y_2)\sim]\}d\mu(h)
\]

for each \( E \in B(H) \). Then \( \mu^* \) is in \( M(H) \) and

\[
F((x_1,x_2) + (y_1,y_2)) = \int_H \exp\{i[(A_{\frac{1}{2}}h,x_1)\sim + (A_{\frac{3}{2}}h,x_2)\sim]\}d\mu^*(h),
\]

that is, \( F((\cdot,\cdot) + (y_1,y_2)) \) belongs to \( F_{A_1,A_2}(B) \). Therefore, by Theorem 2.3, \( F((\cdot,\cdot) + (y_1,y_2)) \) is analytic Feynman integrable and then

\[
I_a^q[F((\cdot,\cdot) + (y_1,y_2))] = \int_H \exp\{-\frac{i}{2} \sum_{k=1}^{2} q_k^{-1} < A_k h, h >\}d\mu^*(h)
\]

\[
= \int_H \exp\{i[(A_{\frac{1}{2}}h,y_1)\sim + (A_{\frac{3}{2}}h,y_2)\sim]\}d\sigma(h)
\]

where the measure \( \sigma \) is defined by

\[
\sigma(E) = \int_E \exp\{-\frac{i}{2} \sum_{k=1}^{2} q_k^{-1} < A_k h, h >\}d\mu(h)
\]

for \( E \in B(H) \), and hence \( G \) is in \( F_{A_1,A_2}(B) \).
THEOREM 3.2. Let $F \in \mathcal{F}_{A_1A_2}(B)$ be given by (2.2) and let $y \in H$. Then

$$\mathcal{I}_q^a[F((x_1,x_2) + (A_1^{\frac{1}{2}}y, A_2^{\frac{1}{2}}y))]$$

(3.2)

$$= \exp\left\{\frac{iq}{2} \left( < A_1y, y > + < A_2y, y > \right) \right\} \mathcal{I}_q^a[F(x_1, x_2) \exp\{-iq[(A_1^{\frac{1}{2}}y, x_1)\sim + (A_2^{\frac{1}{2}}y, x_2)\sim]\}].$$

Proof. Since $F \in \mathcal{F}_{A_1A_2}(B)$ is given by (2.2), we have

$$F(x_1, x_2) \exp\{-iq[(A_1^{\frac{1}{2}}y, x_1)\sim + (A_2^{\frac{1}{2}}y, x_2)\sim]\}$$

$$= \int_H \exp\{i[(A_1^{\frac{1}{2}}(h - qy), x_1)\sim + (A_2^{\frac{1}{2}}(h - qy), x_2)\sim]\} d\mu(h)$$

$$= \int_H \exp\{i[(A_1^{\frac{1}{2}}k, x_1)\sim + (A_2^{\frac{1}{2}}k, x_2)\sim]\} d\mu(k)$$

where $\mu(E) = \mu(E + qy)$ for $E \in B(H)$. Using Theorem 2.3, we obtain

(3.3)

$$\mathcal{I}_q^a[F(x_1, x_2) \exp\{-iq[(A_1^{\frac{1}{2}}y, x_1)\sim + (A_2^{\frac{1}{2}}y, x_2)\sim]\}]$$

$$= \int_H \exp\{-\frac{i}{2q} [< A_1(h - qy), h - qy > + < A_2(h - qy), h - qy >]\} d\mu(h)$$

$$= \exp\{-\frac{iq}{2} [< A_1y, y > + < A_2y, y >]\} \int_H \exp\{-\frac{i}{2q} [< A_1h, h >$$

$$+ < A_2h, h >]\} \exp\{i[< A_1h, y > + < A_2h, y >]\} d\mu(h).$$

By Theorem 3.1 and (3.3), the theorem is proved.

COROLLARY 3.3. Let $F \in \mathcal{F}_{A_1A_2}(B)$ be given by (2.2) and let $y \in H$. Then

$$\mathcal{I}_q^a[F(x_1, x_2)] = \exp\{\frac{iq}{2} ( < A_1y, y > + < A_2y, y >)\} \mathcal{I}_q^a[F((x_1, x_2) + (A_1^{\frac{1}{2}}y, A_2^{\frac{1}{2}}y)) \exp\{iq[(A_1^{\frac{1}{2}}y, x_1)\sim + (A_2^{\frac{1}{2}}y, x_2)\sim]\}].$$
**Remark 3.4.** Let $A_1$ be the identity operator and $A_2 = 0$. Then (3.2) becomes the translation theorem for the analytic Feynman integral on abstract Wiener space, that is,

$$
\int_B^{anf_q} F(x + y) d\nu(x)
$$

(3.4)

$$
= \exp\left\{\frac{iq}{2} |y|^2 \right\} \int_B^{anf_q} F(x) \exp\{-iq(y, x)\} d\nu(x)
$$

which is a well-known result in [7].

Next we introduce one particular Hilbert space [14,15] which has been used in the applications of the Fresnel integral to non-relativistic quantum mechanics. Let $H_N$ be the set of all functions $f : [0,1]^N \to \mathbb{R}$ for which there exists $v$ in $L_2([0,1]^N)$ such that

$$
f(y_1, \ldots, y_N) = \int_0^{y_N} \cdots \int_0^{y_1} v(t_1, \ldots, t_N) dt_1 \cdots dt_N
$$

(3.5)

for all $(y_1, \ldots, y_N) \in [0,1]^N$. And the inner product on $H_N$ is defined by

$$
<f, g> = \int_{[0,1]^N} (\frac{\partial^N f(t_1, \ldots, t_N)}{\partial t_1 \cdots \partial t_N})(\frac{\partial^N g(t_1, \ldots, t_N)}{\partial t_1 \cdots \partial t_N}) dt_1 \cdots dt_N.
$$

(3.6)

Then $H_N$ is a separable Hilbert space. Also it will be helpful to introduce the family of functions $\{f_{(t_1, \ldots, t_N)} : (t_1, \ldots, t_N) \in [0,1]^N\}$ from $H_N$ such that $f_{(t_1, \ldots, t_N)}(y_1, \ldots, y_N) = \prod_{k=1}^N \min\{t_k, y_k\}$. These functions have the reproducing property $<f, f_{(t_1, \ldots, t_N)}> = f(t_1, \ldots, t_N)$ for all $f \in H_N$. Further, it is well-known that $(H_N, C_N, m_N)$ is an example of abstract Wiener space where $C_N = C_N([0,1]^N)$ denotes $N$-parameter Wiener space with Wiener measure $m_N$. Also the above reproducing property carries over to the stochastic inner product (2.1) in the sense that $(x, f_{(t_1, \ldots, t_N)}) = x(t_1, \ldots, t_N)$ for s-a.e. $x \in C_N$.

**Remark 3.5.** Taking $(H, B, \nu) = (H_N, C_N, m_N)$, $A_1 = \text{the identity}$
operator and $A_2 = 0$, we know that (3.2) becomes

$$\int_{C_N}^\text{an} f \quad F(x + y) dm_N(x) = \exp\left\{ \frac{iq}{2} \frac{\partial^N y}{\partial t_1 \cdots \partial t_N} \right\} \int_{C_N}^\text{an} F(x) \quad \exp\left\{ -iq \int_{[0,1]^N} \frac{\partial^N y(t_1, \ldots, t_N)}{\partial t_1 \cdots \partial t_N} dx(t_1, \ldots, t_N) \right\} dm_N(x)$$

(3.7)

which contains Theorem 2 in [4] as a special case $N = 1$.

Finally we end this paper by showing how the analytic Feynman integral can be modified so as to be translation invariant.

Let $f : H \times H \to C$ be a measurable function such that the integral

$$K_f(\lambda_1, \lambda_2) = \int_{H \times H} f(\lambda_1^{\frac{1}{2}} h_1, \lambda_2^{\frac{1}{2}} h_2) d(m \times m)(h_1, h_2)$$

exists for all real $\lambda_1 > 0$ and $\lambda_2 > 0$. If there exists an analytic function $K_f^*(z)$ on $\Omega = \{ z = (z_1, z_2) \in C^2 : Re z_k > 0 \}$ for $k = 1, 2$ such that $K_f^*(\lambda_1, \lambda_2) = K_f(\lambda_1, \lambda_2)$ for all real $\lambda_1 > 0$ and $\lambda_2 > 0$, then we define $K_f^*(z)$ to be the analytic Gauss integral of $f$ over $H \times H$ with parameter $z$, and for $z \in \Omega$ we write

$$I^*_a[f(\cdot, \cdot)] = K_f^*(z).$$

Let $q_1$ and $q_2$ be non-zero real numbers. If the limit

$$\lim_{\tau \to (-i q_1, -i q_2)} I^*_a[f(\cdot, \cdot)] = I^*_a[q(\cdot, \cdot)]$$

exists, $I^*_a[q(\cdot, \cdot)]$ is called the analytic Feynman integral of $f$ over $H \times H$ with parameter $q = (q_1, q_2)$. In particular, if $q_1 = q_2 = q$, we write

$$I^*_a[f(\cdot, \cdot)] = I^*_a[q(\cdot, \cdot)].$$

Using the above definition, the following theorems can be easily obtained from Theorem 2.3 and the same argument as it was used in Theorem 3.1 and Theorem 3.2.
Notes on the analytic Feynman integral

**Theorem 3.6.** Let \( f \in \mathcal{F}_{A_1,A_2}(H) \) be given by

\[
(3.8) \quad f(h_1, h_2) = \int_H \exp\{i[<A_1^\frac{1}{2}h, h_1> + <A_2^\frac{1}{2}h, h_2>]}d\mu(h)
\]

for some \( \mu \in M(H) \). Then \( I^q_\alpha[f(\cdot, \cdot)] \) exists and

\[
I^q_\alpha[f(\cdot, \cdot)] = \int_H \exp\{-i\frac{q}{2} \sum_{k=1}^{2} q_k^{-1} <A_k h, h>\}d\mu(h).
\]

**Theorem 3.7.** Let \( f \in \mathcal{F}_{A_1,A_2}(H) \) be given by (3.8) and let \( y \in H \).

Then \( f((\cdot, \cdot) + (A_1^\frac{1}{2}y, A_2^\frac{1}{2}y)) \in \mathcal{F}_{A_1,A_2}(H) \) and

\[
I^q_\alpha[f((h_1, h_2) + (A_1^\frac{1}{2}y, A_2^\frac{1}{2}y))] = \exp\{i\frac{q}{2}(<A_1 y, y> + <A_2 y, y>)\}
\]

\[
I^q_\alpha[f(h_1, h_2)] \exp\{-i[q(<A_1^\frac{1}{2}y, h_1>) + <A_2^\frac{1}{2}y, h_2>)]\}.
\]

Let \( f \) be an analytic Feynman integrable function over \( H \times H \) for some non-zero real \( q \).

Let

\[
g(h_1, h_2) = \exp\{\frac{iq}{2}(|h_1|^2 + |h_2|^2)\}f(h_1, h_2)
\]

for \((h_1, h_2) \in H \times H\). Then we define the translation invariant analytic Feynman integral of \( g \) with parameter \( q \) by

\[
I^{\text{trans}}_\alpha[g(\cdot, \cdot)] = I^q_\alpha[f(\cdot, \cdot)]
\]

**Theorem 3.8.** Let \( f \in \mathcal{F}_{A_1,A_2}(H) \) and \( g \) be given by (3.8) and (3.9) respectively. Then, for each \( y \in H \), \( g((\cdot, \cdot) + (A_1^\frac{3}{2}y, A_2^\frac{3}{2}y)) \) is translation invariant analytic Feynman integrable over \( H \times H \) with parameter \( q \), and

\[
I^{\text{trans}}_\alpha[g((\cdot, \cdot) + (A_1^\frac{1}{2}y, A_2^\frac{1}{2}y))] = I^{\text{trans}}_\alpha[g(\cdot, \cdot)].
\]
Proof. By (3.9), we have, for \( y \in H \),

\[
g((\cdot, \cdot) + (A_1^\frac{1}{2} y, A_2^\frac{1}{2} y))
\]

\[
= \exp\left\{ \frac{iq}{2} [h_1, h_1] + <h_2, h_2>] \right\} \exp\left\{ -i[q < A_1^\frac{1}{2} y, h_1] + < A_2^\frac{1}{2} y, h_2 >] \right\} f((h_1, h_2) + (A_1^\frac{1}{2} y, A_2^\frac{1}{2} y)).
\]

Since \( f((h_1, h_2) + (A_1^\frac{1}{2} y, A_2^\frac{1}{2} y)) \in \mathcal{F}_{A_1, A_2}(H) \), we know that

\[
f((h_1, h_2) + (A_1^\frac{1}{2} y, A_2^\frac{1}{2} y)) = \int_H \exp\{i[< A_1^\frac{1}{2} k, h_1] + < A_2^\frac{1}{2} k, h_2 >]\} d\mu(k)
\]

for some \( \mu \in M(H) \). Then we can write

\[
g((h_1, h_2) + (A_1^\frac{1}{2} y, A_2^\frac{1}{2} y))
\]

\[
= \exp\left\{ \frac{iq}{2} |h_1|^2 + |h_2|^2] \right\} \int_H \exp\{i[< A_1^\frac{1}{2} h, h_1] + < A_2^\frac{1}{2} h, h_2 >]\} d\mu(h)
\]

where \( \mu(E) = \mu^*(E + qy) \) and \( \mu^*(E) = \exp\left\{ \frac{iq}{2} [h_1, h_1] + <h_2, h_2>] \right\} \int_H \exp\{i[< A_1^\frac{1}{2} k, h_1] + < A_2^\frac{1}{2} k, h_2 >]\} d\mu(k) \) for \( E \in \mathcal{B}(H) \).

Hence \( g((h_1, h_2) + (A_1^\frac{1}{2} y, A_2^\frac{1}{2} y)) \) is translation invariant analytic Feynman integrable.

Applying Theorem 3.7, (3.9) and (3.10), we obtain

\[
\mathcal{I}_a^{t_n}[g((h_1, h_2) + (A_1^\frac{1}{2} y, A_2^\frac{1}{2} y))]
\]

\[
= \mathcal{I}_a^q[\exp\{iq[< A_1^\frac{1}{2} y, h_1] + < A_2^\frac{1}{2} y, h_2 >]\} \exp\left\{ \frac{iq}{2} [A_1 y, y] + < A_2 y, y >] \right\} f((h_1, h_2) + (A_1^\frac{1}{2} y, A_2^\frac{1}{2} y))]
\]

\[
= \exp\left\{ \frac{iq}{2} [A_1 y, y] + < A_2 y, y >] \right\} \mathcal{I}_a^q[f((h_1, h_2) + (A_1^\frac{1}{2} y, A_2^\frac{1}{2} y))]
\]

\[
= \mathcal{I}_a^q[f(\cdot, \cdot)] = \mathcal{I}_a^{t_n}[g(\cdot, \cdot)].
\]
References


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