CERTAIN DISCRIMINATIONS OF PRIME ENDOMORPHISM AND PRIME MATRIX

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Abstract In this paper, for a commutative ring $R$ with an identity, considering the endomorphism ring $\text{End}_R(M)$ of left $R$-module $RM$ which is (quasi-)injective or (quasi-)projective, some discriminations of prime endomorphism were found as follows: each epimorphism with the irreducible(or simple) kernel on a (quasi-)injective module and each monomorphism with maximal image on a (quasi-)projective module are prime. It was shown that for a field $F$, any given square matrix in $\text{Mat}_{m \times n}(F)$ with maximal image and irreducible kernel is a prime matrix, furthermore, any given matrix in $\text{Mat}_{m \times n}(F)$ for any field $F$ can be factored into a product of prime matrices.

1. Introduction

Let $R$ be a commutative ring with an identity and let $R^n$ be the direct product of $n$-copies of $R$, for any natural number $n$.

From the elementary linear algebras, it is well-known that there is an $R$-linear mapping between the set $\text{Mat}_{m \times n}(R)$ of all $m \times n$-matrices and the set $\text{Hom}_R(R^n, R^m)$ of all linear mappings from $R^n$ into $R^m$, where $n, m \in N$ are any natural numbers. In this paper the fact that between $\text{Hom}_R(R^n, R^m)$ and $(\text{Mat}_{m \times n}(R))^t$ there is an $R$-linear mapping too, where $t$ stands for the transpose operator is mostly used. In other words, for an element $(r_1, r_2, \ldots, r_n) \in R^n,$
\[
(r_1 \ r_2 \ \cdots \ r_n) \begin{pmatrix}
  a_{11} & a_{12} & \cdots & a_{1m} \\
  a_{21} & a_{22} & \cdots & a_{2m} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{n1} & a_{n2} & \cdots & a_{nm}
\end{pmatrix} = (r_1, r_2, \cdots, r_n)f
\]

Let \((a_{ij})_{n \times m}\) act on the right side of \(R^n\) for the associated matrix \((a_{ij})_{n \times m}\) with \(f\), denoted by \(Mat(f)\). A module \(RM\) is said to be \(R-\text{quasi-projective}(R-\text{quasi-injective}, \text{resp.})\) if for each epimorphism \(g : RM \to RN\) (for each monomorphism \(f : RK \to RM\), resp.) and for each homomorphism \(\gamma : RM \to RN(\gamma : RK \to RM, \text{resp.})\) there is an \(R-\text{homomorphism such that } \gamma : RM \to RM\) such that \(\gamma = \bar{\gamma}g(\gamma : RM \to RM\text{ such that } \gamma = f\bar{\gamma}, \text{resp.).}\)

Recall that \(R^k\) is an \(R-(\text{quasi-})\text{injective and (quasi-)projective module}\) for any natural number \(k \in \mathbb{N}\). Because we are studying left \(R-\)modules \(RM\), conveniently let the compositions of all mappings be written by the reverse order, in the order as follows:

\[
gh : RK \xrightarrow{g} RM \xrightarrow{h} RN
\]

**Lemma 1.1.** Every monomorphism on any left \(R-(\text{quasi-})\text{injective module } RM\) is right invertible in \(\text{End}_R(M)\).

**Proof.** In the definition of an \(R-(\text{quasi-})\text{injective, replacing } RK\text{ by } RM\text{ and } \gamma\text{ by the identity on } RM\), the proof is established immediately.

**Lemma 1.2.** Every epimorphism on any left \(R-(\text{quasi-})\text{projective module } RM\) is left invertible in \(\text{End}_R(M)\).

**Proof.** In the definition of an \(R-(\text{quasi-})\text{projective, replacing } RN\text{ by } RM\text{ and } \gamma\text{ by the identity on } RM\), the proof is established immediately.

For the reason of the following definition, it will be answered partly in Remark 2.7.

An endomorphism \(g\) is said to be left \(\text{retractable in } \text{End}_R(M)\) if there is an endomorphism \(g' \in \text{End}_R(M)\) such that the restriction \(g'g|_{\text{Img}}\) of the composition \(g'g\) of \(g'\) and \(g\) to the image \(\text{Img}\) of \(g\) is the identity of the image of \(g\), that is, \(g'g|_{\text{Img}} = 1_{\text{Img}}\) the identity endomorphism on \(\text{Img} \leq M\).
DEFINITION 1.3. For a non-unit endomorphism \( p \) of the endomorphism ring \( S = \text{End}_R(M) \), \( p \) is said to be prime if \( p = fg \) for \( f, g \in S \), then \( f \) is right invertible or \( g \) is left retractable in \( \text{End}_R(M) \).

Two endomorphisms \( f, g \in \text{End}_R(M) \) are said to be similar if \( \text{Im}f = \text{Im}g \leq M \). This definition of the similarity of two \( f, g \in \text{End}_R(M) \) (with an \( n \)-dimension vector space \( F \cdot M \) over a field \( F = R \) having a fixed basis) is not the same as the similarity of two associated matrices \( \text{Mat}(f), \text{Mat}(g) \) in \( \text{Mat}_{n \times n}(R) \) with \( f, g \) by \( \text{Mat} : \text{End}_R(M) \rightarrow \text{Mat}_{n \times n}(R) \), i.e., not the same as the similarity of matrices in many general Linear Algebra books. Two endomorphisms \( f, g \in \text{End}_R(M) \) are said to be cosimilar if \( \ker f = \ker g \leq M \).

2. Results

Any commutative ring \( R \) with an identity and \( _RR^n \) are (quasi-)injective (quasi-)projective module for any natural number \( n \). By Lemmas 1.1 and 1.2, the following Theorems 2.1 and 2.2 are obtained easily.

**Theorem 2.1.** All non-unit epimorphisms on any left \( R- \) (quasi-)projective module \( _RM \) are prime.

**Theorem 2.2.** All non-unit monomorphisms on any left \( R- \) (quasi-)injective module \( _RM \) are prime.

A submodule \( N \leq M \) is said to be irreducible (simple) if \( N \) has no non-zero submodule.

**Proposition 2.3.** For a left (quasi-)projective module \( _RM \), if a monomorphism \( f \) in \( \text{End}_R(M) \) has the maximal image \( \text{Im}f \leq M \), then \( f \) is prime.

**Proof.** Suppose that \( f = gh \) for some endomorphisms \( g, h \in \text{End}_R(M) \). Then the maximal submodule \( \text{Im}f = \text{Im}gh \leq \text{Im}h \) implies that \( \text{Im}h = M \) or \( \text{Im}f = \text{Im}h \). If \( \text{Im}h = M \), then \( h \) is left invertible in \( \text{End}_R(M) \) since \( M \) is (quasi-)projective. Hence \( h \) is left retractable. If \( \text{Im}f = \text{Im}h \), then \( h = sf \) and \( f = th \) for some \( s, t \in \text{End}_R(M) \) since \( M \) is (quasi-)projective. Thus \( f = gh = gsf \) and \( (1_M - gs)f = 0 \) follow, where \( 1_M \) denotes the identity mapping on \( M \). Hence \( \text{Im}(1_M - gs)f = 0 \).
0 and $\text{Im}(1_M - gs) \leq \ker f = 0$ implies that $\text{Im}(1_M - gs) = 0$ and $1_M - gs = 0$. Hence $g$ is right invertible in $\text{End}_R(M)$. Therefore $f$ is prime. □

Let a prime monomorphism denote a monomorphism with the maximal image on a (quasi-)projective module.

**Proposition 2.4.** For any left (quasi-)injective $R$-module $R^M$, if an epimorphism $f$ in $\text{End}_R(M)$ has the irreducible kernel $\ker f \leq M$, then $f$ is prime.

**Proof.** By the dual proof of the Proposition 2.3, it is proved.

Let a prime epimorphism denote an epimorphism with the irreducible
(simple) kernel on an (quasi-)injective module.

**Corollary 2.5.** For an endomorphism $g$ and for any prime monomorphism $f_\alpha \in \text{End}_R(M)$ with the (quasi-)projective module $R^M$, if $\text{Im}g \leq \cap_\alpha \text{Im}f_\alpha$, then $f_\alpha$ divides $g$ for each $\alpha$.

**Proof.** Suppose that $\text{Im}f \leq \cap_\alpha \text{Im}f_\alpha$ for some indexed $\{f_\alpha\}_\alpha$. Then the fact $\text{Im}f \leq \text{Im}f_\alpha$, for each $\alpha$ implies that $f = s_\alpha f_\alpha$ for some $s_\alpha \in \text{End}_R(M)$ and for each $\alpha$ since $R^M$ is (quasi-)projective.

**Corollary 2.6.** For an endomorphism $f$ and for any prime epimorphism $f_\alpha \in \text{End}_R(M)$ with (quasi-)injective module $R^M$, if $\ker f \geq \sum_\alpha \ker f_\alpha$, then $f_\alpha$ divides $f$ for each $\alpha$.

**Remark 2.7.** The definition of prime endomorphism (quasi-)injective
and
(quasi-)projective module $R^M$ if $\text{End}_R(M)$ is commutative is the same as the definition of irreducible or prime elements of commutative rings.

Precisely, on a (quasi-)injective and (quasi-)projective module $R^M$ each prime endomorphism $f$ with $f = gh$ implies that $g$ is a unit in $\text{End}_R(M)$ or $h$ is left retractable.

For a right invertible factor $g$ of $f$, there is an $s \in \text{End}_R(M)$ such that $gs = 1_M$. To show that $sg = 1_M$, let's consider the following
diagram including monomorphism $g$ and epimorphism $s$ with the condition $gs = 1_M$:

\[
\begin{array}{ccccccc}
M & \rightarrow & M \\
\alpha \downarrow & & \downarrow g \\
0 & \rightarrow & M & \rightarrow & M & \rightarrow & 0 \\
\downarrow s & & \downarrow \beta & & \downarrow s \\
M & \rightarrow & M & \rightarrow & M & \rightarrow & M
\end{array}
\]

then there are endomorphisms $\alpha, \beta \in \text{End}_R(M)$ such that $g = \alpha s$ and $s = g\beta$ since $RM$ is (quasi-)injective (quasi-)projective. Clearly $\beta\alpha = 1_M$. And hence $sg = (g\beta)(\alpha s) = g(\beta\alpha)s = gs = 1_M$ follows. Therefore $g$ is a unit. Thus if $\text{End}_R(M)$ is commutative, and if a prime endomorphism $f$ has a product $f = gh = hg$, then one of $g$ and $h$ is at least a unit in $\text{End}_R(M)$.

From the above Corollaries 2.5 and 2.6 it isn’t told in general that $f$ has a factorization in terms of the prime epimorphisms or the prime monomorphisms. It depends on the first left endomorphism and on the last right endomorphism. In other words, if $f = sp_\alpha$ (or $f = p_\alpha t$) for some prime epimorphism or prime monomorphism $p_\alpha$. Then we must try to factor out $s$ (or $t$) and so on, respectively.

**Proposition 2.8.** For a left (quasi-)injective and (quasi-)projective module $RM$, if a non-unit endomorphism $f$ has the maximal image $\text{Im}f \leq M$ and the irreducible(simple) kernel $\text{ker} f \leq M$. Then $f$ is prime.

**Proof.** Suppose that $f = gh$ with $\text{Im}g \cap \text{ker} h \neq 0$ or with $\text{Im}g \cap \text{ker} h = 0$. Then $\ker g \subseteq \ker f = g^{-1}(\ker h)$ the preimage of $\ker h$ under $g$ implies that $\ker g = 0$ from $\text{Im}g \cap \ker h \neq 0$. Or we have a case of $\text{Im}g \cap \ker h = 0$ with $f = gh$. If $\text{Im}p \supseteq \text{Im}h = M$, the retractability of $h$ follows immediately. Hence we assume that $\text{Im}h = \text{Im}p \leq M$ is proper in $M$.

We have a monomorphism $g$ which is right invertible in $\text{End}_R(M)$ for the first case. For the case of $\text{Im}g \cap \ker h = 0$, if $\ker h \neq 0$ we have a submodule $\text{Im}g \oplus \ker h \leq M$. Then $\text{Im}f \simeq \text{Im}g \simeq \text{Im}h$ follows.
from \( \ker f = \ker g \) and \( \text{Im} h = \text{Im} p \), where the symbol \( \simeq \) denotes the isomorphic. Hence \( h \) is left retractable on \( \text{Im} h \) through the extendable isomorphisms on a left (quasi-)injective and (quasi-)projective module \( R^M \).

If \( \ker h = 0 \), it follows that a monomorphism \( h \) (which is a unit since \( R^M \) is a left (quasi-)injective and (quasi-)projective module) is left retractable on \( \text{Im} h \leq M \). Therefore \( f \) is a prime endomorphism. \( \square \)

3. Applications

Remind that the ring \( R \) is assumed to be a commutative ring with an identity. Here \( R^n = \prod_{i=1}^n R \) denotes the product of \( \{R_i\}_{1 \leq i \leq n} \) with \( R_i = R \) and \( R^{(n)} \) denotes the direct product of \( n \)-copies of \( R \). Recall the linear algebra theory: there is an \( R \)-linear mapping between the set of all linear mappings from an \( n \)-dimensional vector space \( F^U \) into the \( m \)-dimensional vector space \( F^V \) and the set \( \text{Mat}_{n \times m}(R) = (\text{Mat}_{m \times n}(R))^t \) of \( n \times m \)-matrices whose entries are in \( R \) where \( t \) denotes the transpose operator. For any field \( F \) with identity 1, the following should be noticed:

1. Every maximal submodule of \( F^{(n)} \) is the direct product \( F^{(n-1)} \) and every irreducible(simple) submodule of \( F^{(n)} \) is the direct product \( F^{(1)} \).
2. The direct product \( F^{(n)} \) of \( n \)-copies of any field \( F \) is (quasi-)injective and (quasi-)projective for any \( n \), moreover \( F^{(n)} \) is self-generated and self-cogenerated.
3. Hereafter we assume that each \( k \)-dimensional space \( F^k \) has the standard orthogonal basis
\[
\{e_i = (x_1, \ldots, x_k) \mid x_i = 1, x_j = 0, \text{ for } j \neq i, 1 \leq i, j \leq k \}
\]
for each natural number \( k \in \mathbb{N} \).
4. It is important to remember that every monomorphism and every epimorphism from \( F^k \) into itself \( F^k \) are automorphisms for every \( k \in \mathbb{N} \).

Briefly and conveniently, let's replace the associated linear mapping \( L(A) \) by \( A \) properly.

**Application 3.1.** Let \( A = (a_{ij}) \in \text{Mat}_{n \times n}(F) \) be a matrix with the maximal image \( \text{Im} A = F^{(n-1)} \) and the irreducible(simple) kernel
ker\(A = F^{(1)}\). Then \(A\) is a prime matrix. Furthermore every similar cosimilar matrix to the above matrix \((a_{ij})_{n \times n}\) is also prime.

**Proof.** For any epic or monic matrix \(U\), the associated linear mapping

\[ L(U) : F^n \to F^n \text{ the (quasi-)injective (quasi-)projective } n-\text{dimensional vector space } F^n \text{ over } F \text{ is an automorphism.} \]

Hence each monic matrix and each epic matrix are units. Thus the Proposition 2.8 can apply here to the matrix ring \(Mat_{n \times n}(F)\). Hence we have immediately a prime matrix \(A\) with the maximal image and the irreducible kernel. \(\Box\)

For example, let \(A = (a_{ij}) \in Mat_{n \times n}(F)\) be a matrix such that

\[
\begin{align*}
    a_{jj} &= 1, \text{ for } 1 \leq j \leq n \\
    a_{ik} &= a_{ki} = -1, \text{ for the only one } k, 1 \leq k \leq n, \\
    a_{ij} &= 0, \text{ if } i \neq j \neq k.
\end{align*}
\]

\[
(a_{ij})_{n \times n} = \begin{pmatrix}
    1 & 0 & 0 & \cdots & -1 & \cdots & 0 \\
    0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
    0 & 0 & 1 & \cdots & 0 & \cdots & 0 \\
    \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
    -1 & \cdots & 0 & 1 & \cdots & \cdots & \cdots \\
    0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
    \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1
\end{pmatrix}
\]

is a prime matrix

For this matrix \((a_{ij})_{n \times n}\), \(A = (a_{ij})_{n \times n}\) is an endomorphism with the maximal image

\[ ImA = \{(x_1, \ldots, x_k, \ldots, x_n) \mid x_1 = -x_k, x_t \in F\} \]

\[ = F^{(n-1)} \]

and the irreducible(simple) kernel

\[ \ker A = F^{(1)} \]

\[ = \{ (a_1, 0, \cdots, 0, a_k, 0, \cdots, 0) \mid a_1 = a_k, a_t = 0 \text{ for } i \neq k, 1 \leq i \leq n \}. \]

**Corollary 3.2.** For any matrix \(A = (a_{ij})_{n \times n} \in Mat_{n \times n}(F)\) with a field \(F\) and for any prime matrix \(P_\alpha = (p_{ij})_\alpha \in Mat_{n \times n}(F)\), if
$\text{Im}A \leq \cap \text{Im}P_\alpha$ and if $\ker A \geq \sum \ker P_\alpha$, then $P_\alpha$ divides $A$ for each $\alpha$.

Recalling the item (3) above the Application 3.1, we only consider the standard orthogonal bases of all $F^k$. Then the following are obtained by the associate $\text{Mat}$ preserving composition of linear mappings, that is,

$$\text{Mat}(fg) = \text{Mat}(f)\text{Mat}(g).$$

**Application 3.3.** For $n \in \mathbb{N}$ and for any field $F$ with an identity 1, if $f : F^n \to F^n$ is a prime endomorphism, then the associated matrix $\text{Mat}(f)$ is a prime matrix. Clearly if a square matrix is prime, then its associated linear mapping is also a prime endomorphism.

For two matrices $A$, $C$, we call $A$ an edge factor of $C$ if $C = AB \cdots H$ or $C = H \cdots BA$ for some matrices $B, \cdots, H$.

**Remark 3.4.** For two square matrices $A, B \in \text{Mat}_{n \times n}(F)$, the following are to be read about similar matrices:

1. If $A, B$ are similar in the sense of Linear Algebra, i.e., there is a unit matrix $N \in \text{Mat}_{n \times n}(F)$ such that $A = N^{-1}BN$. Then $A$ is prime if and only if $B$ is prime. However $A$ and $B$ need not be, in general, similar cosimilar in the sense of this paper.

2. For $A, B$ as in (1) and for $C \in \text{Mat}_{n \times n}(F)$, $A$ is a factor of $C$ if and only if $B$ is a factor of $C$, however for an edge factor $A$ of $C$, $B$ need not be an edge factor of $C$ in general.

3. Moreover for similar cosimilar matrices $A, B \in \text{Mat}_{n \times n}(F)$, $A$ is prime if and only if $B$ is prime.

4. If $A, B$ are similar cosimilar and $C \in \text{Mat}_{n \times n}(F)$. Then $A$ is a factor, or an edge factor of $C$ if and only if $B$ is a factor, or an edge factor of $C$, respectively.

A **Method of Factorizing a Square Matrix.** There might be lots of different ways to factorize any given square matrix $A \in \text{Mat}_{n \times n}(F)$.

1. Find prime square matrices $P_\alpha$ such that $\cap \text{Im}P_\alpha \geq \text{Im}A$ and $\ker A \geq \sum \ker P_\alpha$.

2. Select one $P_{\alpha_0}$ of the prime matrices $P_\alpha$.

3. Find a factor matrix $F_{\alpha_0}$ such that $A = F_{\alpha_0}P_{\alpha_0}$ or $A = P_{\alpha_0}F_{\alpha_0}$. 
Certain discriminations of prime endomorphism and prime matrix

(4) *Do the step (1) for the factor matrix $F_{cm}$.*

(5) *After the steps (1) and (4), go to the steps (1) and (4).*

(6) *Select those factors of A and write them properly.*

For further applications of prime matrices with distinct size $n$ by $m$ for $n \neq m$, here some illustrations are given.

(1) For $n \leq m$ and a monomorphism $f : F^m \to F^n$, let $k = m - n$ and let partitionize the associated matrix $\text{Mat}(f)$ by $k$ by $k$, that is, $(\text{Mat}(f)) = (F_{11} \ F_{12})$, where $F_{11} \in \text{Mat}_{k \times k}$ and $F_{12} \in \text{Mat}_{k \times n}$. Then we have a prime matrix $P \in \text{Mat}_{m \times m}(F)$, precisely

$$
P = \begin{pmatrix}
F_{11} & F_{12} \\
D_{kk} & 0_{nn}
\end{pmatrix}
$$

where $D_{kk} = (d_{ij})_{k \times k}$ with

$$
d_{ij} = \begin{cases}
0 & \text{if } i = j = l \text{ for only one } l, 1 \leq l \leq k \\
\delta_{ij} & \text{elsewhere, for the Kronecker's delta } \delta_{ij}
\end{cases}
$$

and where $0_{nn}$ is the zero matrix. This matrix $P = \begin{pmatrix} F_{11} & F_{12} \\
D_{kk} & 0_{nn}\end{pmatrix}$ is a prime factor of $\left(\begin{pmatrix} \text{Mat}(f) \\
0_{km} \end{pmatrix}\right)_{m \times m}$.

(2) For $n \geq m$ and an epimorphism $f : F^n \to F^m$, let $k = n - m$ and let partitionize the associated matrix $\text{Mat}(f)$ by $m$ by $m$, that is $(\text{Mat}(f)) = \begin{pmatrix} F_{11} \\
F_{21} \end{pmatrix}$, where $F_{11} \in \text{Mat}_{m \times m}$ and $F_{21} \in \text{Mat}_{k \times m}$. Then we have a prime matrix $P$ in $\text{Mat}_{n \times n}(F)$ such that $P = \begin{pmatrix} F_{11} & 0_{mk} \\
F_{21} & D_{kk}\end{pmatrix}$ where $D_{kk} = (d_{ij})_{k \times k}$ is as in the above (1) and where $0_{mk}$ is the zero matrix. This matrix $P$ is a prime factor of $\left(\begin{pmatrix} \text{Mat}(f) \\
0_{nk} \end{pmatrix}\right)_{n \times n}$.

References


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