A NOTE ON THE TWO DIMENSIONAL SURFACE IN FOUR DIMENSIONAL EQUIAFFINE SPACE

E. T. Ivlev, O. V. Rozhkova and Hai Gon Je

Abstract. In this paper, we investigate the existence of the two dimensional surface in four dimensional equiaffine space and characterize that surface.

0. Introduction

In [7-9] an invariant clothing of the families of two dimensional and \( m \) dimensional planes in four dimensional and \( 2m \) dimensional equiaffine space \((m > 2)\) respectively, has been carried out. By this construction the case when the family of two dimensional planes envelops some two dimensional surface in the four dimensional space \( A_4 \) and the family of \( m \) dimensional planes envelops some \( m \) dimensional surfaces in \( A_{2m} \), is taken out of consideration.

This article is devoted to an invariant construction of clothings of the families of two dimensional planes in \( A_4 \) enveloping some two dimensional general surface \( S_2 \) that is, in fact, reduced to the study of the two dimensional surface in four dimensional equiaffine space. Therefore, this article is referred to the General Theory of surfaces (see references [3,4,6]).

In this article §1 is devoted to analytical apparatus, in which, in particular, an analytical construction of the canonical frame of the surface \( S_2 \) in \( A_4 \) is brought, in §2 a Basic Theory of the Affine Theory of surfaces in \( A_4 \) is proved. In §3 the clothing of a surface \( S_2 \) is built.

Notation and terminology correspond to the adopted in [1-9].

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1. An analytical fixation of the frame

The surface $S_2$ in $A_4$ is a hodograph of the vector function of two arguments: $\vec{r} = \vec{r}(u, v)$.

If $\omega^i$ and $\omega_i^k$ are the Pfaff's forms from these arguments, the derivative formulas of a mobile reper are to be expressed as

$$d\vec{r} = \omega^i \vec{e}_i, \quad d\vec{e}_i = \omega_i^k \vec{e}_k,$$

where the forms $\omega^i$, $\omega_i^k$ satisfy the structural equations

$$D\omega^i = \omega^j \wedge \omega^i_j, \quad D\omega_i^k = \omega_i^j \wedge \omega_j^k.$$

Here the formulas $\omega_1^i, \omega_2^i, \omega_3^i, \omega_4^i$ by virtue of the condition of equiaffinity

$$(\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4) = 1,$$

satisfy the differential correlation $\omega_1^1 + \omega_2^2 + \omega_3^3 + \omega_4^4 = 0$.

When the frame changes, the main parameters $u, v$ remain constant but the secondary parameters, according to which the Pfaff's forms are denoted usually as $\pi^i$, $\pi_i^k$ and differentiation as $\delta$, change. We have

$$\delta \vec{r} = \pi^i \vec{e}_i, \quad \delta \vec{e}_i = \pi_i^k \vec{e}_k.$$

On placing the beginning of frame in a point of the surface, we obtain

$$\delta \vec{r} = 0, \quad \pi^1 = \pi^2 = \pi^3 = \pi^4 = 0.$$

The forms $\omega^1, \omega^2, \omega^3, \omega^4$ have become the main ones and they depend only on two arguments $u$ and $v$. Consequently, there are two linear dependences between them. Choosing them in the form

$$\omega^3 = 0, \quad \omega^4 = 0,$$

we have $(d\vec{r}, \vec{e}_1, \vec{e}_2) = 0$, that is, the vectors $\vec{e}_1$ and $\vec{e}_2$ will lie inside the tangent plane.

To continue the construction of the reper we differentiate (3) externally, using (2), we obtain $\omega^i \wedge \omega_i^\alpha + \omega^j \wedge \omega_j^\beta = 0, \ (\alpha = 3, 4)$. 
Hence, according to the Cartan's lemma,

\[ \omega^\alpha_a = A^\alpha_{a\beta} \omega^\beta, \quad A^\alpha_{[a\beta]} = 0, \quad (\alpha, \beta, \gamma = 1, 2, \hat{\alpha}, \hat{\beta}, \hat{\gamma} = 3, 4). \]

Differentiating these differential equations externally, we obtain

\[ (dA^\alpha_{a\beta} - A^\alpha_{a\gamma} \omega^\gamma - A^\alpha_{a\gamma} \omega^\gamma + A^\beta_{a\beta} \omega^\delta) \wedge \omega^\beta = 0. \]

Let us consider two linear elements

1. \( \omega^\alpha(d), \omega^k_i(d) \) and \( \omega^\alpha(d) = 0, \)

2. \( \omega^\alpha(\delta) \) and \( \omega^k_i(\delta) = \pi^k_i. \)

Substituting them in (5), we obtain

\[ (\delta A^\alpha_{a\beta} - A^\alpha_{a\gamma} \pi^\gamma - A^\alpha_{a\gamma} \pi^\gamma + A^\beta_{a\beta} \pi^\delta) = 0 \]

The following fixation is possible with the use of (6).

\[ A^3_{12} = 0, A^4_{12} = 0, A^4_{11} = 0, \quad A^3_{22} = 0, A^3_{11} = 1, A^4_{22} = 1, \]

\[ 2\pi^1 - \pi^3 = 0, \quad \pi^1 = 0, \quad \pi^3 = 0, 2\pi^2 - \pi^4 = 0, \quad \pi^2 = 0, \quad \pi^4 = 0. \]

From (4)

\[ \omega^3_1 = \omega^1, \quad \omega^3_2 = 0, \quad \omega^3_1 = 0, \quad \omega^4_2 = \omega^2, \]

after applying the Cartan's lemma, formulas (5) brought to the correlations.

\[ 2\omega^1 - \omega^3_2 = A\omega^1 + B\omega^2, \]
\[ 2\omega^3_2 - \omega^4_1 = A^*\omega^2 + B^*\omega^1, \]
\[ \omega^1 = B\omega^1 + C\omega^2, \]
\[ \omega^2 = B^*\omega^2 + C^*\omega^1, \]
\[ \omega^3 = -C\omega^1 + E\omega^2, \]
\[ \omega^4 = -C^*\omega^2 + E^*\omega^1. \]
We differentiate (9) externally, using (2) and (8). Then, we have

\begin{align}
(dA - A\omega_1^1 - B\omega_1^2 + 3\omega_1^3 + (EE^* + CC^* - 2BB^*)\omega_1^2) \wedge \omega^1 &+ (dB - B\omega_2^2 - A\omega_2^1) \wedge \omega^2 = 0, \\
(dB^* - B^*\omega_1^1 - A^*\omega_1^2) \wedge \omega^1 +
(dA^* - A^*\omega_2^2 - B^*\omega_2^1 + 3\omega_2^3 + (2BB^* - EE^* - CC^*)\omega_2^1) \wedge \omega^2 = 0, \\
(dB - B\omega_2^2 - C\omega_1^2) \wedge \omega^1 + (dC + C\omega_1^1 - B\omega_1^2 - 2C\omega_2^2 + \omega_2^4) \wedge \omega^2 = 0, \\
(dC^* + C^*\omega_2^2 - B^*\omega_1^1 - 2C^*\omega_1^2 + \omega_1^3) \wedge \omega^1 &+ (dB^* - B^*\omega_1^1 - C^*\omega_2^1) \wedge \omega^2 = 0, \\
(-dC + C\omega_1^1 - E\omega_1^2 - \omega_2^4 - C\omega_3^2 + C\omega_4^1) \wedge \omega^1 + (dE - E\omega_2^2 + C\omega_2^1 + E\omega_3^3 - E\omega_4^4) \wedge \omega^2 = 0, \\
(dE^* - E^*\omega_1^1 + C^*\omega_1^2 - E^*\omega_3^2 + E^*\omega_4^2) \wedge \omega^1 &+ (-dC^* + C^*\omega_2^2 - E^*\omega_2^1 - \omega_3^2 + C^*\omega_3^3 - C^*\omega_4^4) \wedge \omega^2 = 0.
\end{align}

Hence, in a similar manner as above, we shall arrive at the following correlations for fixation of the rest of secondary forms.

\begin{align}
\delta A - A\pi_1^1 + 3\pi_3^1 = 0, & \quad \delta B - B\pi_2^2 = 0, \\
\delta B^* - B^*\pi_1^1 = 0, & \quad \delta A^* - A^*\pi_2^2 + 3\pi_4^2 = 0, \\
\delta C + 3C\pi_1^1 + \pi_3^1 = 0, & \quad \delta C^* - 3C^*\pi_2^2 + \pi_3^2 = 0, \\
\delta E - 5E\pi_2^2 = 0, & \quad \delta E^* - 5E^*\pi_1^1 = 0.
\end{align}

Using correlations (11), we carry out the following final fixation of the affine frame on the surface \(S_2\).

\begin{align}
A = 0, B = 1, A^* = 0, B^* = 1, C = 0, C^* = 0, \\
\pi_1^1 = 0, \pi_2^2 = 0, \pi_3^1 = 0, \pi_3^2 = 0, \pi_4^1 = 0, \pi_4^2 = 0.
\end{align}

From (9) we obtain

\begin{align}
\omega_2^1 = \omega^1, & \quad \omega_2^2 = \omega^2, & \quad \omega_3^3 = E\omega^2, & \quad \omega_4^4 = E^*\omega^1, \\
2\omega_1^1 - \omega_3^2 = \omega^2, & \quad \omega_2^2 - \omega_4^4 = \omega^1.
\end{align}
Formulas (10), by virtue of (13) and (12), after applying the Cartan's lemma, brought to the differential equations.

\[(14) \quad \omega^\alpha = A^\alpha_{\alpha\beta} \omega^\beta, \quad \omega^\prime = A^\alpha_{\alpha\beta} \omega^\beta\]

(do not sum in \(\alpha\)), where

\[(15)\]
\[
(dE^* + E^* \omega^2 + A^2_{31} \omega^2 - E^* A^3_{32} \omega^2 + 2E^* A^2_{22} \omega^2 - E^* A^1_{12} \omega^2) \land \omega^1 = 0,
\]
\[
(dE + E \omega^1 - A^1_{42} \omega^1 - EA^1_{41} \omega^1 + 2E A^1_{11} \omega^1 - EA^2_{21} \omega^1) \land \omega^2 = 0,
\]
\[
A^2_{32} + A^1_{41} = 1, \quad A^1_{11} + A^2_{32} = 1, \quad A^3_{32} = 2A^1_{12} - 1, \quad A^4_{41} = 2A^2_{21} - 1,
\]
\[
-3A^1_{32} + EE^* - A^2_{21} + 3 = 0, \quad -3A^2_{41} + EE^* + 3 - A^1_{12} = 0.
\]

2. The basic theorem of the affine theory of surfaces in \(A_4\)

As all the secondary forms have been reduced to zero, the forms \(\omega^k\) are the linear combinations of the forms \(\omega^1\) and \(\omega^2\), which are defined due to formulas (3),(8), (13) and (14). In these combinations the corresponding coefficients satisfy not only correlations (15), but also the structural equations, which follow from (2). Let us notice that equations (13) are derived from (8), and (14) are derived from (13) by means of external differentiation.

**Theorem 1.** The surface \(S_2\) in \(A_4\) exists and is determined with arbitrariness of two functions of two arguments.

**Proof.** Differentiating (14) externally, we obtain

\[(16)\]
\[
dA^1_{31} \land \omega^1 + dA^1_{32} \land \omega^2 =
\]
\[
(2A^1_{31} - A^2_{32} + E^* A^1_{42} - 2A^2_{21} A^1_{32} - 2A^1_{41} A^1_{32} - 2A^1_{12} A^1_{31}) \omega^1 \land \omega^2,
\]
\[
dA^2_{41} \land \omega^1 + dA^2_{42} \land \omega^2 =
\]
Thus, if the given functions

\[
A_{31}, A_{32}, A_{31}^2, A_{32}^2, A_{41}, A_{42}, A_{41}^2, A_{42}^2, A_{11}, A_{12},
A_{21}^2, A_{22}^2, E, E^*,
\]

satisfy (15) and (16), then affinity of the surface \( S_2 \) in \( A_4 \) is given.

The definition of this surface is reduced to integration of the following quite integrable system of derivative equations

\[
\begin{align*}
d\bar{r} &= \omega^1 \bar{e}_1 + \omega^2 \bar{e}_2, \\
d\bar{e}_1 &= (A_{11}^1 \omega^1 + A_{12}^2 \omega^2) \bar{e}_1 + \omega^2 \bar{e}_2 + \omega^1 \bar{e}_3, \\
d\bar{e}_2 &= \omega^1 \bar{e}_1 + (A_{21}^2 \omega^1 + A_{22}^2 \omega^2) \bar{e}_2 + \omega^2 \bar{e}_4, \\
d\bar{e}_3 &= (A_{31}^1 \omega^1 + A_{32}^2 \omega^2) \bar{e}_1 + (A_{31}^2 \omega^1 + A_{32}^2 \omega^2) \bar{e}_2 + \omega^3 \bar{e}_3 + E^* \omega^1 \bar{e}_4, \\
d\bar{e}_4 &= (A_{41}^1 \omega^1 + A_{42}^2 \omega^2) \bar{e}_1 + (A_{41}^2 \omega^1 + A_{42}^2 \omega^2) \bar{e}_2 + E \omega^2 \bar{e}_3 + \omega_4^4 \bar{e}_4, \\
\omega_3 &= 2 \omega_{11}^1 \omega^1 + (2 \omega_{12}^2 - 1) \omega^2, \\
\omega_4 &= 2 \omega_{22}^2 \omega^2 + (2 \omega_{21}^2 - 1) \omega^1.
\end{align*}
\]

For the system (17) according to Bachvalov's Theorem, we obtain that the arbitrariness of the existence of surface \( S_2 \) in \( A_4 \) is equal to two functions of two arguments.

**Remark.** The same arbitrariness is to be received on considering (15)–(16).
3. The focal (conjugate) lines on the surface $S_2$

The focal hyperplanes. Rationing of the vectors $\vec{e}_3$ and $\vec{e}_4$.

3.1. The focal straight lines $l_1$ and $l_2$

We shall put

\begin{equation}
L_2 = (\vec{A}, \vec{e}_1, \vec{e}_2) = (\vec{r}, \vec{e}_1, \vec{e}_2)
\end{equation}

the tangent plane to $S_2$ at the point $\vec{A}$.

Let the point in $L_2$ with the radius vector

\begin{equation}
\vec{X} = \vec{r} + x^\alpha \vec{e}_\alpha \in L_2
\end{equation}

be a focus, that is, describes a line with a tangent belonged to $L_2$ along a (focal) line on $S_2$ [8]. Then $(d\vec{X}, \vec{e}_1, \vec{e}_2) = 0$, which, by virtue of (1), (3) and (4), leads to the correlations.

\begin{equation}
x^\alpha A^\alpha_{\beta \gamma} \omega^\beta = 0, \ (\alpha, \beta = 1, 2; \ \hat{\alpha}, \hat{\beta} = 3, 4)
\end{equation}

This system has the non-trivial solutions in $\omega^\alpha$ if and only if

\begin{equation}
det[x^\alpha A^\alpha_{\beta \gamma}] = \\
= \left| \begin{array}{cc}
x^\alpha A^3_{\alpha 1} & x^\alpha A^3_{\alpha 2} \\
x^\beta A^3_{\beta 1} & x^\beta A^3_{\beta 2}
\end{array} \right| \\
= (A^3_{\alpha 1} A^4_{\beta 2} - A^3_{\alpha 2} A^4_{\beta 1}) x^\alpha x^\beta \\
= (A^3_{11} A^4_{12} - A^3_{12} A^4_{11})(x^1)^2 + (A^3_{11} A^4_{22} - A^3_{22} A^4_{11})x^1 x^2 \\
+ (A^3_{21} A^4_{22} - A^3_{22} A^4_{21})(x^2)^2 = 0.
\end{equation}

Thus, to each point $\vec{A} \in S_2$ in $A_4$ the plane $L_2$ correspond two focal lines $l_1$ and $l_2$. The tangent line (focal or conjugate) at the point $\vec{A}$ on $S_2$ corresponds to each of such straight lines by virtue of (20).
3.2. The focal hyperplanes $\Gamma_3^1$ and $\Gamma_3^2$

Let us take up the hyperplane at each point $\tilde{A} \in S_2$

(22)  
$$x : x_{\alpha}x^{\tilde{\alpha}} = 0,$$

passing through $L_2$. Let this hyperplane be a focal hyperplane, that is, the plane containing $L_2$ which is near it along some (focal) line on $S_2$ [8]. From $d(\tilde{A}, \tilde{e}_1, \tilde{e}_2) = (...) \tilde{e}_\alpha + \omega^\alpha_1(\tilde{A}, \tilde{e}_\alpha, \tilde{e}_2) + \omega^\alpha_2(\tilde{A}, \tilde{e}_1, \tilde{e}_\alpha)$ by virtue of (22), we obtain

(23)  
$$x_{\alpha}A_{\alpha\beta}^\tilde{\alpha}\omega^\beta = 0.$$

This system has the non-trivial solutions in $\omega^\alpha$ if and only if

(24)  
$$\text{det}[x_{\alpha}A^\tilde{\alpha}_{\alpha\beta}]$$

$$= \left| x_{\alpha}A_{11}^\tilde{\alpha} x_{\alpha}A_{12}^\tilde{\alpha} x_{\beta}A_{22}^\beta \right|$$

$$= (A_{11}^\tilde{\alpha}A_{22}^\beta - A_{12}^\tilde{\alpha}A_{12}^\beta)x_{\alpha}x_{\beta}$$

$$= (A_{11}^3A_{22}^3 - A_{12}^3A_{12}^3)(x_3)^2 + (A_{11}^3A_{22}^4 + A_{22}^3A_{12}^4 - 2A_{12}^3A_{12}^4)x_3x_4$$

$$+ (A_{11}^4A_{22}^3 - A_{12}^4A_{12}^3)(x_4)^2$$

$$= 0.$$

Thus, to each point $\tilde{A} \in S_2$ in $A_4$ correspond two focal (tangent) [9] hyperplanes $\Gamma_3^1$ and $\Gamma_3^2$, which, by virtue of (20) and (23) contain $L_2$ and $L_2'$ along the corresponding focal (conjugate) lines on $S_2$. In this case the plane $L_2$ intersects with its contiguous $L_2'$ along the corresponding focal line on $S_2$ with the tangent $l_\alpha$ in the straight line $l_\alpha$.

3.3. The characteristics of the fixation carried out

From (20) and (24) we notice that the following expression will be the discriminant of these quadratic equations:

(25)  
$$\Delta = (A_{11}^3A_{22}^4 - A_{22}^3A_{11}^4)^2$$

$$- 4(A_{21}^3A_{22}^4 - A_{21}^4A_{22}^3)(A_{11}^3A_{12}^4 - A_{12}^3A_{11}^4).$$
In this paragraph one considers the case, when
\begin{equation}
\Delta \neq 0
\end{equation}
on the surface \(S_2\) in \(A_4\). In this case each of quadratic equations (21) and (24) will have two different solutions.

It follows from (21)–(26) that the fixation of the affine frame \(\{\vec{r}, \vec{e}_i\}\) of the surface \(S_2\) in \(A_4\), carried out according to formulas (7), is characterized by
\begin{equation}
\begin{align*}
l_1 &= (\vec{A}, \vec{e}_1), \\
l_2 &= (\vec{A}, \vec{e}_2), \\
\Gamma^1_3 &= (\vec{A}, \vec{e}_1, \vec{e}_2, \vec{e}_3), \\
\Gamma^2_3 &= (\vec{A}, \vec{e}_1, \vec{e}_2, \vec{e}_4)
\end{align*}
\end{equation}
so that, the focal lines on \(S_2\) are the coordinate lines and the corresponding tangents:
\begin{equation}
l_1 : \omega^2 = 0, \quad l_2 : \omega^1 = 0.
\end{equation}

### 3.4. Focuses on the straight lines \(l_1\) and \(l_2\)

Let the points
\begin{equation}
\vec{t} = \vec{A} + t\vec{e}_1 \in l_1, \quad \vec{r} = \vec{A} + r\vec{e}_2 \in l_2
\end{equation}
be focuses of the lines \(l_1\) and \(l_2\) respectively. From \((d\vec{t}, \vec{e}_1) = 0, (d\vec{r}, \vec{e}_2) = 0\) by virtue of (1),(3),(8) and (9), we obtain
\begin{equation}
\begin{align*}
l_1 : \omega^2 + t(C^*\omega^1 + B^*\omega^2) &= 0, \\
t\omega^1 &= 0, \\
l_2 : \omega^1 + r(B\omega^1 + C\omega^2) &= 0, \\
r\omega^2 &= 0.
\end{align*}
\end{equation}
Hence, the following formulas are focuses and focal directions
\begin{equation}
\begin{align*}
l_1 : 1) \quad &\vec{A} : \omega^2 = 0, \\
&2) \quad \vec{r}_1 = \vec{A} - \frac{1}{B^*} \vec{e}_1 : \omega^1 = 0; \\
l_2 : 1) \quad &\vec{A} : \omega^1 = 0, \\
&2) \quad \vec{r}_1 = \vec{A} - \frac{1}{B} \vec{e}_2 : \omega^2 = 0.
\end{align*}
\end{equation}
It follows from (31) that by fixation (12) \( (B = 1, \ B^* = 1) \) the vectors \( \vec{e}_1 \) and \( \vec{e}_2 \) are normalized, so that the points 

\[
\begin{align*}
\vec{t}_1 &= \vec{A} - \vec{e}_1 \in l_1, \\
\vec{t}_1' &= \vec{A} - \vec{e}_2 \in l_2
\end{align*}
\]

are focuses of the rays \( l_1 \) and \( l_2 \) respectively. Under the circumstances, the case when \( B = 0 \) (resp. \( B^* = 0 \)), the focus \( t_1 \) (resp. \( t_1' \)) is not an eigen point, that is, a congruence of the straight lines \( l_2 \) (resp. \( l_1 \)) is cylindrical, is taken out of consideration.

## 3.5. The characteristics of the hyperplanes \( \Gamma_3^1 \) and \( \Gamma_3^2 \)

Let the points

\[
\begin{align*}
\vec{X}_1 &= \vec{A} + x^1 \vec{e}_1 + x^2 \vec{e}_2 + x^3 \vec{e}_3 \in \Gamma_3^1, \\
\vec{X}_2 &= \vec{A} + x^1 \vec{e}_1 + x^2 \vec{e}_2 + x^4 \vec{e}_4 \in \Gamma_3^2
\end{align*}
\]

be the current points of the hyperplanes \( \Gamma_3^1 \) and \( \Gamma_3^2 \).

From

\[
(d\vec{X}_1, \vec{e}_1, \vec{e}_2, \vec{e}_3) = 0, \ (d\vec{X}_2, \vec{e}_1, \vec{e}_2, \vec{e}_4) = 0
\]

by virtue of (1), (3), (8), (9) and (27) we obtain

\[
\begin{align*}
x^2 \omega^2 + x^3 (-C^* \omega^2 + E^* \omega^1) &= 0, \\
x^1 \omega^1 + x^4 (-C \omega^1 + E \omega^2) &= 0,
\end{align*}
\]

respectively. It follows from (35) that the next characteristics are the corresponding one of the corresponding hyperplanes along some lines:
2-dimensional surface in 4-dimensional equiaffine space

Table 1

<table>
<thead>
<tr>
<th>for hyperplane $\Gamma_3^1$</th>
<th>for hyperplane $\Gamma_3^2$</th>
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</thead>
<tbody>
<tr>
<td>1) $\omega^1 = 0$, $x^2 = x^4 C^*$, $x^4 = 0$ $\iff$ $\rho_1 = (\vec{A}, \vec{e}_1, C \vec{e}_2 + \vec{e}_3)$</td>
<td>1) $\omega^2 = 0$, $x^1 = x^4 C$, $x^3 = 0$ $\iff$ $\rho_2 = (\vec{A}, \vec{e}_2, C \vec{e}_1 + \vec{e}_4)$</td>
</tr>
<tr>
<td>2) $\omega^2 = 0$, $x^3 = 0$ $\iff$ $L_2 = (\vec{A}, \vec{e}_1, \vec{e}_2) (E^* \neq 0)$</td>
<td>2) $\omega^1 = 0$, $x^4 = 0$ $\iff$ $L_2 = (\vec{A}, \vec{e}_1, \vec{e}_2) (E \neq 0)$</td>
</tr>
<tr>
<td>3) characteristic element $x^2 = x^3 = 0$ $\iff$ $l_1 = (\vec{A}, \vec{e}_1) = \rho_1 \cap L_2$</td>
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</tr>
<tr>
<td>4) $E^* = 0$, $x^2 - C^* x^3 = 0$, $x^4 = 0$</td>
<td>4) $E = 0$, $x^1 - C x^4 = 0$, $x^3 = 0$</td>
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<tr>
<td>characteristic element</td>
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<tr>
<td>When $\omega^2 = 0$</td>
<td>$\Gamma_3^1 \parallel (\Gamma_3^2)$</td>
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<td></td>
<td>When $\omega^1 = 0$</td>
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3.6. The hypercone $K^0_2$

Let us take up the point with the radius vector

$$\vec{X} = \vec{A} + x^\alpha \vec{e}_\alpha + x^{\bar{\alpha}} \vec{e}_{\bar{\alpha}} \in A_4$$

We shall put

$$\vec{X}_1 = \vec{A} + x^\alpha \vec{e}_\alpha + x^3 \vec{e}_3 = \text{Pr}_{\Gamma_3^1} \vec{X},$$

$$\vec{X}_2 = \vec{A} + x^\alpha \vec{e}_\alpha + x^4 \vec{e}_4 = \text{Pr}_{\Gamma_3^2} \vec{X}$$

Let the point $\vec{X}$ be such a point that $\vec{X}_1$ and $\vec{X}_2$ describe the characteristics of the hyperplanes $\Gamma_3^1$ and $\Gamma_3^2$ along the corresponding lines. From (34), we obtain (35)

This system has the non-trivial solutions if and only if $x^1$ satisfy the equation:

$$K^0_2 : x^1 x^2 - C x^2 x^4 - C^* x^1 x^3 - (EE^* - CC^*) x^3 x^4 = 0.$$ 

Thus, the totality of all points $\vec{X} \in A_4$, which are satisfied the point $\vec{A} \in S_2$, so that the corresponding points (37) lie inside the corresponding characteristical hyperplanes $\Gamma_3^1$ and $\Gamma_3^2$, forms a second order hypercone $K^0_2$ in $A_4$ with the vertex at the point $\vec{A}$. This hypercone is defined by equation (38).
It follows from (38) that the plane

\[ \Gamma_2 = (\bar{A}, \bar{e}_2 C^* + \bar{e}_3, C\bar{e}_1 + \bar{e}_4) \]

is polarly associated with the plane \( L_2 \) in \( K_2^0 \).

It follows from table 1 and (39) that after fixation (12) \( (C = 0, C^* = 0) \)

\[ \rho_1 = (\bar{A}, \bar{e}_1, \bar{e}_3), \rho_2 = (\bar{A}, \bar{e}_2, \bar{e}_4), \Gamma_2 = (\bar{A}, \bar{e}_3, \bar{e}_4) \]

Hence,

\[ l_3 = (\bar{A}, \bar{e}_3) = \rho_1 \cap \Gamma_2, \]
\[ l_4 = (\bar{A}, \bar{e}_4) = \rho_2 \cap \Gamma_2. \]

Then the plane

\[ P_2 = l_3 \cup l_4 \]

can be clothings plane of surface \( S_2 \) at a point \( \bar{A} : \)

\[ P_2 \cap \Gamma_2 = \bar{A}, \quad P_2 \cup \Gamma_2 = A_4 \]

Taking into consideration (38) and \( C = 0, C^* = 0 \), we notice that the hypercone \( K_2^0 \) is defined by the equation

\[ K_2^0 : \ x^1 x^2 - EE^* x^3 x^4 = 0. \]

**Theorem 2.** The surface \( S_2 \) in \( A_4 \) of a class \( E = 0 \) (or \( E^* = 0 \)) is characterized by that a hypercone \( K_2^0 \) is degenerated into two hyperplanes

\[ L_3^1 = (\bar{A}, \bar{e}_2, \bar{e}_3, \bar{e}_4) = l_2 \cup \Gamma_2, \]
\[ L_3^2 = (\bar{A}, \bar{e}_1, \bar{e}_3, \bar{e}_4) = l_2 \cup \Gamma_2. \]

**Proof.** From the equations (43), (41) and (42) the validity of the equation (44) follows.
2-dimensional surface in 4-dimensional equiaffine space

References


E. T. Ivlev and O. V. Rozhkova
Department of Higher Mathematics
Tomsk Polytechnic University
Tomsk 634034, Russia

Hai Gon Je
Department of Mathematics
University of Ulsan
Ulsan, 680-749, Korea