PROPERTIES OF PSEUDOCONFORMAL MAPPINGS IN COMPLEX BANACH SPACES

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1. Introduction

T. Higuchi[1] obtained the distribution theorem of holomorphic mappings in several complex variables. P. Liczberski[3] and T. Matsuno[4] investigated the starlikeness of holomorphic mappings in complex vector spaces, separately. And H. J. Kim and K. H. Shon[2] obtained some properties of starlikeness for pseudoconformal mappings in complex Banach spaces. For \((z_1, \ldots, z_n) = z \in \mathbb{C}^n\), define \(|z| = \max_{1 \leq i \leq n} |z_i|\) and let \(D_r = \{z \in \mathbb{C}^n : |z| < r\}\) and \(D = D_1\). Let \(\mathcal{F}\) be the family of \(w : D \to \mathbb{C}^n\) which are holomorphic and satisfy \(w(0) = 0\), \(\text{Re} \left[ \frac{w_i(z)}{z_i} \right] \geq 0\) when \(|z| = |z_i| > 0\), \((1 \leq i \leq n)\), where \(w = (w_1, \ldots, w_n)\).

In this paper, we investigate some properties of starlike mappings with respect to pseudoconformal mappings in complex Banach spaces.

2. Preliminaries

**Definition 2.1.** A holomorphic mapping \(f : D \to \mathbb{C}^n\) is starlike if \(f\) is univalent, \(f(0) = 0\) and \(sf(D) \subset f(D)\) for all \(s \in I = [0, 1]\).

**Definition 2.2** For a system of \(n\) holomorphic functions \(f_j = \ldots\)

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\[ f_j(z) \ (j = 1, 2, \cdots, n), \]

\[
\det \frac{\partial f}{\partial z} = \begin{vmatrix}
\frac{\partial f_1}{\partial z_1} & \cdots & \frac{\partial f_1}{\partial z_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial f_n}{\partial z_1} & \cdots & \frac{\partial f_n}{\partial z_n}
\end{vmatrix} \neq 0
\]

then we call \( f \) a pseudoconformal mapping.

From Theorems 1 and 2 of T. J. Suffridge[5], we have the following theorem.

**Theorem 2.3** The mapping \( f : D \rightarrow \mathbb{C}^n \) is starlike if and only if there exists \( w \in \mathcal{F} \) such that a pseudoconformal mapping \( f = Jw \), where \( f \) and \( w \) are written as column vectors and \( f(0) = 0 \).

**Definition 2.4** If \( f : D \rightarrow \mathbb{C}^n \) is a biholomorphic map of \( D \) onto a convex domain, we say that \( f \) is convex.

T. J. Suffridge[5] proved that for the pseudoconformal mapping \( f : D \rightarrow \mathbb{C}^n \) being biholomorphic and \( f(0) = 0 \), the mapping \( f \) is convex if and only if there exists \( f \) which is univalent of \( D \) onto convex domains such that \( f(z) = T(f_1(z_1), f_2(z_2), \cdots, f_n(z_n)) \), where \( T \) is a nonsingular linear transformation.

3. Starlike mapping in complex Banach spaces

Let \( X \) and \( Y \) be complex Banach spaces and let \( B = \{ x \in X : \| x \| < 1 \} \). For \( 0 \neq x \in X \), let \( T(x) \) be the collection of all continuous real linear functionals \( x^* \) on \( X \) satisfying \( x^*(x) = x \) and \( x^*(y) \leq \| y \| \) for all \( y \in X \). Let \( \mathcal{F}_0(B) \) be the class of mappings \( w : B \rightarrow X \) which are holomorphic, and satisfy \( w(0) = 0 \), and \( x^*(w(x)) \geq 0 \) when \( 0 \neq x \in B \) and \( x^* \in T(x) \). Further let \( \mathcal{F}(B) \) be the class of \( w \in \mathcal{F}_0(B) \) which satisfy \( x^*(w(x)) > 0 \) when \( 0 \neq x \in B \) and \( x^* \in T(x) \).

We can define a starlike map in the complex Banach spaces like a definition of a starlike map in \( \S2 \). That is, a holomorphic mapping \( f : B \rightarrow Y \) is starlike if \( f \) is one-to-one, \( f(0) = 0 \), and \( sf(B) \subset f(B) \) for all \( s \in I \).
THEOREM 3.1[6]. Suppose \( f : B \to Y \) is starlike and that \( f^{-1} \) is holomorphic on an open subset \( f(B) \) of \( Y \). There exists \( w \in \mathcal{F}(B) \) such that \( f(x) = Df(x)w(x) \).

THEOREM 3.2[6]. Let \( f : B \to Y \) be holomorphic and \( f(0) = 0 \). Assume \( Df(x) \) has a bounded inverse for each \( x \in B \) and for some \( w \in \mathcal{F}(B) \), \( f(x) = Df(x)w(x) \). Then \( f \) is starlike.

EXAMPLE 3.3. Define \( f : B \to Y = I^3 \) by \( f(x) = (ax_1, bx_2, cx_3) \) where \( a, b, c \) are arbitrary constants, and \( ||x||^3 = |x_1|^3 + |x_2|^3 + |x_3|^3 \).

Then \( \frac{f(x)}{Df(x)} = w(x) \) where \( w(x) = (x_1, x_2, x_3) \). But for \( 0 \leq t \leq 1 \), let \( v(x, y, t) : B \to B \) be the restriction of the linear map having matrix

\[
\begin{pmatrix}
1 - t & \sqrt{1 - t^2} - 1 & \sqrt{1 - t^2} - 1 \\
\sqrt{1 - t^2} - 1 & 1 - t & \sqrt{1 - t^2} - 1 \\
\sqrt{1 - t^2} - 1 & \sqrt{1 - t^2} - 1 & 1 - t
\end{pmatrix}.
\]

Then \( f \) is starlike.

Let \( \mathcal{K}_0(B) \) be the class of all functions \( w : B \times B \times B \to X \) which are holomorphic in each variable and satisfy \( w(x, x, x) = 0 \) and \( x^*(w(x, y, z)) \geq 0 \) if \( x^* \in T(x) \) and \( \max\{||y||, ||z||\} \leq ||x|| \). Let \( \mathcal{K}(B) \) be the collection of all \( w \in \mathcal{K}_0(B) \) which satisfy \( x^*(w(x, y, z)) > 0 \) when \( x^* \in T(x) \) and \( \max\{||y||, ||z||\} < ||x|| \). The technique of the following theorem is based on the method in T J Suffridge[6].

THEOREM 3.4 If \( w \in \mathcal{K}_0(B) \) and \( |\alpha| < 1 \) then \( \frac{1}{|\alpha|} w(\alpha x, \alpha y, \alpha z) \in \mathcal{K}_0(B) \) (the limit value at \( \alpha = 0 \) is \( Dw(0, 0, 0)(x, y, z) \)). Furthermore if \( x^* \in T(x) \), \( 0 \neq x \in B \) and \( \max\{||y||, ||z||\} \leq ||x|| \), then \( x^*(w(x, y, z)) = 0 \) if and only if \( x^*(Dw(0, 0, 0)) = 0 \).

Proof. For \( 0 < |\alpha| < 1 \), \( x^* \in T(x) \), define \( x_{\alpha}^* \) by

\[
x_{\alpha}^*((x, y, z)) = x^*\left(\frac{\alpha(x, y, z)}{\alpha}\right)
\]

for all \( (x, y, z) \in X \times X \times X \). Then \( x_{\alpha}^* \in T(\alpha x) \). Thus,

\[
0 \leq \frac{1}{|\alpha|} x_{\alpha}^* (w(\alpha x, \alpha y, \alpha z)) = \frac{1}{|\alpha|} x^* \left(\frac{w(\alpha x, \alpha y, \alpha z)}{\alpha}\right)
\]
\[ x^* \left( \frac{w(\alpha x, \alpha y, \alpha z)}{\alpha} \right) = x^* \left( \frac{w(\alpha x, \alpha y, \alpha z)}{\alpha} \right). \]

Since \( x^* \) is continuous, we have

\[ \frac{1}{\alpha} w(\alpha x, \alpha y, \alpha z) \in K_0(B) \]

for \(|\alpha| < 1\). Since \( x^*((x, y, z)) = \text{Re}[x^*((x, y, z) - iw^*(i(x, y, z))] \) is the real part of a continuous complex linear functional

\[ x^* \left( \frac{w(\alpha x, \alpha y, \alpha z)}{\alpha} \right) \]

is nonnegative harmonic of \( \alpha \) for fixed \((x, y, z)\) and \(|\alpha| < \frac{1}{\|x, y, z\|} \).

Since

\[ \frac{1}{\alpha} w(\alpha x, \alpha y, \alpha z) \in K_0(B), \]

we have

\[ x^* \left( \frac{(\alpha x, \alpha y, \alpha z)}{\alpha} \right) \geq 0 \]

if \( x^* \in T(x) \). Hence \( w \) is holomorphic and so

\[ x^* \left( \frac{w(\alpha x, \alpha y, \alpha z)}{\alpha} \right) \]

is harmonic. Therefore

\[ x^* \left( \frac{w(\alpha x, \alpha y, \alpha z)}{\alpha} \right) > 0 \]

or

\[ x^* \left( \frac{w(\alpha x, \alpha y, \alpha z)}{\alpha} \right) \equiv 0 \]

for fixed \((x, y, z)\). Hence we have \( x^*(Dw(0, 0, 0)(x, y, z)) \equiv 0 \).
Pseudoconformal mappings in complex Banach space

References


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