EXPERIMENTAL RESULTS OF W-CYCLE MULTIGRID FOR PLANAR LINEAR ELASTICITY

JAECHIL YOO

Abstract. In [3], Franca and Stenberg developed several Galerkin least squares methods for the solution of the problem of linear elasticity. That work concerned itself only with the error estimates of the method. It did not address the related problem of finding effective methods for the solution of the associated linear systems. In this work, we present computational experiments of W-cycle multigrid method. Computational experiments show that the convergence is uniform as the parameter, \( \nu \), goes to 1/2.

1. Introduction

Let \( \Omega \) be a bounded convex polygonal domain in \( \mathbb{R}^2 \) and \( \partial \Omega \) be the boundary of \( \Omega \). The pure displacement boundary value problem for planar linear elasticity is given in the form

\[
\begin{align*}
2\mu \{ \nabla \cdot \varepsilon(u) + \frac{\nu}{1 - 2\nu} \nabla \cdot u \} + f &= 0 \quad \text{in } \Omega, \\
n &= 0 \quad \text{on } \partial \Omega
\end{align*}
\]

Here \( u = (u_1, u_2) \) denotes the displacement, \( f = (f_1, f_2) \) is the body force, \( \nu \) is Poisson's ratio and \( \mu \) is the shear modulus given by \( \mu = E/(2(1 + \nu)) \) where \( E \) is the Young's modulus. Instead of using Poisson's ratio \( \nu \) and Young's elasticity modulus \( E \), we can also work with the Lamé constants \( \lambda \) and \( \mu \). These constants are related to each other by the following equations:

\[
\lambda = \frac{E\nu}{(1 + \nu)(1 - 2\nu)}, \quad \nu = \frac{\lambda}{2(\lambda + \mu)}.
\]
\[ \mu = \frac{E}{2(1 + \nu)} , \quad E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} . \]

We restrict Poisson's ratio to \( 0 \leq \nu < 1/2 \) where the upper limit corresponds to an incompressible material.

Throughout this paper, we use mesh parameter \( h_k \) and grid level \( k \) which may vary from occurrence to occurrence.

We define various standard differential operators as follows, see [2]:

\[ \nabla \cdot \mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} , \]

\[ \nabla \cdot \mathbf{\tau} = \left( \frac{\partial \tau_{11}}{\partial x} + \frac{\partial \tau_{12}}{\partial y} \right), \quad \nabla \mathbf{v} = \left( \frac{\partial v_1}{\partial x} \quad \frac{\partial v_1}{\partial y} \right), \]

\[ \mathbf{\tau} : \mathbf{\eta} = \sum_{i=1}^{2} \sum_{j=1}^{2} \tau_{ij} \eta_{ij} , \quad \text{and} \quad \varepsilon(\mathbf{v}) = \frac{1}{2} \left[ \nabla \mathbf{v} + (\nabla \mathbf{v})^t \right] . \]

Let \( H^m(\Omega) \) denote the usual Sobolev space of functions with \( L^2(\Omega) \) derivatives up to order \( m \). \( H^m(\Omega) \) is equipped with the norm

\[ ||v||_{H^m(\Omega)} := \left( \int_{\Omega} \sum_{|\alpha| \leq m} |\partial^\alpha v|^2 \, dxdy \right)^{\frac{1}{2}} . \]

We use the following convention for the Sobolev seminorms:

\[ |v|_{H^m(\Omega)} := \left( \int_{\Omega} \sum_{|\alpha| = m} |\partial^\alpha v|^2 \, dxdy \right)^{\frac{1}{2}} . \]

Let \( H^m_0(\Omega) = \{ v \in H^m(\Omega) : v|_{\partial \Omega} = 0 \} \).

It is well known that for \( f \in L^2(\Omega) \), equation (1) has a unique solution \( u \in H^2(\Omega) \cap H^1_0(\Omega) \), see [4].

There is a great deal of literature dealing with approximation schemes for the equations of linear elasticity. To avoid the locking phenomenon in linear elasticity problems, there are several different approaches:
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nonconforming finite element methods, the methods of reduced/selected integration, first order least squares methods, and Galerkin least squares methods. For all of these approaches, mixed finite element methods involving a pair of finite element spaces are commonly used and we have to solve large linear systems arising from the finite element discretizations. With the usual mixed finite element methods, the system is indefinite and hence the problem poses difficulties.

In recent years, modern iterative methods such as multigrid and domain decomposition methods have been applied to mixed finite element methods. Among those iterative methods, the multigrid method has been one of the most popular and fastest methods. So we study the multigrid method to solve the large sparse linear systems derived from the Galerkin least squares method for the pure displacement boundary value problem.

It is well-known that one way of driving stabilized mixed finite element methods is to combine the classical Galerkin formulation with least-squares forms of the differential equations. (See [3] and references therein). An advantage of this method is that the class of finite element spaces that can be used is considerably enlarged, hence the methods are easily incorporated into existing finite element codes. In this paper, we present a scheme of W-cycle multigrid method to solve the linear system arising from P-1 conforming finite element method for the mixed formulation of the pure displacement boundary value problem as in [1], [5] and [6]. We give the computational results of W-cycle multigrid method with \( \alpha/4 \) at the coarse grid and with the constant \( \alpha \), where \( \alpha \) is the stabilization parameter in the Galerkin least squares method. Moreover, we show that the number of iterations for the W-cycle multigrid methods is reduced by a half when we take twice as many smoothings in the algorithm and also reduced by a half when we cut the mesh size by a half. S. Brenner [1] reports very similar results for the pure displacement boundary value problem with the nonconforming finite element method.

This paper is organized as follows. We explain the conforming finite element method in section 2. We discuss the W-cycle multigrid algorithm in section 3. The computational results are presented in section 4.
2. The finite element method

For simplicity, we assume that $2\mu = 1$. Let $p = -\frac{1}{\epsilon} \nabla \cdot u$, where $\epsilon = (1 - 2\nu)/\nu$. Then (1) is equivalent to

$$- \nabla \cdot \epsilon(u) + \nabla p = f \text{ in } \Omega,$$

(2)

$$\epsilon p + \nabla \cdot u = 0 \text{ in } \Omega,$$

$$u = 0 \text{ on } \partial \Omega.$$

Hence, we have the following weak formulation:

Find $(u, p) \in H^1_0(\Omega) \times L^2(\Omega)$ such that

$$\int_{\Omega} \epsilon(u) : \epsilon(v) \, dx \, dy - \int_{\Omega} (\nabla \cdot v) p \, dx \, dy = \int_{\Omega} f \cdot v \, dx \, dy,$$

(3)

$$\epsilon \int_{\Omega} pq \, dx \, dy + \int_{\Omega} (\nabla \cdot u) q \, dx \, dy = 0, \quad \forall q \in L^2(\Omega).$$

Let $T^k$ be a family of triangulations of $\Omega$, where $T^{k+1}$ is obtained by connecting the midpoints of the edges of the triangles in $T^k$. Let $h_T = \text{diam}(T)$ for each $T \in T^k$ and $h_k = \max_{T \in T^k} h_T$, then $h_k = 2h_{k+1}$. Now let's define the conforming finite element spaces for our multigrid method.

$$V_k := \{ v \in C^0(\Omega) ; v|_T \text{ is linear for all } T \in T^k \text{ and } v|_{\partial \Omega} = 0 \},$$

$$P_k := \{ q \in C^0(\Omega) ; q|_T \text{ is linear for all } T \in T^k \} \text{ and}$$

$$\tilde{P}_k := \{ q \in C^0(\Omega) ; q|_T \text{ is linear for all } T \in T^k \text{ and } \int_{\Omega} q \, dx = 0 \}.$$

Then the discretized problem for (3) is the following:

Find $(u_k, p_k) \in V_k \times \tilde{P}_k$ such that

$$B_k((u_k, p_k), (v_k, q_k)) = F_f(v_k, q_k) \quad \forall (v_k, q_k) \in V_k \times \tilde{P}_k$$

(4)
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where

\[
\mathcal{B}_k\left( (u_k, p_k), (v_k, q_k) \right) \\
= \int_\Omega \varepsilon(u_k) : \varepsilon(v_k) \, dx dy - \int_\Omega (\nabla \cdot u_k) q_k \, dx dy - \int_\Omega (\nabla \cdot v_k) p_k \, dx dy \\
- \alpha \sum_{T \in T^k} h_T^2 \int_T \left( -\nabla \cdot \varepsilon(u_k) + \nabla p_k \right) \cdot \left( -\nabla \cdot \varepsilon(v_k) + \nabla q_k \right) \, dx dy \\
- \epsilon \int_\Omega p_k q_k \, dx dy
\]

and

\[
\mathcal{F}_f(v_k, q_k) = \int_\Omega f \cdot v_k \, dx dy - \alpha \sum_{T \in T^k} h_T^2 \int_T f \cdot \left( -\nabla \cdot \varepsilon(v_k) + \nabla q_k \right) \, dx dy.
\]

Note that the bilinear form \(B_k\) is symmetric and indefinite.

3. Multigird algorithm

In this section, we discuss the W-cycle multigrid algorithm.

In order to define the fine-to-coarse operator \(I_k^{k-1}\), we introduce the following mesh-dependent inner product:

\[
\left( (u, p), (v, q) \right)_k := (u, v)_{L^2(\Omega)} + h_k^2 (p, q)_{L^2(\Omega)}.
\]

Then \(I_k^{k-1} : V_k \times P_k \rightarrow V_{k-1} \times P_{k-1}\) is defined by

\[
\left( I_k^{k-1}(u, p), (v, q) \right)_{k-1} = \left( (u, p), (v, q) \right)_k
\]

for all \((u, p) \in V_k \times P_k\) and \((v, q) \in V_{k-1} \times P_{k-1}\).

Define \(B_k : V_k \times P_k \rightarrow V_k \times P_k\) by

\[
\left( B_k(u, p), (v, q) \right)_k = \mathcal{B}_k\left( (u, p), (v, q) \right),
\]

for all \((u, p), (v, q) \in V_k \times P_k\).
Theorem 1. (i) Given \((u, p) \in V_k \times P_k\),

\[(u, p) \in V_k \times \tilde{P}_k \Leftrightarrow \((u, p), (0, 1)\)^k = 0.

(ii) \(I_k^{k-1} : V_k \times \tilde{P}_k \rightarrow V_{k-1} \times \tilde{P}_{k-1}\).

Proof. See [1].

Theorem 2. The subspace \(V_k \times \tilde{P}_k\) is invariant under \(B_k\).

Proof. See [1].

Theorem 3. The spectral radius of \(B_k\) is at most \(Ch_k^{-2}\).

Proof. See [1].

Because of the result of Theorem 3 and indefiniteness of the system, the usual iterative methods are not appropriated to solve our linear system.

The mesh-dependent norms on \(V_k \times \tilde{P}_k\) are defined as follows

\[\|(u, p)\|_{s, k} := \sqrt{\((B_k^2)^s/2(u, p), (u, p)\)^k}\] for all \((u, p) \in V_k \times \tilde{P}_k\).

Note that \(B_k\) is nonsingular and symmetric, hence \(B_k^{-2}\) is positive definite with respect to \((\cdot, \cdot)_k\). Therefore, this norm is well-defined for each \(s \in R\). Moreover,

\[\|(u, p)\|_{0, k} := \sqrt{\|u\|_{L^2(\Omega)}^2 + h_k^2 \|p\|_{L^2(\Omega)}^2}\] for all \((u, p) \in V_k \times \tilde{P}_k\),

\[\|B_k((u, p), (v, q))\| \leq \|(u, p)\|_{2, k} \|(v, q)\|_{0, k}\] for all \((u, p), (v, q) \in V_k \times \tilde{P}_k\),

and

\[\|(u, p)\|_{2, k} = \sup_{(v, q) \in V_k \times \tilde{P}_k \setminus \{(0, 0)\}} \frac{|B_k((u, p), (v, q))|}{\|(v, q)\|_{0, k}}\] for all \((u, p) \in V_k \times \tilde{P}_k\).
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Let

\[
B^*_k - 1 \left( (u, p), (v, q) \right) = \int_{\Omega} \varepsilon(u) : \varepsilon(v) \, dx \, dy - \int_{\Omega} (\nabla \cdot u) q \, dx \, dy - \int_{\Omega} (\nabla \cdot v) p \, dx \, dy - \frac{\alpha}{4} \sum_{T \in T^{k-1}} h_T^2 \left( - \nabla \cdot \varepsilon(u) + \nabla p, - \nabla \cdot \varepsilon(v) + \nabla q \right)_{L^2(T)} - \varepsilon \int_{\Omega} pq \, dx \, dy
\]

and

\[
F^*_f (v, q) = \int_{\Omega} f \cdot v \, dx \, dy - \frac{\alpha}{4} \sum_{T \in T^{k-1}} h_T^2 \left( f, - \nabla \cdot \varepsilon(v) + \nabla q \right)_{L^2(T)}
\]

Note that \( B^*_k - 1 \) and \( F^*_f \) are different from \( B_{k-1} \) and \( F_f \). The difference is in the least squares term. We divide the stabilization parameter \( \alpha \) by 4 to define \( B^*_k - 1 \) and \( F^*_f \).

Define \( \tilde{P}^{k-1}_k : V_k \times \tilde{P}_k \rightarrow V_{k-1} \times \tilde{P}_{k-1} \) by

\[
B^*_k - 1 \left( \tilde{P}^{k-1}_k (u, p), (v, q) \right) = B_k \left( (u, p), (v, q) \right)
\]

for all \( (u, p) \in V_k \times \tilde{P}_k \) and \( (v, q) \in V_{k-1} \times \tilde{P}_{k-1} \).

Now we describe the \( k \)-th level iteration scheme of the conforming W-cycle multigrid algorithm. The \( k \)-th level iteration with initial iterate \( (y_0, z_0) \) yields \( CMG \left( k, (y_0, z_0), (w, r) \right) \) as a conforming approximate solution to the following problem.

Find \( (y, z) \in V_k \times \tilde{P}_k \) such that

\[
B_k (y, z) = (w, r), \quad \text{where} \quad (w, r) \in V_k \times \tilde{P}_k.
\]

For \( k = 1 \), \( CMG \left( 1, (y_0, z_0), (w, r) \right) \) is the solution obtained from a direct method. In other words,

\[
CMG \left( 1, (y_0, z_0), (w, r) \right) = (B_1)^{-1} (w, r).
\]
For $k > 1$, there are two steps.

**Smoothing step**: Let $(y_m, z_m) \in V_k \times \tilde{P}_k$ be defined recursively by the initial iterate $(y_0, z_0)$ and the equations

$$(y_l, z_l) = (y_{l-1}, z_{l-1}) + \frac{1}{\Lambda_k^2} B_k((w, r) - B_k(y_{l-1}, z_{l-1})), \quad 1 \leq l \leq m,$$

where $\Lambda_k := Ch_k^{-2}$ is greater than or equal to the spectral radius of $B_k$, and $m$ is the number of smoothings.

**Correction step**: The coarser-grid correction in $V_k \times \tilde{P}_k$ is obtained by applying the $(k - 1)$-th level conforming iteration. More precisely,

$$(\bar{v}_0, q_0) = (0, 0) \quad \text{and} \quad (v_i, q_i) = CMG(k - 1, (v_0, q_0), (\bar{w}, \bar{r})), \quad i = 1, 2$$

where $(\bar{w}, \bar{r}) \in V_{k-1} \times \tilde{P}_{k-1}$ is defined by $(\bar{w}, \bar{r}) := I^{k-1}_k((w, r) - B_k(y_m, z_m))$.

Then $CMG(k, (y_0, z_0), (w, r)) = (y_m, z_m) + I^{k-1}_k(v_2, q_2)$.

**Remark 1.** In the smoothing step, we use $B_k$ instead of the restriction of $B_k$. Because the space $V_k \times P_k$ has a natural coordinate system which consists of the values of piecewise linear functions at mesh points on the triangles. In view of Theorem 1 and Theorem 2, the result of the smoothing step and the correction step belongs to $V_k \times \tilde{P}_k$. Therefore, in the actual implementation of the multigrid method, we use only the natural coordinate system of $V_k \times P_k$. Note that $B_k$ is represented by a sparse banded matrix and $B_k$ is not invertible.

4. Experimental results

We apply the $W$-cycle multigrid algorithm to the pure displacement boundary value problem (2) studied in [1]. The domain $\Omega$ is the unit square, and the body force $f = (f_1, f_2)$ is taken to be as follows:

$$f_1 = \pi^2 [2 \sin 2\pi y (1 - 2 \cos 2\pi x) - 0.5 \cos \pi (x + y) + \frac{\epsilon}{\epsilon + 2} \sin \pi x \sin \pi y];$$

$$f_2 = \pi^2 [2 \sin 2\pi x (1 - 2 \cos 2\pi y) - 0.5 \cos \pi (x + y) + \frac{\epsilon}{\epsilon + 2} \sin \pi x \sin \pi y].$$
The exact solution \( u = (u_1, u_2) \) is

\[
\begin{align*}
u_1 &= \sin 2\pi y (-1 + \cos 2\pi x) + \frac{e}{\epsilon + 2} \sin \pi x \sin \pi y, \\
u_2 &= \sin 2\pi x (1 - \cos 2\pi y) + \frac{e}{\epsilon + 2} \sin \pi x \sin \pi y.
\end{align*}
\]

The programs execute until the discrete \( L^2 \) relative error is less than 5% of the initial error. We use the initial iterates, \( u^0 = (u_1^0, u_2^0) = (0, 0) \) and \( p^0 = 0 \). The computations were done in double-precision arithmetic for various \( \alpha \)'s, smoothing steps and Poisson's ratio \( \nu \)'s. The numbers in the columns represent the number of iterations to achieve an \( L^2 \) relative error of less than 5% in the displacement.

We know that the number of iterations for the \( W \)-cycle multigrid is reduced in half when we take twice as many smoothings and cut in half when we have the mesh size by a half. We also observe that our multigrid is robust for the moderate \( \alpha \)'s in that the convergence is uniform as the parameter, Poisson’s ratio \( \nu \), goes to 1/2.

Also, we give the numerical experiments with the fixed \( \alpha \) for all levels and with \( \alpha/4 \) at the coarse grid for \( W \)-cycle multigrid methods. A very attractive feature of using the fixed \( \alpha \) for all levels in our CMG algorithm is its inherent simplicity, the bilinear form at the coarse grid is the same form at the fine grid. In other word, the structure of the linear system at the coarse grid is same as that of the linear system at the fine grid. The numerical experiments show that the number of iterations of \( W \)-cycle multigrid method is nearly same in both cases with \( \alpha \) fixed and \( \alpha \) modified.

Note that the size of our linear system is 12675 by 12675 for the case of \( N = 64 \) and 3267 by 3267 for the case of \( N = 32 \).

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Table 1: \( \alpha/4 \) at the coarse grid and \( \nu = 0.3 \)
Table 2: $\alpha/4$ at the coarse grid and $\nu = 0.45$

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Table 3: $\alpha/4$ at the coarse grid and $\nu = 0.495$

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Table 4: $\alpha/4$ at the coarse grid and $\nu = 0.4995$

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Table 5: Fixed $\alpha$ for all levels and $\nu = 0.3$
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### Table 6: Fixed $\alpha$ for all levels and $\nu = 0.45$

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### Table 7: Fixed $\alpha$ for all levels and $\nu = 0.495$

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</table>

### Table 8: Fixed $\alpha$ for all levels and $\nu = 0.4995$

<table>
<thead>
<tr>
<th>smo</th>
<th>$N = 32$</th>
<th>$N = 64$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\alpha = 0.3$</td>
<td>$\alpha = 0.1$</td>
</tr>
<tr>
<td>1</td>
<td>1173</td>
<td>1100</td>
</tr>
<tr>
<td>2</td>
<td>587</td>
<td>550</td>
</tr>
<tr>
<td>3</td>
<td>391</td>
<td>367</td>
</tr>
<tr>
<td>4</td>
<td>294</td>
<td>275</td>
</tr>
</tbody>
</table>

### References


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