A GEOMETRIC CRITERION FOR MEMBERSHIP IN NEW CLASSES $A_{1,1}^2(r)$

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1. Introduction

Let $H$ be a separable, infinite dimensional, complex Hilbert space and let $L(H)$ denote the algebra of all bounded linear operators on $H$. A dual algebra is a subalgebra of $L(H)$ that contains the identity operator $1_H$ and is closed in the ultraweak operator topology on $L(H)$. For $T \in L(H)$, let $A_T$ denote the smallest subalgebra of $L(H)$ that contains $T$ and $1_H$ and is closed in the ultraweak operator topology. Moreover, let $Q_{A_T}$ denote the quotient space $C_1(H)/^\perp A_T$, where $C_1(H)$ is the trace class ideal in $L(H)$ under the trace norm, and $^\perp A_T$ denotes the preannihilator of $A_T$ in $C_1(H)$. For a brief notation, we shall denote $Q_{A_T}$ by $Q_T$. One knows that $A_T$ is the dual space of $Q_T$ and that the duality is given by

$$ (1) \quad \langle A, [L] \rangle = tr(AL), \quad A \in A_T, \quad [L] \in Q_T. $$

The Banach space $Q_T$ is called a predual of $A_T$. For $x$ and $y$ in $H$, we can write $x \otimes y$ for the rank one operator in $C_1(H)$ defined by

$$ (2) \quad (x \otimes y)(u) = (u, y)x, \quad \forall u \in H. $$

The theory of dual algebras is applied to the study of invariant subspaces, dilation theory, and reflexivity. The classes $A_{m,n}$ (to be defined in Section 2) were defined by Bercovici-Foias-Pearcy in [3]. Also these classes are closely related to the study of the theory of dual algebras.

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Especially, B. Chevreau and C. Pearcy [7] defined the properties $E_{\theta,\gamma}$ (to be defined in Section 2), and B. Chevreau, C. Exner and C. Pearcy [6] obtained some new sufficient conditions for membership in the class $A_{1,\mathfrak{m}}$ (to be defined in Section 2) concerning the properties $E_{\theta,\gamma}$. In this paper, we construct new classes and obtain a geometric criterion for membership in the classes $A_{m,n}^l$ (to be defined in Section 3).

2. Notation and preliminaries

The notation and terminology employed herein agree with those in [4], [5], [7], [12]. We shall denote by $D$ the open unit disc in the complex plane $\mathbb{C}$, and we write $T$ for the boundary of $D$. The space $L^p = L^p(T), 1 \leq p \leq \infty$, is the usual Lebesgue function space relative to normalized Lebesgue measure $m$ on $T$. The space $H^p = H^p(T), 1 \leq p \leq \infty$, is the usual Hardy space. It is well–known that the space $H^\infty$ is the dual space of $L^1/H^1_0$, where

$$H^1_0 = \{ f \in L^1 : \int_0^{2\pi} f(e^{it})e^{int}dt = 0, \text{ for } n = 0,1,2,\ldots \},$$

and the duality is given by the pairing

$$\langle f, [g] \rangle = \int_T fgdm \text{ for } f \in H^\infty, \ [g] \in L^1/H^1_0.$$

Recall that any contraction $T$ can be written as a direct sum $T = T_1 \oplus T_2$, where $T_1$ is a completely nonunitary contraction and $T_2$ is a unitary operator. If $T_2$ is absolutely continuous or acts on the space $(0), T$ will be called an absolutely continuous contraction. The following Foiaş-Sz.Nagy functional calculus provides a good relationship between the function space $H^\infty$ and a dual algebra $\mathcal{A}_T$.

**Theorem 2.1** [4, Theorem 4.1]. *Let $T$ be an absolutely continuous contraction in $\mathcal{L}(\mathcal{H})$. Then there is an algebra homomorphism $\Phi_T : H^\infty \to \mathcal{A}_T$ defined by $\Phi_T(f) = f(T)$ such that*

(a) $\Phi_T(1) = 1_{\mathcal{H}}, \Phi_T(\xi) = T_t$,
(b) $\|\Phi_T(f)\| \leq \|f\|_{H^\infty}, \ f \in H^\infty$,
(c) $\Phi_T$ is continuous if both $H^\infty$ and $\mathcal{A}_T$ are given their weak* topologies,*
(d) the range of \( \Phi_T \) is weak* dense in \( \mathcal{A}_T \),
(e) there exists a bounded, linear, one-to-one map \( \phi_T : Q_T \to L^1/H_0^1 \) such that \( \Phi_T^* = \Phi_T \), and
(f) if \( \Phi_T \) is an isometry, then \( \Phi_T \) is a weak* homeomorphism of \( H^\infty \) onto \( \mathcal{A}_T \) and \( \phi_T \) is an isometry of \( Q_T \) onto \( L^1/H_0^1 \).

**Definition 2.2** [3]. Let \( \mathcal{A} \subset \mathcal{L}(\mathcal{H}) \) be a dual algebra and let \( m \) and \( n \) be any cardinal numbers such that \( 1 \leq m, n \leq \aleph_0 \). A dual algebra \( \mathcal{A} \) will be said to have property \((A_{m,n})\) if \( m \times n \) system of simultaneous equations of the form

\[
[x_i \otimes y_j] = [L_{i,j}], \quad 0 \leq i < m, 0 \leq j < n,
\]

where \( \{[L_{i,j}]\}_{0 \leq i < m} \) is an arbitrary \( m \times n \) array from \( Q_A \), has a solution \( \{x_i\}_{0 \leq i < m}, \{y_j\}_{0 \leq j < n} \) consisting of a pair of sequences of vectors from \( \mathcal{H} \). Furthermore, if \( m \) and \( n \) are positive integers and \( r \) is a fixed real number satisfying \( r \geq 1 \), a dual algebra \( \mathcal{A} \) (with property \((A_{m,n})\)) is said to have property \((A_{m,n}(r))\) if for every \( s > r \) and every \( m \times n \) array \( \{[L_{i,j}]\}_{0 \leq i < m} \) from \( Q_A \) such that the rows and columns of the matrix \( \{[L_{i,j}]\}_{0 \leq i < m} \) are summable, there exist sequences \( \{x_i\}_{0 \leq i < m} \) and \( \{y_j\}_{0 \leq j < n} \) from \( \mathcal{H} \) that satisfy (5) and also satisfy the following conditions:

\[
\|x_i\|^2 \leq s \sum_{0 \leq j < n} \|[L_{i,j}]\|, \quad 0 \leq i < m,
\]

and

\[
\|y_j\|^2 \leq s \sum_{0 \leq i < m} \|[L_{i,j}]\|, \quad 0 \leq j < n.
\]

Finally, a dual algebra \( \mathcal{A} \subset \mathcal{L}(\mathcal{H}) \) has property \((A_{m,\aleph_0}(r))\) (for some real number \( r \geq 1 \)) if for every \( s > r \) and every array \( \{[L_{i,j}]\}_{0 \leq i < m, 0 \leq j < \infty} \) from \( Q_A \) with summable rows, there exist sequences \( \{x_i\}_{0 \leq i < m} \) and \( \{y_j\}_{0 \leq j < \infty} \) from \( \mathcal{H} \) that satisfy (5) and (6a, b) with the replacement of \( n \) by \( \aleph_0 \). Properties \((A_{\aleph_0,\aleph_0}(r))\) and \((A_{\aleph_0,\aleph_0}(r))\) are defined similarly. For brief notation, we shall denote \((A_{n,n})\) by \((A_n)\). Furthermore, if \( m \) and \( n \) are cardinal numbers such that \( 1 \leq m, n \leq \aleph_0 \), we denote by \( A_{m,n} = A_{m,n}(\mathcal{H}) \) the set of all \( T \) in \( \mathcal{A}(\mathcal{H}) \) such that the singly generated dual algebra \( \mathcal{A}_T \) has property \((A_{m,n})\).
Definition 2.3 [7]. Suppose $A \subset \mathcal{L}(\mathcal{H})$ is a dual algebra and $0 \leq \theta < \gamma \leq 1$. We denote by $\mathcal{E}_\theta(A)$ (resp. $\mathcal{E}_\delta(A)$) the set of all $[L]$ in $Q_A$ such that there exist sequences $\{x_i\}_{i=1}^\infty$ and $\{y_i\}_{i=1}^\infty$ of vectors from $\mathcal{H}$ satisfying

(a) \( \limsup_{i \to \infty} \|[x_i \otimes y_i] - [L]\| \leq \theta \),

(b) \( \|x_i\| \leq 1, \|y_i\| \leq 1, 1 \leq i < \infty \),

(c') \( \|[x_i \otimes z]\| \to 0 \) for all $z$ in $\mathcal{H}$ (resp. $(c') \|z \otimes y_i\| \to 0$ for all $z$ in $\mathcal{H}$), and

(d') \( \{y_i\} \) converges weakly to zero (resp. $(d') \{x_i\}$ converges weakly to zero).

For $0 \leq \theta < \gamma \leq 1$, the dual algebra $A$ is said to have property $E_{\theta, \gamma}$ (resp. $E_{\delta, \gamma}$) if the closed absolutely convex hull of the set $\mathcal{E}_\theta(A)$ (resp. $\mathcal{E}_\delta(A)$) contains the closed ball $B_{0, \gamma}$ of radius $\gamma$ centered at the origin in $Q_A$:

\[
\overline{\text{co}}(\mathcal{E}_\theta(A)) \supset \{[L] \in Q_A : \|[L]\| \leq \gamma\} = B_{0, \gamma}.
\]

(resp. \( \overline{\text{co}}(\mathcal{E}_\delta(A)) \supset B_{0, \gamma} \))

To establish our results, it will be convenient to use the minimal coisometric extension theorem [12]: every contraction $T$ in $\mathcal{L}(\mathcal{H})$ has a minimal coisometric extension $B = B_T$ that is unique up to unitary equivalence. Given such $T$ and $B$, one knows that there exists a canonical decomposition of the isometry $B^*$ as

\[
B^* = S \oplus R^*
\]

corresponding to a decomposition of the space

\[
\mathcal{K} = S \oplus \mathcal{R},
\]

where, if $S \neq (0)$, $S$ is a unilateral shift operator of some multiplicity in $\mathcal{L}(S)$, and, if $\mathcal{R} \neq (0)$, $R$ is a unitary operator in $\mathcal{L}(\mathcal{R})$. Of course, either $S$ or $\mathcal{R}$ may be $(0)$ (\([7]\))
**Lemma 2.4** [7, Lemma 3.2]. If $T$ is an absolutely continuous contraction in $\mathcal{L}(\mathcal{H})$ with minimal coisometric extension $B$ in $\mathcal{L}(\mathcal{K})$, and the subspace $\mathcal{R}$ of $\mathcal{K}$ in (9) is nonzero, then the unitary operator $R$ in (8) is absolutely continuous.

**Lemma 2.5** [7, Lemma 3.5]. Suppose $T \in A(\mathcal{K})$ and has minimal coisometric extension $B$ in $\mathcal{L}(\mathcal{K})$. Then $B \in A(\mathcal{K}), \Phi_T \circ \Phi_B^{-1}$ is an isometry and weak* homeomorphism from $\mathcal{A}_B$ onto $\mathcal{A}_T$, and $j = \varphi_B^{-1} \circ \varphi_T$ is a linear isometry of $Q_T$ onto $Q_B$. Moreover,

\[(10) \quad j([C_\lambda]_T) = [C_\lambda]_B, \quad \lambda \in D,\]

and

\[(11) \quad j([x \otimes y]_T) = [x \otimes y]_B, \quad x, y \in \mathcal{H}.\]

**Lemma 2.6** [7, Lemma 3.6]. If $T$ belongs to $A(\mathcal{H})$ and has minimal coisometric extension $B$ in $\mathcal{L}(\mathcal{K})$, $x, y \in \mathcal{H}$, and $w, z \in \mathcal{K}$, then

\[(12) \quad \|[x \otimes y]_T\| = \|[x \otimes y]_B\|,\]

\[(13) \quad [x \otimes z]_B = [x \otimes Pz]_B,\]

and

\[(14) \quad [w \otimes z]_B = [Qw \otimes Qz]_B + [Aw \otimes Az]_B.\]

**Lemma 2.7** [7, Lemma 3.7]. Suppose $T \in A(\mathcal{H})$ and has minimal coisometric extension $B$ in $\mathcal{L}(\mathcal{K})$, and $\{x_n\}_{n=1}^{\infty}$ is a sequence from $\mathcal{H}$ such that

\[(15) \quad \|[x_n \otimes y]_T\| \to 0, \quad \forall y \in \mathcal{H},\]

then we have

\[(16) \quad \|[x_n \otimes z]_B\| \to 0, \quad \forall z \in \mathcal{K},\]

\[(17) \quad \|[Qx_n \otimes z]_B\| \to 0, \quad \forall z \in \mathcal{K},\]

and

\[(18) \quad \|[Ax_n \otimes z]_B\| \to 0, \quad \forall z \in \mathcal{K}.\]
Lemma 2.8 [7, Lemma 3.8]. Suppose \( T \in \mathfrak{A}(\mathcal{H}) \) and has \( B \) in \( \mathcal{L}(\mathcal{K}) \) for its minimal coisometric extension. If \( \{z_n\} \) is any sequence in \( \mathcal{K} \) that converges weakly to zero, then

\[
\| [w \otimes z_n]_B \| \to 0, \quad \forall w \in S.
\]

Suppose \( U \) is an absolutely continuous unitary operator in \( \mathcal{L}(\mathcal{N}) \) with spectral measure \( E_U \), and let \( \mu \) be a scalar spectral measure for \( U \). Then one knows, via the absolute continuity, that there exists a Borel set \( \Sigma \subset \mathbb{T} \) such that \( \mu \) is equivalent to Lebesgue measure \( m|_{\Sigma} \) (where this measure is defined to be zero on Borel subsets of \( \mathbb{T}\backslash \Sigma \)). For any vectors \( x \) and \( y \) in \( \mathcal{N} \), let us denote by \( \mu_{x,y} \) the complex measure on \( \mathbb{T} \) defined by

\[
\mu_{x,y}(B) = (E_U(B)x, y)
\]

for every Borel subset \( B \) of \( \mathbb{T} \). Obviously all of these complex measures \( \mu_{x,y} \) are absolutely continuous with respect to the measure \( m|_{\Sigma} \). Therefore, for each pair \( x, y \in \mathcal{N} \), there is a function in \( L^1(\Sigma) \), which is denoted by \( x^U \cdot y \) or \( x \cdot y \), that is the Radon-Nikodym derivatives of \( \mu_{x,y} \) with respect to \( m|_{\Sigma} \). We thus have, of course,

\[
(l(U)x, y) = \int_{\mathbb{T}} l \, d\mu_{x,y} = \int_{\Sigma} l(x \cdot y) \, dm, \quad l \in L^\infty(\Sigma).
\]

Lemma 2.9 [7, Lemma 3.9]. Suppose \( T \in \mathfrak{A}(\mathcal{H}) \) and has \( B = S^* \oplus R \) as its minimal coisometric extension, with \( R \neq (0) \). Then, for every pair of vectors \( w, z \in \mathcal{R} \), we have

\[
[w R z] = \varphi_B([w \otimes z]_B).
\]

Lemma 2.10. For \( k = 1, 2 \), suppose \( T_k \) belongs to \( \mathfrak{A}(\mathcal{H}) \) and has minimal coisometric extension \( B_k = S^*_k \oplus R_k \) in \( \mathcal{L}(\mathcal{K}) \) with \( R_k \neq (0) \). Let \( \Sigma_k \subset \mathbb{T} \) and \( \mathcal{R}_{0,k} \subset \mathcal{R}_k \) be as in [7, Proposition 3.10], and denote the projection of \( \mathcal{K} \) onto \( \mathcal{R}_{0,k} \) by \( A_{0,k} \). Let also \( \epsilon \) and \( \rho \) be arbitrary
real numbers such that $\epsilon > 0$ and $0 < \rho < 1$. If $a_1 \in \mathcal{H}, b_k \in R_k$ and $h_k \in L^1(\Sigma_k)$ are given, and we write $h_{1,k} = (A_k a_1 \cdot b_k) + h_k$, then there exist, for each $k = 1, 2$, $u_k \in \mathcal{H}$ and $c_k \in R_k$ such that

\begin{equation}
\|h_{1,k} - A_k(a_1 + u_k) \cdot c_k\|_1 < \epsilon,
\end{equation}

\begin{equation}
\|Q u_k\| < \epsilon,
\end{equation}

\begin{equation}
\|(A_k - A_{0,k})u_k\| < \epsilon,
\end{equation}

\begin{equation}
\|u_k\| \leq 2\|h_k\|_1^{1/2},
\end{equation}

\begin{equation}
\|c_k\| \leq (1/\rho)\{\|b_k\| + \|h_k\|_1^{1/2}\},
\end{equation}

and

\begin{equation}
c_k - b_k \in R_{0,k},
\end{equation}

where the notation $\|\cdot\|_1$ indicates the norm on $L^1(\Sigma)$.

**Proof.** It is clear from [7, Theorem 3.11].

We shall employ the notation $C_0 = C_0(\mathcal{H})$ for the class of all (completely nonunitary) contractions $T$ in $\mathcal{L}(\mathcal{H})$ such that the sequences $\{T^{*n}\}$ converges to zero in the strong operator topology and is denoted by, as usual, $C_0 = (C_0)^*$, and $N$ is denoted by the set of all natural numbers.

**Lemma 2.11** [8, Theorem 2.1]. Suppose $\{T_k\}_{k=1}^{\infty}$ is any sequence of operators contained in the class $A_{R_0} \cap C_0$, $\{[L_k]_{T_k}\}_{k=1}^{\infty}$ is an arbitrary sequence (where $[L_k]_{T_k} \in Q_{T_k}$), and $\{\epsilon_k\}_{k=1}^{\infty}$ is any sequence of positive numbers. Then there exists a dense set $D \subset \mathcal{H}$ such that for every $x$ in $D$, there exists a sequence $\{y_k^x\}_{k=1}^{\infty} \subset \mathcal{H}$ satisfying

\begin{equation}
[x \otimes y_k^x]_{T_k} = [L_k]_{T_k}, \quad k \in \mathbb{N},
\end{equation}

and

\begin{equation}
\|y_k^x\| > \epsilon_k, \quad k \in \mathbb{N}.
\end{equation}
3. Main results

From the idea of lemma 2.11, we construct new classes as following:

**Definition 3.1.** Let $m, n$ and $l$ be any cardinal numbers such that $1 \leq m, n, l \leq \aleph_0$. We denote by $\mathbb{A}_{m,n}(\mathcal{H})$ the class of all sets $\{T_k\}_{k=1}^l$ such that $T_k$ belong to $A(\mathcal{H})$ for all $k = 1, 2, \ldots, l$ and that every $m \times n \times l$ system of simultaneous equations of the form

$$(31) \quad [x_i \otimes y_j^{(k)}]_{T_k} = [L_{ij}^{(k)}]_{T_k},$$

where $\{[L_{ij}^{(k)}]_{T_k}\}_{0 \leq i < m}$ is an arbitrary $m \times n$ array from $Q_{T_k}$ for each $1 \leq k \leq l$, has a solution $\{x_i\}_{0 \leq i < m}, \{y_j^{(k)}\}_{0 \leq j < n}$ consisting of a pair of sequences of vectors from $\mathcal{H}$. Furthermore, if $m$ and $n$ are positive integers and $r$ is a fixed real number satisfying $r > 1$, then we denoted by $(\mathbb{A}_{m,n}(r))$ the class of all sets $\{T_k\}_{k=1}^l$ such that $T_k$ belong to $A(\mathcal{H})$ for all $k = 1, 2, \ldots, l$ and that every $m \times n \times l$ system of simultaneous equations of the form (31) has a solution $\{x_i\}_{0 \leq i < m}, \{y_j^{(k)}\}_{0 \leq j < n}$ consisting of a pair of sequences of vectors from $\mathcal{H}$ and also satisfy the following conditions:

$$(32) \quad \|x_i\|^2 \leq s \sum_{0 \leq j < n} \|[L_{ij}^{(k)}]_{T_k}\|, \quad 0 \leq i < m, \quad 1 \leq k \leq l$$

and

$$(33) \quad \|y_j^{(k)}\|^2 \leq s \sum_{0 \leq i < m} \|[L_{ij}^{(k)}]_{T_k}\|, \quad 0 \leq j < n, \quad 1 \leq k \leq l.$$
Lemma 3.3. Suppose \( m, n \) and \( l \) are cardinal numbers such that \( 1 \leq m, n, l \leq R_0 \) and \( T_k \in A(\mathcal{H}) \) has minimal coisometric extension \( B_k \) in \( \mathcal{L}(\mathcal{S}_k \oplus \mathcal{R}_k) \) for \( k, 1 \leq k \leq l \). Then \( \{T_k\}_{k=1}^{l} \in A_{m,n}^l \) if and only if for \( \{|[L^{(k)}_{ij}]T_k|\}_0 \leq j \leq n \leq i \leq m \subset Q_{T_k}, 1 \leq k \leq l \), there exists a Cauchy sequence \( \{x_{i,p}\}_{p=1}^{\infty} \) in \( \mathcal{H} \) and sequences \( \{w_{j,p}^{(k)}\}_{p=1}^{\infty} \) in \( \mathcal{S}_k \) and \( \{b_{j,p}^{(k)}\}_{p=1}^{\infty} \) in \( \mathcal{R}_k \) such that \( \{w_{j,p}^{(k)} + b_{j,p}^{(k)}\} \) is bounded and \( \|((\varphi_{B_k}^{-1} \circ \varphi_{T_k})(|[L^{(k)}_{ij}]T_k|) - [x_{i,p} \otimes (w_{j,p}^{(k)} + b_{j,p}^{(k)})]B_k]\| \to 0 \).

Proof. It is clear from [7, Proposition 4.7].

Convention. In the following theorems we assume that \( \mathcal{R}_k \) are either simultaneously (0) or not (0).

Theorem 3.4. For \( k = 1, 2, \) suppose that \( T_k \in A(\mathcal{H}) \) has minimal coisometric extension \( B_k \) in \( C_0(\mathcal{K}) \), and \( A_{T_k} \) has property \( E_{\theta, \gamma}^\prime \) for some \( 0 < \theta < \gamma \leq 1 \). Suppose also that, for each \( k = 1, 2, \) \( 0 < \rho < 1 \), \( |L_k| \in Q_{B_k}, a \in \mathcal{H}, w_k \in \mathcal{S}_k, b_k \in \mathcal{R}_k \), and \( \delta > 0 \) are given such that

\[
\max_k \{\|[L_k]_{B_k} - [a \otimes (w_k + b_k)]_{B_k}\|\} < \delta.
\]

Then there exist \( \hat{a} \in \mathcal{H}, \hat{w}_k \in \mathcal{S}_k, \hat{b}_k \in \mathcal{R}_k, \) \( k = 1, 2 \), such that

\[
\max_{k=1,2} \{\|[L_k]_{B_k} - [\hat{a} \otimes (\hat{w}_k + \hat{b}_k)]_{B_k}\|\} < (\theta/\gamma)\delta,
\]

and

\[
\|\hat{a} - a\| < 6(\delta/\gamma)^{1/2}, \quad \|\hat{w}_k - w_k\| < (\delta/\gamma)^{1/2} \\
\|\hat{b}_k\| < 1/\rho\{\|b_k\| + (\delta/\gamma)^{1/2}\}, \quad k = 1, 2.
\]

Proof. Of course, either of the spaces \( \mathcal{S}_k \) or \( \mathcal{R}_k \) may be zero, for all \( k \), but the proof is unchanged in these special cases. Let

\[
[D_k]_{B_k} = [L_k]_{B_k} - [a \otimes (w_k + b_k)]_{B_k}
\]
and set \( d = \max_k \{ \| [D_k]_{B_k} \| \} \), so, \( 0 < d < \delta \). We may assume that
\( d > 0 \), since otherwise we can simply take \( \hat{a} = a, \hat{w}_k = w_k \) and \( \hat{b}_k = b_k \)
for each \( k = 1, 2 \). And, we choose \( \epsilon > 0 \) such that

\[(\theta / \gamma) d + \epsilon < (\theta / \gamma) \delta.\]

With \( j \) as in Lemma 2.5, note that \( \| (\gamma / d)^{-1}( [D_k]_{B_k} ) \| < \gamma \), and thus, by hypothesis, for each \( k = 1, 2 \), there exist \( N \in \mathbb{N} \), elements \([P_{1,k}], \ldots, [P_{N,k}]\) from \( \mathcal{E}_\delta^\gamma(A_{T_k}) \), and scalars \( \tilde{\alpha}_{1,k}, \ldots, \tilde{\alpha}_{N,k} \) such that

\[\| (\gamma / d)^{-1}( [D_k]_{B_k} ) - \sum_{i=1}^{N} \tilde{\alpha}_{i,k} [P_{i,k}]_{T_k} \| < (\epsilon / 2)(\gamma / d),\]

and \( \sum_{i=1}^{N} |\tilde{\alpha}_{i,k}| < 1, k = 1, 2 \). Upon setting \( \alpha_{i,k} = (d / \gamma) \tilde{\alpha}_{i,k} \), for each \( i, k \), we obtain, by multiplying (39) by \( d / \gamma \),

\[\| (D_k^\gamma)^{-1} ([D_k]_{B_k} ) - \sum_{i=1}^{N} \alpha_{i,k} [P_{i,k}]_{T_k} \| < (\epsilon / 2), \quad k = 1, 2,\]

and

\[\sum_{i=1}^{N} |\alpha_{i,k}| < d / \gamma, \quad k = 1, 2.\]

For each \( i = 1, \ldots, N \), by definition of \( \mathcal{E}_\delta^\gamma(A_{T_k}) \), there exist sequences \( \{x_{n_i,(k)}^{(i,k)}\}_{n_i=1,k=1}^{\infty,2} \) and \( \{y_{n_i,(k)}^{(i,k)}\}_{n_i=1,k=1}^{\infty,2} \) in the unit ball of \( \mathcal{H} \) such that

\[\| [P_{i,k}]_{T_k} - [x_{n_i,(k)}^{(i,k)} \otimes y_{n_i,(k)}^{(i,k)}]_{T_k} \| < \theta + (\epsilon / 2)(\gamma / d), \quad n_i, \in \mathbb{N},\]

\[\lim_{n_i \to \infty} \| [x_{n_i,(k)}^{(i,k)} \otimes z]_{T_k} \| = 0, \quad \forall z \in \mathcal{H},\]

and

\[\{y_{n_i,(k)}^{(i,k)}\}_{n_i=1}^{\infty},\]
converges weakly to zero for each \( k = 1, 2 \). By (40) and (42), we get, for any choice of the \( N \)-tuple \( \nu = (n_1, \cdots, n_N) \),

\[
\| j^{-1}([D_k]_{B_k}) - \sum_{i=1}^{N} \alpha_{i,k}[x_{n_i}^{(i,k)} \otimes y_{n_i}^{(i,k)}]_{T_k} \| < \epsilon/2 + (d/\gamma)\{\theta + (\epsilon/2)(\gamma/d)\} = \epsilon + (d\theta/\gamma)
\]

and, we obtain, using (11),

\[
\|[D_k]_{B_k} - \sum_{i=1}^{N} \alpha_{i,k}[x_{n_i}^{(i,k)} \otimes y_{n_i}^{(i,k)}]_{B_k} \| < \epsilon + (d\theta/\gamma)
\]

for every choice of \( \nu \). Take \( \tau > 0 \) such that

\[
(\theta \delta/\gamma) - \{(d\theta/\gamma) + \epsilon\} = 5\tau.
\]

Using (14) and (37) we may combine (46) and (47) to yield

\[
\|[L_k]_{B_k} - [Qa \otimes w_k]_{B_k} - (A)\| < (\theta \delta/\gamma) - 5\tau,
\]

where

\[
(A) = \sum_{i=1}^{N} \alpha_{i,k}[Qx_{n_i}^{(i,k)} \otimes Qy_{n_i}^{(i,k)}]_{B_k} - [M_k(\nu)]_{B_k}
\]

and

\[
[M_k(\nu)]_{B_k} = [Aa \otimes b_k]_{B_k} + \sum_{i=1}^{N} \alpha_{i,k}[Ax_{n_i}^{(i,k)} \otimes Ay_{n_i}^{(i,k)}]_{B_k}
\]

for every choice of \( \nu \). Let us define, for arbitrary \( \nu = (n_1, \cdots, n_N) \),

\[
u^{(k)} = \sum_{i=1}^{N} \beta_{i,k} x_{n_i}^{(i,k)}, \quad \nu^{(k)} = \sum_{i=1}^{N} \beta_{i,k} y_{n_i}^{(i,k)}.
\]
where \((\beta^{(k)}_i)^2 = \alpha_{i,k}\) for \(i = 1, \cdots, N, \ k = 1, 2\). Then, for every choice of \(\nu\),

\[
[Q(a + u_{\nu}) \otimes (w_k + Qv^{(k)}_{\nu})]_{B_k} = [Qa \otimes w_k]_{B_k} + [Qu_{\nu} \otimes w_k]_{B_k} + [Qa \otimes Qv^{(k)}_{\nu}]_{B_k} + [Qu_{\nu} \otimes Qv^{(k)}_{\nu}]_{B_k}, \ k = 1, 2,
\]

and

\[
\|[Qu_{\nu} \otimes Qv^{(k)}_{\nu}]_{B_k}\| \\
\leq \sum_{i=1}^{N} |\alpha_{i,k}| \|[Qx^{(i,k)}_{n_i} \otimes Qy^{(i,k)}_{n_i}]_{B_k}\| + \sum_{i,j=1 \atop i \neq j}^{N} |\beta^{(k)}_{i} \beta^{(k)}_{j}| \|[Qx^{(i,k)}_{n_i} \otimes Qy^{(j,k)}_{n_j}]_{B_k}\| + \sum_{k_1=1}^{2} \sum_{i,j=1}^{N} |\beta^{(k_1)}_{i} \beta^{(k)}_{j}| \|[Qx^{(i,k_1)}_{n_i} \otimes Qy^{(j,k)}_{n_j}]_{B_k}\|, \ k = 1, 2.
\]

Thus we see from (51), (52), and \(B_k \in C_0, \ k = 1, 2\), it suffices to choose the indices \(n^0_1, \cdots, n^0_N\) (one at a time, in the indicated order) sufficiently large that for \(\nu_0 = (n^0_1, \cdots, n^0_N)\) the following properties are valid:

\[
\|[Qa \otimes Qv^{(k)}_{\nu_0}]_{B_k}\| < \tau/4,
\]

\[
\|[Qu_{\nu_0} \otimes w_k]_{B_k}\| < \tau/4,
\]

\[
\sum_{i,j=1 \atop i \neq j}^{N} |\beta^{(k)}_{i} \beta^{(k)}_{j}| \|[Qx^{(i,k_1)}_{n^0_i} \otimes Qy^{(j,k)}_{n^0_j}]_{B_k}\| < \tau/4,
\]

\[
\sum_{k_1=1}^{2} \sum_{i,j=1}^{N} |\beta^{(k_1)}_{i} \beta^{(k)}_{j}| \|[Qx^{(i,k_1)}_{n^0_i} \otimes Qy^{(j,k)}_{n^0_j}]_{B_k}\| < \tau/4.
\]
A geometric criterion for membership in new classes $A_{1,1}^2(r)$

(57) \[ \| [A u_\nu_o \otimes b_k]_{B_k} \| < \tau, \]

and

(58) \[ \| u_\nu_o \|^2 < 2\delta / \gamma, \quad \| v_{\nu_o}^{(k)} \|^2 < \delta / \gamma, \quad k = 1, 2. \]

Therefore, by combining (51)-(55), we obtain for each $k = 1, 2$,

(59) \[ \| [Q a \otimes w_k]_{B_k} + \sum_{i=1}^{N} \alpha_{i,k} [Q x_{n_i, o}^{(i,k)} \otimes Q y_{n_i, o}^{(i,k)}]_{B_k} \]

\[ - [Q (a + u_\nu_o) \otimes (w_k + Q v_{\nu_o}^{(k)})]_{B_k} \| < \tau. \]

We next define

(60) \[ a_1 = a + u_\nu_o, \quad \tilde{w}_k = w_k + Q v_{\nu_o}^{(k)}, \quad k = 1, 2, \]

and conclude from (60), (59) and (48) that

(61) \[ \| [L_k]_{B_k} - [Q a_1 \otimes \tilde{w}_k]_{B_k} - [M_k(\nu_o)]_{B_k} \| < (\theta \delta / \gamma) - 4\tau, \]

\[ k = 1, 2. \]

Moreover, if in $[M_k(\nu_o)]_{B_k}$ we replace $a$ by $a_1$, and so define, for $k = 1, 2$,

(62) \[ [M_k^{(1)}(\nu_o)]_{B_k} = [A a_1 \otimes b_k]_{B_k} + \sum_{i=1}^{N} [A x_{n_i, o}^{(i,k)} \otimes A y_{n_i, o}^{(i,k)}]_{B_k}, \]

then by (49), (57), (60), and (61) we have

(63) \[ \| [L_k]_{B_k} - [Q a_1 \otimes \tilde{w}_k]_{B_k} - [M_k^{(1)}(\nu_o)]_{B_k} \| < (\theta \delta / \gamma) - 3\tau, \]

\[ k = 1, 2. \]

Now suppose that $\mathcal{R}_k = (0)$, for all $k = 1, 2$. Then $b_k = 0$, $[M_k^{(1)}(\nu_o)]_{B_k} = 0$, $Q a_1 = a_1$, and

\[ \| [L_k]_{B_k} - [a_1 \otimes \tilde{w}_k]_{B_k} \| < (\theta \delta / \gamma) - 3\tau, \quad k = 1, 2. \]
Then, by (60) and (58), we have
\[ ||a - a_1|| < (2\delta/\gamma)^{1/2} \quad \text{and} \quad ||w_k - \tilde{w}_k|| < (\delta/\gamma)^{1/2}, \quad k = 1, 2, \]
so (with \( \tilde{b}_k = 0 \)) the proof in this case is complete.
Hence we may suppose that \( \mathcal{R}_k \neq (0), \quad k = 1, 2, \) we let \( \Sigma_k \subset \mathbb{T} \) be as in Lemma 2.10, and we prepare to apply Lemma 2.10 to deal with the term \( [M_k^{(1)}(\nu_o)]_{B_k} \) in (63). By (62) and Lemma 2.9 we have
\[
(64) \quad \varphi_{B_k}([M_k^{(1)}(\nu_o)]_{B_k}) = [Aa_1 \cdot b_k] + \sum_{i=1}^{N} \alpha_i,k [Ax_{\nu_0}^{(i,k)} \cdot Ay_{\nu_0}^{(i,k)}].
\]
Thus we define the function \( h_k, \quad k = 1, 2, \) in \( L^1(\Sigma_k) \) to be
\[
h_k = \sum_{i=1}^{N} \alpha_i,k (Ax_{\nu_0}^{(i,k)} \cdot Ay_{\nu_0}^{(i,k)}).
\]
We note from (21) and (41) that \( ||h_k||_1 \leq \delta/\gamma, \) and we set \( \epsilon' = \{\tau/(2(||w'|| + 1))\}(<\tau) \) where \( ||w'|| = \max_{k=1,2} ||\tilde{w}_k||. \) With \( a_1 \) and \( b_k, k = 1, 2, \) as in (64), an application of Lemma 2.10 yields the existence of \( \tilde{u}_k \in \mathcal{H} \) and \( c_k \in \mathcal{R}_k, \quad k = 1, 2, \) such that
\[
(65) \quad ||Aa_1 \cdot b_k + \sum_{i=1}^{N} \alpha_i,k (Ax_{\nu_0}^{(i,k)} \cdot Ay_{\nu_0}^{(i,k)}) - A(a_1 + \sum_{k=1}^{2} \tilde{u}_k) \cdot c_k ||_1 < \epsilon' + \tau < 2\tau, \quad k = 1, 2,
\]
\[
(66) \quad ||Q(\sum_{k=1}^{2} \tilde{u}_k)|| < \tau/(||w'|| + 1),
\]
\[
(67) \quad ||\sum_{k=1}^{2} \tilde{u}_k|| \leq 4(\delta/\gamma)^{1/2},
\]
\[
(68) \quad ||c_k|| \leq (1/\rho)\{||b_k|| + ||h_k||^{1/2}\} < \{||b_k|| + (\delta/\gamma)^{1/2}\}, \quad k = 1, 2.
\]
A geometric criterion for membership in new classes $A_{1,1}^2(\tau)$

Since $L^1(\Sigma_k) \subset L^1(\mathbb{T})$ and the norm in $L^1(\mathbb{T})$ dominates the norm in $(L^1/H^1_0)(\mathbb{T})$, we obtain using (62), (22), and (65),

$$\|M_k^{(1)}(\nu_o)|_{B_k} - [A(a_1 + \sum_{k=1}^{2} \tilde{u}_k) \otimes c_k]|_{B_k} \| < 2\tau, \quad k = 1, 2. \quad (69)$$

Thus from (63) and (69) we get

$$\|L_k|_{B_k} - [Qa_1 + \tilde{w}_k]_{B_k} - [A(a_1 + \sum_{k=1}^{2} \tilde{u}_k) \otimes c_k]|_{B_k} \|
\leq \|L_k|_{B_k} - [Qa_1 + \tilde{w}_k]_{B_k} - [M_k^{(1)}(\nu_o)]_{B_k} \|
+ \|M_k^{(1)}(\nu_o)|_{B_k} - [A(a_1 + \sum_{k=1}^{2} \tilde{u}_k) \otimes c_k]|_{B_k} \|
< \langle \theta \delta / \gamma \rangle - 3\tau + 2\tau = \langle \theta \delta / \gamma \rangle - \tau, \quad k = 1, 2, \quad (70)$$

and since, by (66), we have

$$\|Q(\sum_{k=1}^{2} \tilde{u}_k) \otimes \tilde{w}_k|_{B_k} \| \leq \|Q(\sum_{k=1}^{2} \tilde{u}_k)\| \cdot \|\tilde{w}_k\|
< (\tau/(\|w'\| + 1))\|w'\| < \tau, \quad k = 1, 2, \quad (71)$$

the inequality (70) yields

$$\|L_k|_{B_k} - [Q(a_1 + \sum_{k=1}^{2} \tilde{u}_k) \otimes \tilde{w}_k]_{B_k} - [A(a_1 + \sum_{k=1}^{2} \tilde{u}_k) \otimes c_k]|_{B_k} \|
< \langle \theta \delta / \gamma \rangle, \quad k = 1, 2. \quad (72)$$

Since $\tilde{w}_k \in S_k$ and $c_k \in R_k$, by using (14) one can rewrite (72) as

$$\|L_k|_{B_k} - [(a_1 + \sum_{k=1}^{2} \tilde{u}_k) \otimes (\tilde{w}_k + c_k)]_{B_k} \| < \langle \theta \delta / \gamma \rangle, \quad k = 1, 2.$$
So if we define

\[ \hat{a} = a_1 + \sum_{k=1}^{2} \bar{u}_k = a + u_{\nu_0} + \bar{u}_1 + \bar{u}_2 \]

\[ \hat{b}_k = c_k, \quad \hat{w}_k = \bar{w}_k, \quad k = 1, 2, \]

then (35) is satisfied. Moreover,

\[ \| \hat{a} - a \| \leq \| u_{\nu_0} \| + \| \sum_{k=1}^{2} \bar{u}_k \| < (2\delta/\gamma)^{1/2} + 4(\delta/\gamma)^{1/2} = 6(\delta/\gamma)^{1/2}, \]

from (58) and (67), so the first inequality in (35) is satisfied. Furthermore, from (60) and (58) we have

\[ \| \hat{w}_k - w_k \| \leq \| Q u_{\nu_0}^{(k)} \| < (\delta/\gamma)^{1/2}, \quad k = 1, 2. \]

Finally,

\[ \| \hat{b}_k \| = \| c_k \| < (1/\rho) \{ \| b_k \| + (\delta/\gamma)^{1/2} \}, \quad k = 1, 2. \]

We are ready to prove main theorems.

**Theorem 3.5.** For \( k = 1, 2 \), suppose that \( T_k (\in A(\mathcal{H})) \) has minimal coisometric extension \( B_k \) in \( C_0(\mathcal{K}) \), and \( A_{T_k} \) has property \( E_{\theta, \gamma}' \) for some \( 0 \leq \theta < \gamma \leq 1 \). Suppose also that \( \delta > 0, \) \( [L_k] \in Q_{T_k}, \) \( a \in \mathcal{H}, \) \( w_k \in S_k \) and \( b_k \in R_k \) are given such that

(73) \[ \max_{k=1,2} \{ \| [L_k]T_k - [a \otimes P(w_k + b_k)]T_k \| \} < \delta, \]

where \( P \) is the projection of \( \mathcal{K} \) onto the subspace \( \mathcal{H} \). Then there exist \( \hat{a} \in \mathcal{H}, \) \( \hat{w}_k \in S_k \) and \( \hat{b}_k \in R_k \) such that

(74) \[ [L_k]T_k = [\hat{a} \otimes P(\hat{w}_k + \hat{b}_k)]T_k, \quad k = 1, 2, \]

(75) \[ \| \hat{a} - a \| < 6(\delta/\gamma)^{1/2}(1/\{1 - (\theta/\gamma)^{1/2}\}), \]

(76) \[ \| \hat{w}_k - w_k \| < (\delta/\gamma)^{1/2}(1/\{1 - (\theta/\gamma)^{1/2}\}), \quad k = 1, 2, \]

and

(77) \[ \| \hat{b}_k \| < 2\| b_k \| + 2(\delta/\gamma)^{1/2}(1/\{1 - (\theta/\gamma)^{1/2}\}), \quad k = 1, 2 \]
Proof. Since if \( \mathcal{A}_T k \) has property \( E^*_\theta, \gamma \) for each \( k \), it also has property \( E^{T}_{\theta, \gamma} \) for all \( 0 < \theta < \gamma \), the right-hand side of (75), (76), and (77) are continuous functions of \( \theta \) and \( \delta \), it suffices to treat the case \( 0 < \theta < \gamma \). Suppose now that (73) holds, let \( \{ s_n \} \) be a sequence of positive numbers strictly decreasing to \( 3/4 \) such that \( s_1 = 1 \), and define \( \rho_n = (s_{n+1}/s_n), \ n \in \mathbb{N} \). Set

\[
[\hat{L}_k]_{B_k} = \varphi^{-1}_{B_k} \circ \varphi_{T_k}([L_k]), \quad k = 1, 2.
\]

Then we have, by (73), (11), (12), and (13),

\[
\max_{k=1,2} \{ \|[\hat{L}_k]_{B_k} - [a \otimes (w_k + b_k)]_{B_k}\| \} < \delta.
\]

We now set

\[
a = a_1, \ w_k = w_{1,k}, \ b_k = b_{1,k}, \quad k = 1, 2,
\]

and apply Theorem 3.4 to obtain \( a_2 \in \mathcal{H}, \ w_{2,k} \in S_k \) and \( b_{2,k} \in \mathcal{R}_k, \ k = 1, 2, \) such that

\[
\max_{k=1,2} \{ \|[\hat{L}_k]_{B_k} - [a_2 \otimes (w_{2,k} + b_{2,k})]_{B_k}\| \} < (\theta/\gamma)\delta,
\]

\[
\|a_2 - a_1\| < 6(\delta/\gamma)^{1/2}, \quad \|w_{2,k} - w_{1,k}\| < (\delta/\gamma)^{1/2},
\]

\[
\|b_{2,k}\| < (1/\rho_1)\{\|b_{1,k}\| + (\delta/\gamma)^{1/2}\}, \quad k = 1, 2.
\]

Suppose now that vectors \( \{ a_p \}_{p=1}^n \) in \( \mathcal{H} \), \( \{ w_{p,k} \}_{p=1}^n \) in \( S_k \), and \( \{ b_{p,k} \}_{p=1}^n \) in \( \mathcal{R}_k \), have been chosen so that for \( p = 2, \cdots, n, \ k = 1, 2, \)

\[
\max_{k=1,2} \{ \|[\hat{L}_k]_{B_k} - [a_p \otimes (w_{p,k} + b_{p,k})]_{B_k}\| \} < (\theta/\gamma)^{p-1}\delta,
\]

\[
\|a_p - a_{p-1}\| < 6(\delta/\gamma)^{1/2}(\theta/\gamma)^{(p-2)/2},
\]

\[
\|w_{p,k} - w_{p-1,k}\| < (\delta/\gamma)^{1/2}(\theta/\gamma)^{(p-2)/2},
\]

\[
(\theta/\gamma)^{(p-2)/2},
\]
and

\[ (84_p) \quad \|b_{p,k}\| < (1/\rho(p-1)}\{\|b_{p-1,k}\| + (\delta/\gamma)^{1/2}(\theta/\gamma)^{(p-2)/2}\}. \]

Then, applying Theorem 3.4, we deduce the existence of vectors \(a_{n+1}\) in \(\mathcal{H}\), \(w_{n+1,k}\) in \(S_k\), and \(b_{n+1,k}\) in \(\mathcal{R}_k\) such that the inequalities \((81)_{n+1}\), \((82)_{n+1}\), \((83)_{n+1}\), and \((84)_{n+1}\) are valid. Therefore, by induction, there exist sequences \(\{a_n\}_{n=1}^{\infty}\) in \(\mathcal{H}\), \(\{w_{n,k}\}_{n=1}^{\infty}\) in \(S_k\), \(k = 1, 2\), and \(\{b_{n,k}\}_{n=1}^{\infty}\) in \(\mathcal{R}_k\), \(k = 1, 2\), satisfying the appropriate inequalities for all \(n \in \mathbb{N}\), and it is clear from \((82)_p\) and \((83)_p\) that \(\{a_n\}\) and \(\{w_{n,k}\}\) are Cauchy, for each \(k = 1, 2\). Define

\[ \hat{a} = \lim_{n \to \infty} a_n, \]
\[ \hat{w}_k = \lim_{n \to \infty} w_{n,k}, \quad k = 1, 2, \]

and observe that since

\[ \|\hat{a} - a\| = \left\| \sum_{p=2}^{\infty} (a_p - a_{p-1}) \right\| \]
\[ \leq \sum_{p=2}^{\infty} \|a_p - a_{p-1}\| \]
\[ = 6(\delta/\gamma)^{1/2}(1/(1 - (\theta/\gamma)^{1/2})) \],

and

\[ \|\hat{w}_k - w_k\| < (\delta/\gamma)^{1/2}(1/(1 - (\theta/\gamma)^{1/2})) \],

inequalities \((75)\) and \((76)\) are satisfied. Furthermore, by iterating \((84)_p\), we see that

\[ \frac{1}{2} \|b_{n,k}\| \leq s_n \|b_{n,k}\| \]
\[ \leq \|b_k\| + (\delta/\gamma)^{1/2} \sum_{p=1}^{n-1} s_p(\theta/\gamma)^{(p-1)/2}, \]

and therefore that

\[ \|b_{n,k}\| \leq 2\|b_k\| + 2(\delta/\gamma)^{1/2}(1/(1 - (\theta/\gamma)^{1/2})), \quad n \in \mathbb{N}, \quad k = 1, 2. \]

Thus the sequence \(\{b_{n,k}\}\) is bounded and w.l.o.g., we may suppose that \(\{b_{n,k}\}\) converges weakly to \(\hat{b}_k\). Hence

\[ \|\hat{b}_k\| \leq 2\|b_k\| + 2(\delta/\gamma)^{1/2}(1/(1 - (\theta/\gamma)^{1/2})), \quad k = 1, 2, \]

which establishes \((77)\). That \((74)\) is valid now follows from \((81)_p\) as in the proof of Lemma 3.3.
Theorem 3.6. Under the hypotheses of Theorem 3.5, suppose that $\|[L_1]T_1\|$ and $\|[L_2]T_2\|$ are equal. Then, we have $\{T_1, T_2\} \in A_{1,1}^2(\tau(\theta, \gamma))$, where

$$r(\theta, \gamma) = (18/\gamma)(1/(1 - (\theta/\gamma)^{1/2}))^2.$$  

Proof. By theorem 3.5, the set $\{T_1, T_2\}$ certainly belongs to some $A_{1,1}^2(\tau)$. To see that $\tau$ may be taken to be as in (85), let $\epsilon > 0$ and set $a = 0$, $w_k = 0$, $b_k = 0$ and $\delta = \max_{k=1,2} \{\|[L_k]r_k\|\} + \epsilon$ in (73). Then from (75), (76) and (77), we see that

$$\|\hat{\alpha}\| \|[\hat{\rho}(\hat{w}_k + \hat{b}_k)\|$$

$$\leq \|\hat{\alpha}\| (\|\hat{w}_k\| + \|\hat{b}_k\|)$$

$$< 6(\delta/\gamma)^{1/2}(1/(1 - (\theta/\gamma)^{1/2}))$$

$$[(\delta/\gamma)^{1/2}(1/(1 - (\theta/\gamma)^{1/2})) + 2(\delta/\gamma)^{1/2}(1/(1 - (\theta/\gamma)^{1/2}))]$$

$$= 18(\delta/\gamma)(1/(1 - (\theta/\gamma)^{1/2}))^2$$

$$= (18/\gamma)(1/(1 - (\theta/\gamma)^{1/2}))^2(\|[L_k]r_k\| + \epsilon),$$

by the hypothesis. Therefore, we have

$$\{T_1, T_2\} \in A_{1,1}^2(\tau(\theta, \gamma)),$$

where

$$r(\theta, \gamma) = (18/\gamma)(1/(1 - (\theta/\gamma)^{1/2}))^2.$$  

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